Time minimization problem on the Heisenberg group with admissible control in a half-disk

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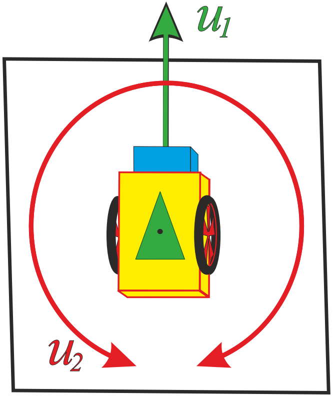
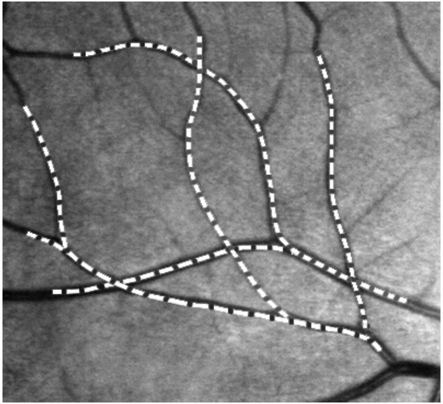
Outline of the Talk

1. Motivation
2. Preliminaries
3. History of the problem
4. Statement of the problem
5. Existence of the solution
6. Pontryagin maximum principle
7. Expression of the extremals
8. Optimal synthesis

Motivation

- Model example of nonholonomic system.
- Nilpotent approximation for a car-like robot.
- Extraction of salient curves in images on curved surfaces.





History of the Problem

- (B. Gaveau, 1977)
  Statement of the Dido problem (sub-Riemannian problem on the Heisenberg group).
- (R.W. Brockett, 1980)
  Optimal control formulation, sub-Riemannian sphere.
- (A.M. Vershik, V.Ya. Gershkovich, 1987)
  Compete analysis of the Dido problem.
- (A.O. Chernyshev, A.P. Mashtakov, 2021, Sirius)
  Extremal trajectories on the Heisenberg group with a positive control.
- (this work)
  Structure of optimal synthesis.

Preliminaries

- The Heisenberg group $\mathrm{H}_3 =: M \simeq \mathbb{R}^2_{x,y} \times \mathfrak{so}_2 \ni Q$, $M \simeq \mathbb{R}^3_{x,y,z} \ni q$:

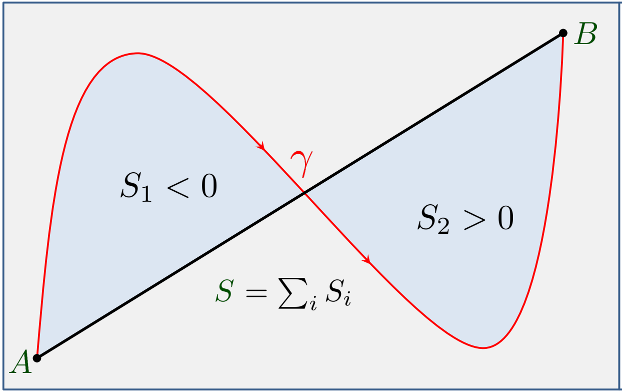
$$QQ' = \big((x,y),Z\big)\,\big((x',y'),Z'\big) = \big((x+x',y+y'),Z+Z'+(x,y)\wedge(x',y')\big)\,.$$

  where $v \wedge w = v \otimes w^T - w \otimes v^T$.
  The Lie algebra $\mathfrak{h}_3 = \mathrm{span}(X_1, X_2, X_3)$, where

$$X_1 = \partial_x - \frac{y}{2}\,\partial_z, \quad X_2 = \partial_y + \frac{x}{2}\,\partial_z, \quad X_3 = \partial z.$$

- By given a dynamics on M, an extremal trajectory is called a trajectory that satisfies the necessary optimality condition — Pontryagin maximum principle (PMP).
- The wavefront is a set of all points in configuration space M, reachable by all the extremal trajectories in a fixed time T.
- Cut point is a point, where the extremal trajectory loses its optimality.

The Dido Problem — Classical Sub-Riemannian Problem on H_3



Given

$A, B \in \mathbb{R}^2$

$S \in \mathbb{R}$

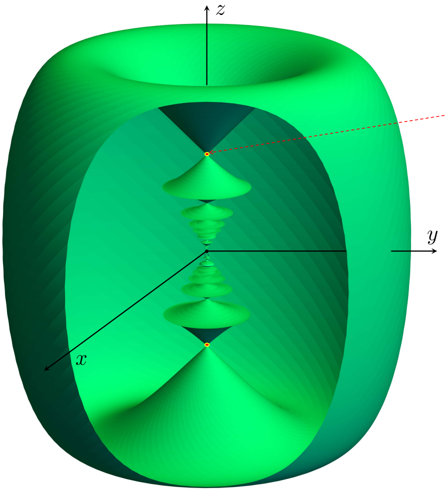
Find γ

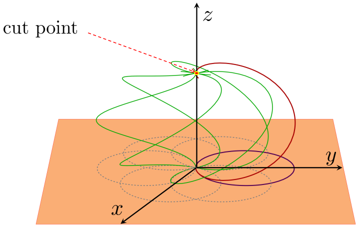
$l(\gamma) \to \min$

$$\dot{x}(t) = u_1(t),\ \dot{y}(t) = u_2(t),\ \dot{z}(t) = \frac{1}{2}\left(x(t)u_2(t) - y(t)u_1(t)\right),\ t \in [0, T]$$

$$(x, y)(0) = A,\ z(0) = 0,\ (x, y)(T) = B,\ z(T) = S,\ \ u_1^2 + u_2^2 \leq 1,\ T \to \min\,.$$

Classical Result: Solution to the Dido Problem



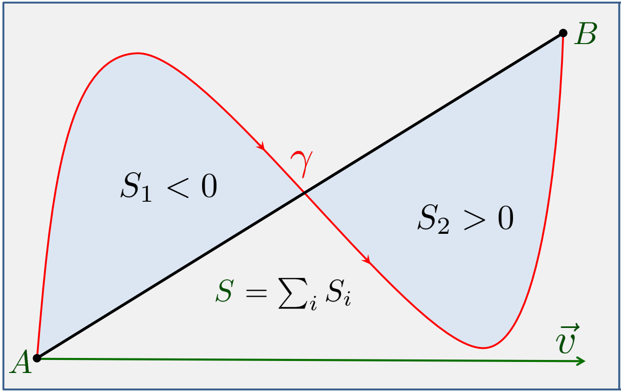


$(x(t), y(t))$ — arcs of circles

$(x(t), y(t), z(t))$ — arcs of helices

$x = 0, y = 0, z \neq 0$ — cut locus

Formulation of the Modified Dido Problem (New)



Given
A,$B \in \mathbb{R}^2$
$S \in \mathbb{R}$
$\vec{v} \in \mathbb{R}^2$

Find γ
monotone
along $\vec{v}$
$l(\gamma) \to \min$

$$\dot{x}(t) = u_1(t),\ \dot{y}(t) = u_2(t),\ \dot{z}(t) = \frac{1}{2}\left(x(t)u_2(t) - y(t)u_1(t)\right),\ t \in [0, T]$$

$$(x, y)(0) = A,\ z(0) = 0,\ (x, y)(T) = B,\ z(T) = S,\ \ u_1^2 + u_2^2 \leq 1, \mathbf{u_1 \geq 0},\ T \to \min .$$

Formal Statement of the Problem

Consider the following control system (dynamics):

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2, \\ \dot{z} = \frac{1}{2}\left(x u_2 - y u_1\right), \end{cases} \qquad \left| \qquad \begin{array}{l} (x, y, z) = q \in M, \\ u_1^2 + u_2^2 \leq 1, \\ u_1 \geq 0. \end{array} \right.$$

By given $q_0 = (0, 0, 0)$, $q_1 \in M$ we aim to find the controls $u_1(t)$, $u_2(t)$ such that the corresponding trajectory $\gamma : [0, T] \to M$ transfers the system from q_0 to q_1 by minimal time

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min .$$

Here u_i are $L^\infty([0, T], \mathbb{R})$, and γ is a Lipschitzian curve on M.

Existence of the solution

Theorem 1. In the time minimization problem for the left-invariant control system on the Heisenberg group with admissible control in a half-disk, there exists an optimal trajectory that transfers the system from the identity to any configuration of the admissible set

$$\mathcal{A} = \{q \in \mathbb{R}^3 \,|\, x > 0\} \cup \{q \in \mathbb{R}^3 \,|\, x = 0,\ z = 0\}.$$

Proof by construction. Let $(x_0, y_0, z_0) \in \mathcal{A}$. *Control in two steps:*
1) $u_1 \geq 0, u_2 \in \mathbb{R}$ *s.t.,* $(0,0,0) \to (x_0, y', 0)$,
2) $u_1 = 0,\ u_2 = f(t)$ *s.t.,* $y_0 = \int_0^{t_1} f(t)dt + y',\ z_0 = \frac{x_0}{2}\int_0^{t_1} f(t)dt,$ *where* $y' = y_0 - \frac{2z_0}{x_0}$.
Existence of optimal trajectories is guaranteed by the Filippov theorem due to compactness and convexity of the set of admissible control.

The control system is not globally controllable $\mathcal{A} \neq \mathrm{H}_3$.
$x(t) = \int_0^t u_1(\tau)\,\mathrm{d}\,\tau \geq 0$ for $t > 0$.

Pontryagin Maximum Principle (PMP)

A necessary condition of optimality is given by PMP.

- Denote $(p_1, p_2, p_3) \in T_q^* M \simeq \mathbb{R}^3$. The Pontryagin function is given by

$$H_u = u_1(p_1 - p_3 \frac{y}{2}) + u_2(p_2 + p_3 \frac{x}{2}).$$

- Let $(u(t), q(t))$, $t \in [0, T]$ be an optimal process. Then the following conditions hold:
  - Hamiltonian system $\dot{p} = -\dfrac{\partial H_u}{\partial q}$, $\dot{q} = \dfrac{\partial H_u}{\partial p}$;
  - Maximum condition $H := \max\limits_{\bar{u} \in U} H_{\bar{u}}(p(t), q(t)) = H_{u(t)}(p(t), q(t)) \in \{0, 1\}$;
  - Non-triviality condition $p_1^2 + p_2^2 + p_3^2 + H^2 \neq 0$.

Pontryagin Maximum Principle (PMP)

Introduce left-invariant Hamiltonians $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$:

$$h_1 = p_1 - p_3 \frac{y}{2}, \quad h_2 = p_2 + p_3 \frac{x}{2}, \quad h_3 = p_3.$$

The Pontryagin function reads as $H_u = u_1 h_1 + u_2 h_2$.
The Hamiltonian system is given by

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2, \\ \dot{z} = \frac{1}{2}\left(x u_2 - y u_1\right), \end{cases} \qquad \begin{cases} \dot{h}_1 = -u_2 h_3, \\ \dot{h}_2 = u_1 h_3, \\ \dot{h}_3 = 0. \end{cases}$$

The nontriviality condition implies that if $h_1 = h_2 = 0$ then the extremal is trivial.

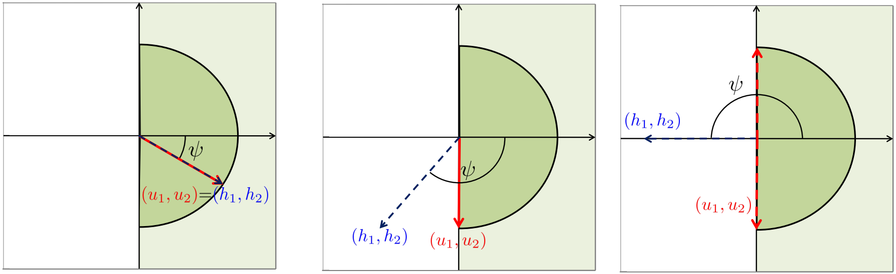
Pontryagin Maximum Principle (PMP)

Let $h_1 = \rho \cos \psi$, $h_2 = \rho \sin \psi$, $\psi \in (-\pi, \pi]$, $\rho > 0$.

The maximum condition implies the following:

- For $\psi = \pi$ we have $H = 0$, $u_1 = 0$, $u_2 \in [-1, 1]$.
- For $|\psi| \in (\frac{\pi}{2}, \pi)$ we have $H = |h_2|$, $u_1 = 0$, $u_2 = \operatorname{sign} h_2$.
- For $|\psi| \leq \frac{\pi}{2}$ we have $H = \sqrt{h_1^2 + h_2^2}$, $u_1 = \cos \psi$, $u_2 = \sin \psi$.

Note, $H = 0$ iff $\psi = \pi$. Thus, **the abnormal extremals satisfy** $u_1 = 0$, $u_2 \in [-1, 1]$.



Abnormal extremals

Theorem 2. Abnormal extremal control exists when $h_1 < 0, h_2 = 0$ and has a form $u_1(t) = 0$, $u_2(t) \in I = [-1, 1]$ — arbitrary $L_\infty([0, T], I)$ function that satisfies the condition

$$h_{10} - h_{30}\, U_2(t) < 0, \text{ where } U_2(t) = \int_0^t u_2(\tau) d\tau, \qquad t \in [0, T].$$

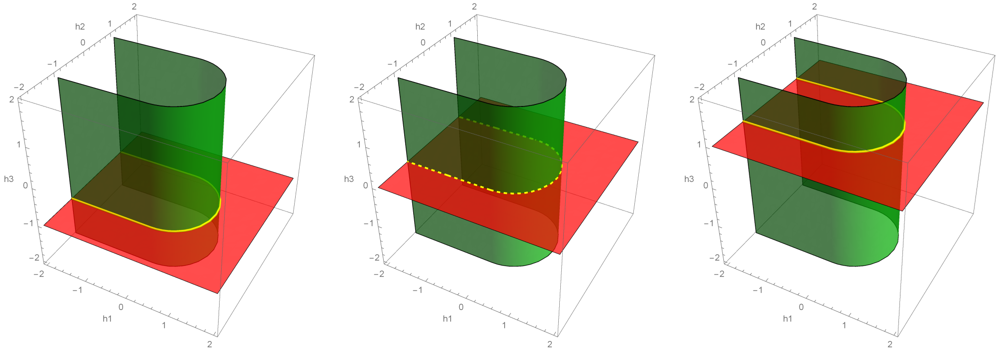
Theorem 3. Abnormal extremal trajectories have a form

$$x(t) = 0, \quad y(t) = U_2(t), \quad z(t) = 0.$$

Theorem 4. Abnormal optimal trajectories have a form

$$x(t) = 0, \quad y(t) = \pm t, \quad z(t) = 0.$$

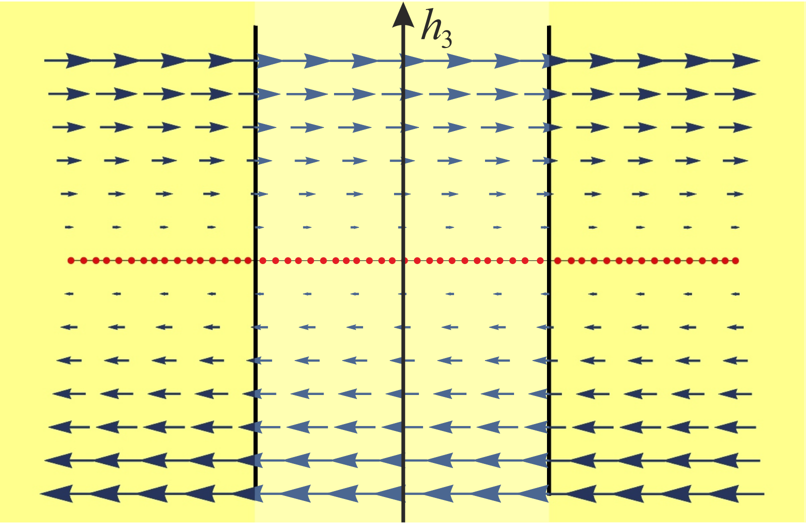
First Integrals of the Normal Hamiltonian System



Hamiltonian $H = \begin{cases} |h_2|, & \text{for } |\psi| \in (\frac{\pi}{2}, \pi), \\ \sqrt{h_1^2 + h_2^2}, & \text{for } |\psi| \leq \frac{\pi}{2}. \end{cases}$

Casimir $E = h_3$.

Phase Portrait on the Level Surface of the Hamiltonian



Normal Hamiltonian System

For $h_{10} < 0$:

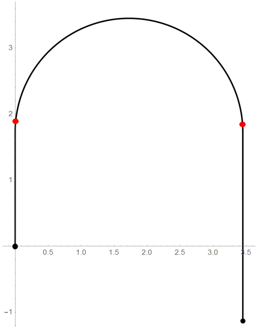
$$\begin{cases} \dot{x} = 0, & x(t_0) = x_0, \\ \dot{y} = h_2, & y(t_0) = y_0, \\ \dot{z} = \frac{x_0 h_2}{2}, & z(t_0) = z_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = 0, & h_2(t_0) = h_2^0 = \pm 1, \\ \dot{h}_3 = 0, & h_3(t_0) = h_{30}. \end{cases}$$

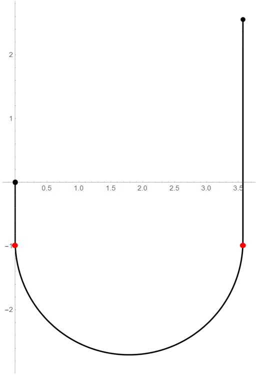
For $h_{10} \geq 0$:

$$\begin{cases} \dot{x} = h_1, & x(t_0) = 0, \\ \dot{y} = h_2, & y(t_0) = y_0, \\ \dot{z} = \frac{1}{2}(x h_2 - y h_1), & z(t_0) = 0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = h_1 h_3, & h_2(t_0) = h_{20}, \\ \dot{h}_3 = 0, & h_3(t_0) = h_{30}. \end{cases}$$

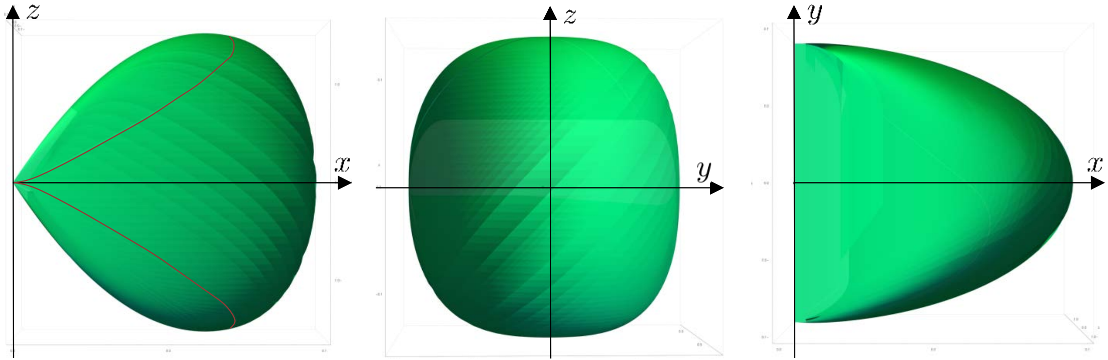
Structure of Optimal Synthesis

Theorem. For any $q \in \mathcal{A}$, there exists a unique optimal trajectory, arriving at q.
The optimal trajectory (x, y) consists of three segments (possibly zero length):
1) segment of a line parallel to O_y; 2) arc of a circle; 3) segment of a line parallel to O_y.
Proof relies on monotonicity of the function z.





Picture of the Wavefront



Conclusion

Summary:

- Left-invariant time minimization problem in H_3 with admissible controls in a half-disk.
- Applications in robotics and image processing.
- Proof of existence of optimal control.
- Necessary optimality condition — PMP.
- Qualitative analysis of dynamics.
- Explicit formulas for optimal controls and trajectories.
- Structure of optimal synthesis.

Plans:

- Explicit optimal synthesis.
- Similar problems in SO_3 and SL_2.

Thank you for your attention!