Time minimization problem on the Heisenberg group with admissible control in a half-disk

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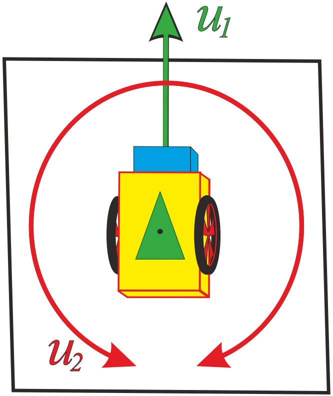
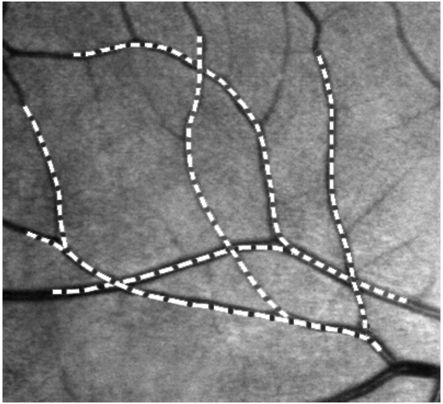
The International Conference on Differential Equations and Dynamical Systems
Suzdal, 30 Jun 2022

Outline of the Talk

Motivation

- Model example of nonholonomic system.
- Nilpotent approximation for a car-like robot.
- Extraction of salient curves in images on curved surfaces.





History of the Problem

- (B. Gaveau, 1977)
  Statement of the Dido problem (sub-Riemannian problem on the Heisenberg group).
- (R.W. Brockett, 1980)
  Optimal control formulation, sub-Riemannian sphere.
- (A.M. Vershik, V.Ya. Gershkovich, 1987)
  Compete analysis of the Dido problem.
- (A.O. Chernyshev, A.P. Mashtakov, 2021, Sirius)
  Extremal trajectories on the Heisenberg group with a positive control.
- (this work)
  Structure of optimal synthesis.

Preliminaries

- The Heisenberg group $\mathrm{H}_3 =: M \simeq \mathbb{R}^2_{x,y} \times \mathfrak{so}_2 \ni Q$, $M \simeq \mathbb{R}^3_{x,y,z} \ni q$:

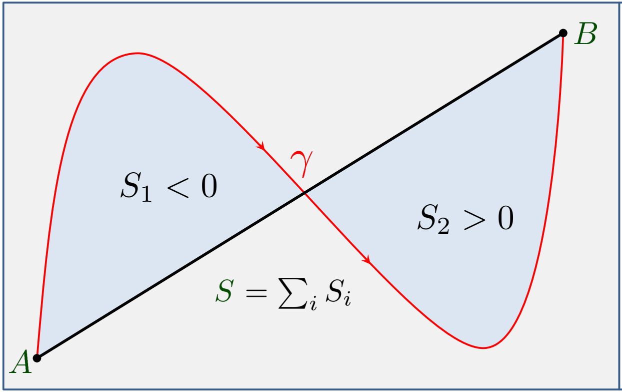
$$QQ' = \left((x,y),Z\right)\left((x',y'),Z'\right) = \left((x+x',y+y'),Z+Z'+(x,y)\wedge(x',y')\right).$$

where $v \wedge w = v \otimes w^T - w \otimes v^T$.
The Lie algebra $\mathfrak{h}_3 = \mathrm{span}(X_1, X_2, X_3)$, where

$$X_1 = \partial_x - \frac{y}{2}\partial_z, \quad X_2 = \partial_y + \frac{x}{2}\partial_z, \quad X_3 = \partial z.$$

- By given a dynamics on M, an extremal trajectory is called a trajectory that satisfies the necessary optimality condition — Pontryagin maximum principle (PMP).
- The wavefront is a set of all points in configuration space M, reachable by all the extremal trajectories in a fixed time T.
- Cut point is a point, where the extremal trajectory loses its optimality.

The Dido Problem — Classical Sub-Riemannian Problem on H_3



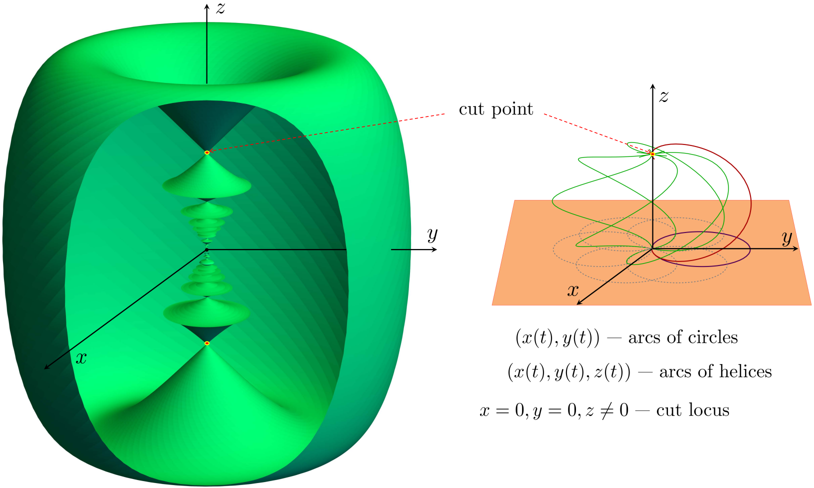
Given
$A, B \in \mathbb{R}^2$
$S \in \mathbb{R}$

Find γ
$l(\gamma) \to \min$

$$\dot{x}(t) = u_1(t),\ \dot{y}(t) = u_2(t),\ \dot{z}(t) = \frac{1}{2}\left(x(t)u_2(t) - y(t)u_1(t)\right),\ t \in [0, T]$$

$$(x, y)(0) = A,\ z(0) = 0,\ (x, y)(T) = B,\ z(T) = S,\ \ u_1^2 + u_2^2 \leq 1,\ T \to \min.$$

Classical Result: Solution to the Dido Problem

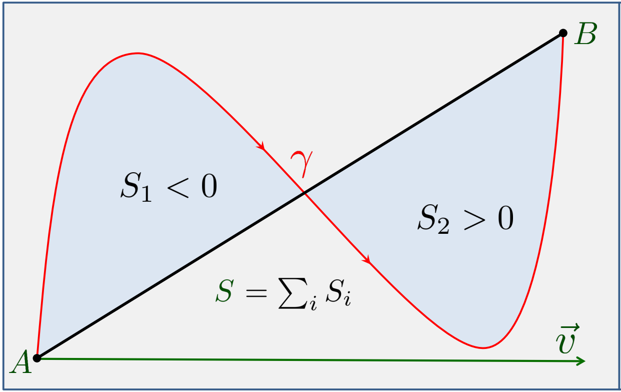


$(x(t), y(t))$ — arcs of circles

$(x(t), y(t), z(t))$ — arcs of helices

$x = 0, y = 0, z \neq 0$ — cut locus

Formulation of the Modified Dido Problem (New)



Given

$A, B \in \mathbb{R}^2$

$S \in \mathbb{R}$

$\vec{v} \in \mathbb{R}^2$

Find γ monotone along $\vec{v}$

$l(\gamma) \to \min$

$$\dot{x}(t) = u_1(t),\ \dot{y}(t) = u_2(t),\ \dot{z}(t) = \frac{1}{2}\left(x(t)u_2(t) - y(t)u_1(t)\right),\ t \in [0, T]$$

$$(x, y)(0) = A,\ z(0) = 0,\ (x, y)(T) = B,\ z(T) = S,\ \ u_1^2 + u_2^2 \le 1, \mathbf{u_1 \ge 0},\ T \to \min .$$

Formal Statement of the Problem

Consider the following control system (dynamics):

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2, \\ \dot{z} = \frac{1}{2}\left(x u_2 - y u_1\right), \end{cases} \qquad \left| \qquad \begin{array}{l} (x, y, z) = q \in M, \\ u_1^2 + u_2^2 \leq 1, \\ u_1 \geq 0. \end{array} \right.$$

By given $q_0 = (0, 0, 0)$, $q_1 \in M$ we aim to find the controls $u_1(t)$, $u_2(t)$ such that the corresponding trajectory $\gamma : [0, T] \to M$ transfers the system from q_0 to q_1 by minimal time

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min.$$

Here u_i are $L^\infty([0, T], \mathbb{R})$, and γ is a Lipschitzian curve on M.

Existence of the solution

Theorem 1. In the time minimization problem for the left-invariant control system on the Heisenberg group with admissible control in a half-disk, there exists an optimal trajectory that transfers the system from the identity to any configuration of the admissible set

$$\mathcal{A} = \{q \in \mathbb{R}^3 \,|\, x > 0\} \cup \{q \in \mathbb{R}^3 \,|\, x = 0,\ z = 0\}.$$

Proof by construction. Let $(x_0, y_0, z_0) \in \mathcal{A}$. *Control in two steps:*
1) $u_1 \geq 0, u_2 \in \mathbb{R}$ *s.t.,* $(0,0,0) \to (x_0, y', 0)$,
2) $u_1 = 0,\ u_2 = f(t)$ *s.t.,* $y_0 = \int_0^{t_1} f(t)dt + y',\ z_0 = \frac{x_0}{2}\int_0^{t_1} f(t)dt,$ *where* $y' = y_0 - \frac{2z_0}{x_0}$.
Existence of optimal trajectories is guaranteed by the Filippov theorem due to compactness and convexity of the set of admissible control.

The control system is not globally controllable $\mathcal{A} \neq \mathrm{H}_3$.
$x(t) = \int_0^t u_1(\tau)\,\mathrm{d}\,\tau \geq 0$ for $t > 0$.

Pontryagin Maximum Principle (PMP)

A necessary condition of optimality is given by PMP.

- Denote $(p_1, p_2, p_3) \in T_q^* M \simeq \mathbb{R}^3$. The Pontryagin function is given by

$$H_u = u_1(p_1 - p_3 \frac{y}{2}) + u_2(p_2 + p_3 \frac{x}{2}).$$

- Let $(u(t), q(t))$, $t \in [0, T]$ be an optimal process. Then the following conditions hold:
  - Hamiltonian system $\dot{p} = -\frac{\partial H_u}{\partial q}$, $\dot{q} = \frac{\partial H_u}{\partial p}$;
  - Maximum condition $H := \max\limits_{\bar{u} \in U} H_{\bar{u}}(p(t), q(t)) = H_{u(t)}(p(t), q(t)) \in \{0, 1\}$;
  - Non-triviality condition $p_1^2 + p_2^2 + p_3^2 + H^2 \neq 0$.

Pontryagin Maximum Principle (PMP)

Introduce left-invariant Hamiltonians $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$:

$$h_1 = p_1 - p_3 \frac{y}{2}, \quad h_2 = p_2 + p_3 \frac{x}{2}, \quad h_3 = p_3.$$

The Pontryagin function reads as $H_u = u_1 h_1 + u_2 h_2$.
The Hamiltonian system is given by

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2, \\ \dot{z} = \frac{1}{2}\left(x u_2 - y u_1\right), \end{cases} \qquad \begin{cases} \dot{h}_1 = -u_2 h_3, \\ \dot{h}_2 = u_1 h_3, \\ \dot{h}_3 = 0. \end{cases}$$

The nontriviality condition implies that if $h_1 = h_2 = 0$ then the extremal is trivial.

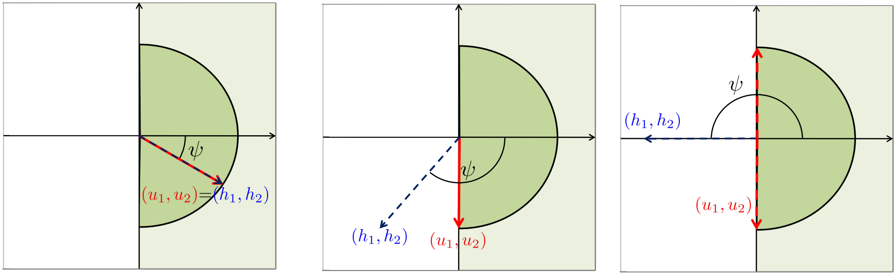
Pontryagin Maximum Principle (PMP)

Let $h_1 = \rho\cos\psi$, $h_2 = \rho\sin\psi$, $\psi \in (-\pi, \pi]$, $\rho > 0$.

The maximum condition implies the following:

- For $\psi = \pi$ we have $H = 0$, $u_1 = 0$, $u_2 \in [-1, 1]$.
- For $|\psi| \in (\frac{\pi}{2}, \pi)$ we have $H = |h_2|$, $u_1 = 0$, $u_2 = \operatorname{sign} h_2$.
- For $|\psi| \leq \frac{\pi}{2}$ we have $H = \sqrt{h_1^2 + h_2^2}$, $u_1 = \cos\psi$, $u_2 = \sin\psi$.

Note, $H = 0$ iff $\psi = \pi$. Thus, **the abnormal extremals satisfy** $u_1 = 0$, $u_2 \in [-1, 1]$.



Abnormal extremals

Theorem 2. Abnormal extremal control exists when $h_1 < 0, h_2 = 0$ and has a form $u_1(t) = 0$, $u_2(t) \in I = [-1, 1]$ — arbitrary $L_\infty([0, T], I)$ function that satisfies the condition

$$h_{10} - h_{30}\, U_2(t) < 0, \text{ where } U_2(t) = \int_0^t u_2(\tau) d\tau, \qquad t \in [0, T].$$

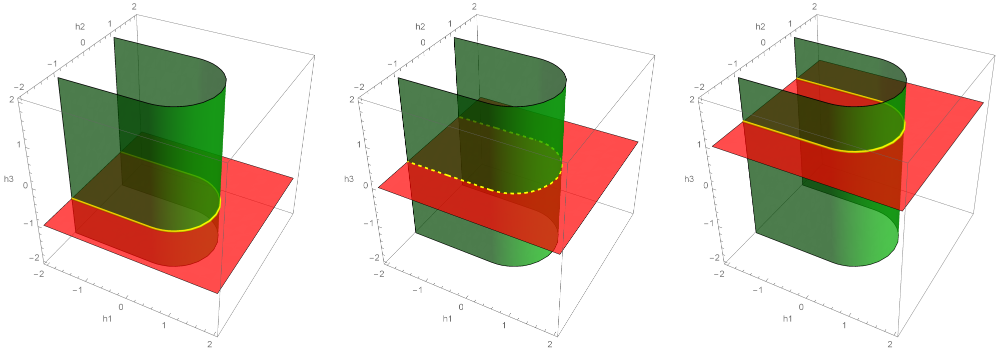
Theorem 3. Abnormal extremal trajectories have a form

$$x(t) = 0, \quad y(t) = U_2(t), \quad z(t) = 0.$$

Theorem 4. Abnormal optimal trajectories have a form

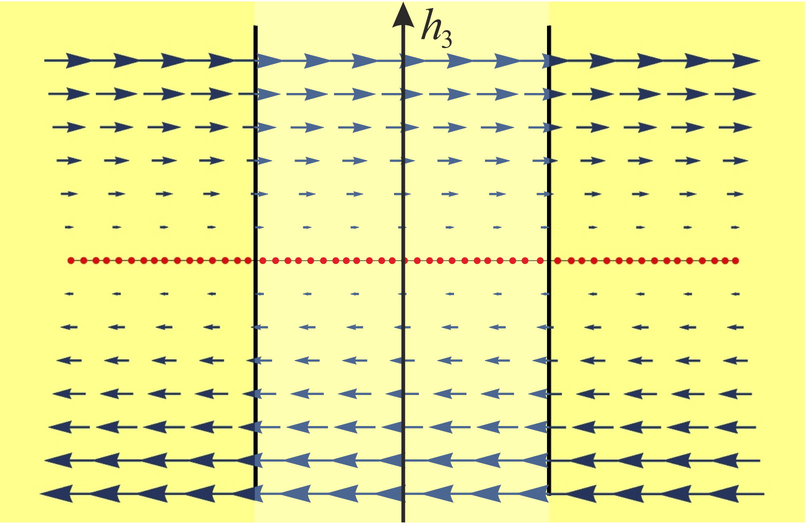
$$x(t) = 0, \quad y(t) = \pm t, \quad z(t) = 0.$$

First Integrals of the Normal Hamiltonian System



Hamiltonian $H = \begin{cases} |h_2|, & \text{for } |\psi| \in (\frac{\pi}{2}, \pi), \\ \sqrt{h_1^2 + h_2^2}, & \text{for } |\psi| \leq \frac{\pi}{2}. \end{cases}$ Casimir $E = h_3$.

Phase Portrait on the Level Surface of the Hamiltonian



Normal Hamiltonian System

For $h_{10} < 0$:

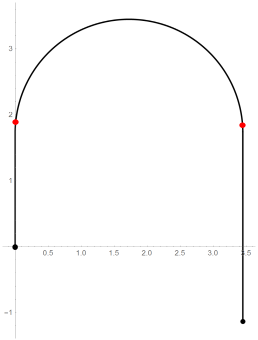
$$\begin{cases} \dot{x} = 0, & x(t_0) = x_0, \\ \dot{y} = h_2, & y(t_0) = y_0, \\ \dot{z} = \frac{x_0 h_2}{2}, & z(t_0) = z_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = 0, & h_2(t_0) = h_2^0 = \pm 1, \\ \dot{h}_3 = 0, & h_3(t_0) = h_{30}. \end{cases}$$

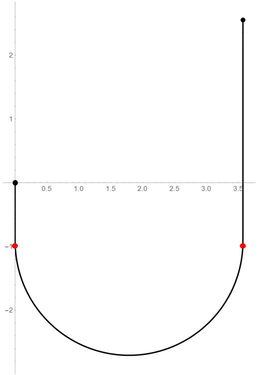
For $h_{10} \geq 0$:

$$\begin{cases} \dot{x} = h_1, & x(t_0) = 0, \\ \dot{y} = h_2, & y(t_0) = y_0, \\ \dot{z} = \frac{1}{2}(x h_2 - y h_1), & z(t_0) = 0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = h_1 h_3, & h_2(t_0) = h_{20}, \\ \dot{h}_3 = 0, & h_3(t_0) = h_{30}. \end{cases}$$

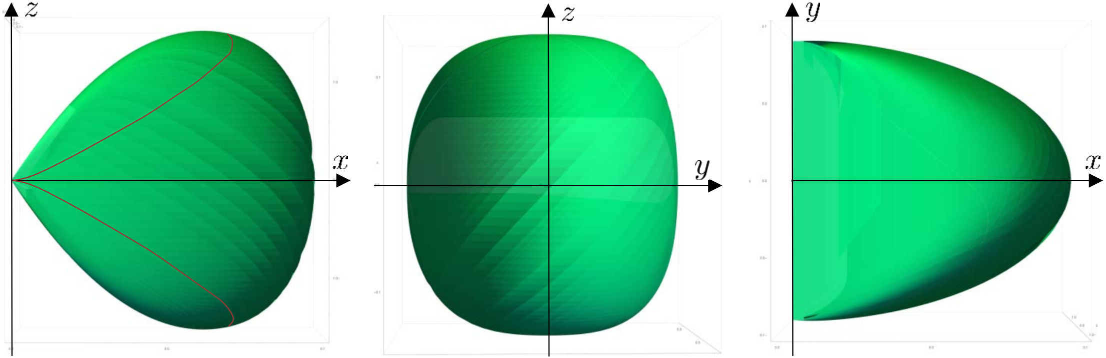
Structure of Optimal Synthesis

Theorem. For any $q \in \mathcal{A}$, there exists a unique optimal trajectory, arriving at q.
The optimal trajectory (x, y) consists of three segments (possibly zero length):
1) segment of a line parallel to O_y; 2) arc of a circle; 3) segment of a line parallel to O_y.
Proof relies on monotonicity of the function z.





Picture of the Wavefront



Conclusion

Summary:

- Left-invariant time minimization problem in H_3 with admissible controls in a half-disk.
- Applications in robotics and image processing.
- Proof of existence of optimal control.
- Necessary optimality condition — PMP.
- Qualitative analysis of dynamics.
- Explicit formulas for optimal controls and trajectories.
- Structure of optimal synthesis.

Plans:

- Explicit optimal synthesis.
- Similar problems in SO_3 and SL_2.

Thank you for your attention!