Extremal trajectories in the left-invariant time minimization problem on the group of motions of a plane with admissible control in a circular sector

Alexey Mashtakov and **Yuri Sachkov**

A.K. Ailamazyan Program Systems Institute of RAS

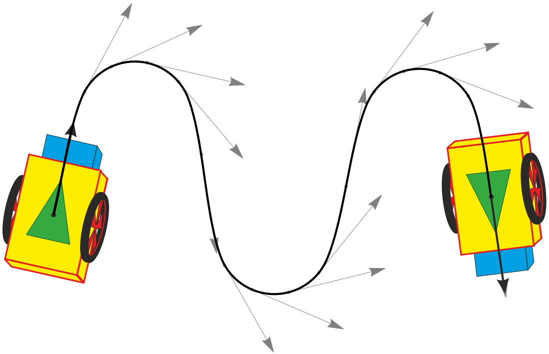
XVI International Conference «Stability and Oscillations of Nonlinear Control Systems» (Pyatnitskiy's Conference), Moscow, 2 Jun 2022

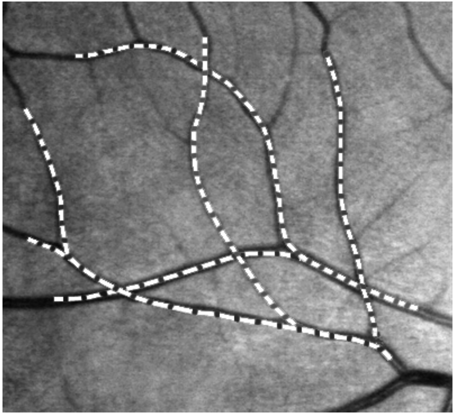
Outline of the Talk

1. Motivation
2. Preliminaries
3. Statement of the problem
4. Existence of the solution
5. Pontryagin maximum principle
6. Expression of the extremals

Motivation: Applications in robotics and image processing

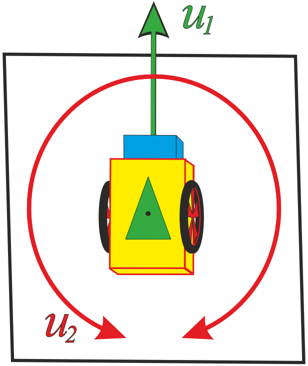
- Motion planning problem for a car-like mobile robot that can move forward and turn.
- Extraction of salient curves in images. E.g. vessel tracking on images of human retina.

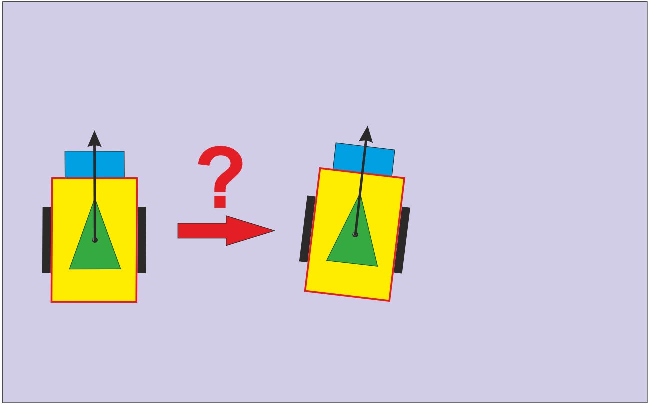




Informal Problem Formulation

Model of a car on a plane. Configuration is determined by its position and orientation. Two controls: tangential u_1 and angular u_2 velocity, $(u_1, u_2) \in U$. By given configurations q_0 and q_1 to find a trajectory that transfers the system from q_0 to q_1 by minimal time.





Preliminaries

- The group of motions of a plane $\mathrm{SE}_2 \equiv M \simeq \mathbb{R}^2_{x,y} \times S^1_\theta \ni q$:

$$qq' = \big((x,y),\theta\big)\big((x',y'),\theta'\big) = \big(R_\theta(x',y') + (x,y), \theta + \theta'\big).$$

  where R_θ is a counter-clockwise planar rotation on angle θ.
  The Lie algebra $\mathrm{se}_2 = \mathrm{span}(X_1, X_2, X_3)$, where

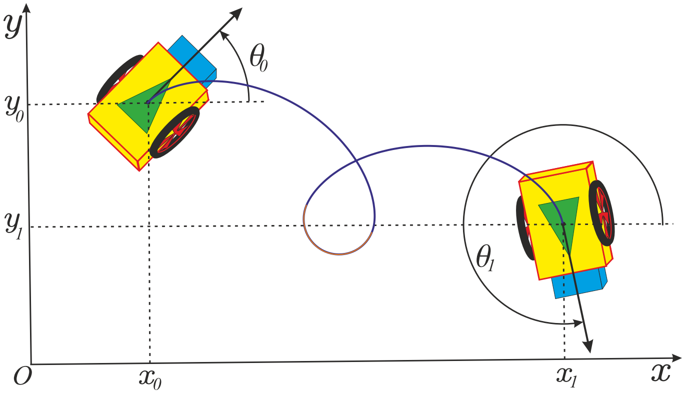
$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \partial_\theta, \quad X_3 = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}.$$

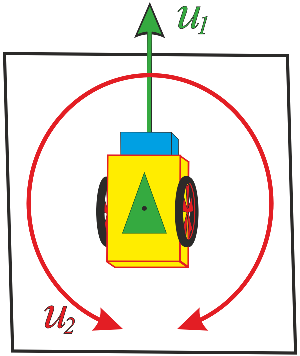
- By given a dynamics on M, an extremal trajectory is called a trajectory that satisfies the optimality condition — Pontryagin maximum principle (PMP).

Model of a Car on a Plane

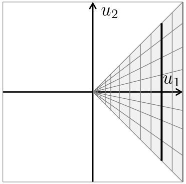
Configuration space: $q \in \mathrm{SE}_2 \simeq \mathbb{R}^2_{x,y} \times S^1_\theta$.

Dynamics: $\dot q = u_1 X_1(q) + u_2 X_2(q) \quad \Leftrightarrow \quad \{\dot x = u_1 \cos\theta,\ \dot y = u_1 \sin\theta,\ \dot\theta = u_2\}$.

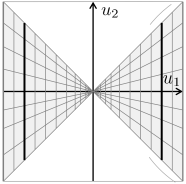




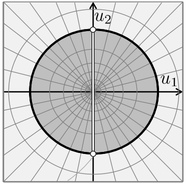
Set of Admissible Controls



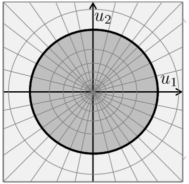
Dubins (1957)



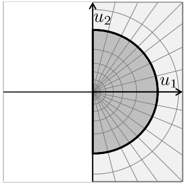
Reeds-Shepp (1990)



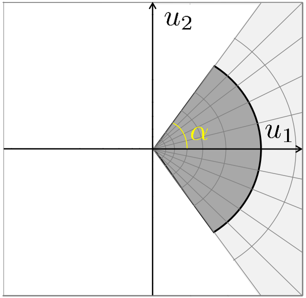
Berestovskii (1994)



Sachkov (2010)



Duits (2018)



This work (2022)

Formal Statement of the Problem

Consider the following control system (dynamics):

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \left| \qquad \begin{array}{l} (x, y, \theta) = q \in \mathrm{SE}_2 = M, \\ u_1 = r\cos\phi,\ u_2 = r\sin\phi, \\ 0 \le r \le 1,\ |\phi| \le \alpha,\ 0 < \alpha < \frac{\pi}{2}. \end{array} \right.$$

By given q_0, $q_1 \in M$ we aim to find the controls $u_1(t)$, $u_2(t)$ such that the corresponding trajectory $\gamma : [0, T] \to M$ transfers the system from q_0 to q_1 by minimal time

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min.$$

Here u_i are $L^\infty([0, T], \mathbb{R})$, and γ is a Lipschitzian curve on M.

Existence of the solution

Theorem. In the time minimization problem for the left-invariant control system on the group of motions of a plane with admissible control in a circular sector with a convex central angle, there always exists an optimal trajectory that transfers the system from an arbitrary given initial configuration to an arbitrary given final configuration.

Proof of global controllability is based on Lie saturation method.
Let $\hat{\mathcal{F}} = \{ X_1 + wX_2 | \, |w| \leq \tan\alpha \}$. *The vector field* $X_1 + wX_2$, $w \neq 0$ *has a periodic trajectory. Thus,* $-(X_1 + wX_2) \in \mathrm{LS}(\hat{F})$. *It implies* $-wX_2 = -(X_1 + wX_2) + X_1 \in \mathrm{LS}(\hat{F})$. *Consequently,* $\pm X_2 \in \mathrm{LS}(\hat{F})$, *and, thus,* $\pm X_1 \in \mathrm{LS}(\hat{F})$. *Hence,* $\mathrm{LS}(\hat{\mathcal{F}}) = \mathrm{Lie}(X_1, X_2)$.
Existence of optimal trajectories is guaranteed by the Filippov theorem due to compactness and convexity of the set of admissible control and global controllability.

The control system is not small time controllable.
$x(t) = \int_0^t u_1(\tau) \cos\theta(\tau) \, \mathrm{d}\,\tau > 0$ for small $t > 0$.

Pontryagin Maximum Principle (PMP)

A necessary condition of optimality is given by PMP.

- Denote $(p_1, p_2, p_3) \in T_q^*M \simeq \mathbb{R}^3$. The Pontryagin function is given by

$$H_u = u_1(p_1 \cos\theta + p_2 \sin\theta) + u_2 p_3.$$

- Let $(u(t), q(t))$, $t \in [0, T]$ be an optimal process. Then the following conditions hold:
  - Hamiltonian system $\dot{p} = -\dfrac{\partial H_u}{\partial q}$, $\dot{q} = \dfrac{\partial H_u}{\partial p}$;
  - Maximum condition $H = \max\limits_{u \in U} H_u(p(t), q(t)) \in \{0, 1\}$;
  - Non-triviality condition $p_1^2 + p_2^2 + p_3^2 + H^2 \neq 0$.

Pontryagin Maximum Principle (PMP)

Introduce left-invariant Hamiltonians $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$:

$$h_1 = p_1 \cos\theta + p_2 \sin\theta, \quad h_2 = p_3, \quad h_3 = p_1 \sin\theta - p_2 \cos\theta.$$

The Pontryagin function reads as $H_u = u_1 h_1 + u_2 h_2$.
The Hamiltonian system is given by

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \begin{cases} \dot{h}_1 = -u_2 h_3, \\ \dot{h}_2 = u_1 h_3, \\ \dot{h}_3 = u_2 h_1. \end{cases}$$

The nontriviality condition implies that if $h_1 = h_2 = 0$ then the extremal is trivial.

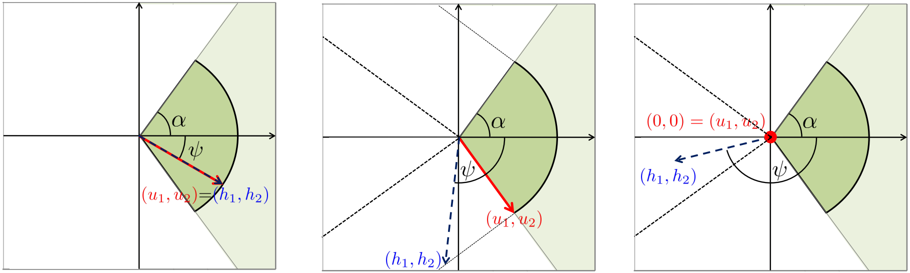
Pontryagin Maximum Principle (PMP)

Let $h_1 = \rho \cos \psi$, $h_2 = \rho \sin \psi$, $\psi \in (-\pi, \pi]$, $\rho > 0$.

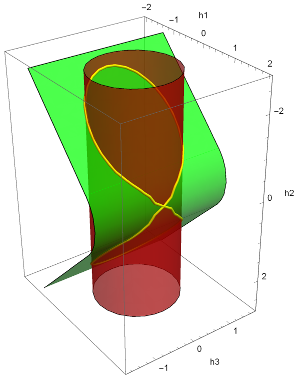
The maximum condition implies the following:

- For $|\psi| \in [\frac{\pi}{2} + \alpha, \pi]$ we have $H = 0$, $u_1 = u_2 = 0$.
- For $\pm\psi \in (\alpha, \frac{\pi}{2} + \alpha)$ we have $H = h_1 \cos \alpha \pm h_2 \sin \alpha$, $u_1 = \cos \alpha$, $u_2 = \pm \sin \alpha$.
- For $|\psi| \leq \alpha$ we have $H = \sqrt{h_1^2 + h_2^2}$, $u_1 = \cos \psi$, $u_2 = \sin \psi$.

We see that $H = 0$ iff $|\psi| \in [\frac{\pi}{2} + \alpha, \pi]$. Thus, **the abnormal extremals are trivial**.



First Integrals of the Hamiltonian System



Hamiltonian $H = \begin{cases} h_1 \cos\alpha + |h_2| \sin\alpha, & \text{for } |\psi| > \alpha, \\ \sqrt{h_1^2 + h_2^2}, & \text{for } |\psi| \leq \alpha. \end{cases}$ Casimir $E = h_1^2 + h_3^2$.

Reduction of the Hamiltonian System via Convex Trigonometry

The polar set to U is $U^o = \{\,(h_1,h_2)\in\mathbb{R}^{2*} \,\big|\, u_1h_1+u_2h_2\le 1\ \forall (u_1,u_2)\in U)$:

$$U^o = \left\{ (\overbrace{\rho\cos\psi}^{=h_1}, \overbrace{\rho\sin\psi}^{=h_2}) \;\middle|\; \begin{array}{l} \text{for } |\psi|\le\alpha : \rho\in[0,1], \\ \text{for } \alpha<\psi<\alpha+\frac{\pi}{2} :\ h_1\cos\alpha+h_2\sin\alpha\le 1, \\ \text{for } -\alpha-\frac{\pi}{2}<\psi<-\alpha :\ h_1\cos\alpha-h_2\sin\alpha\ge 1 \end{array} \right\}$$

The corresponding functions of convex trigonometry are

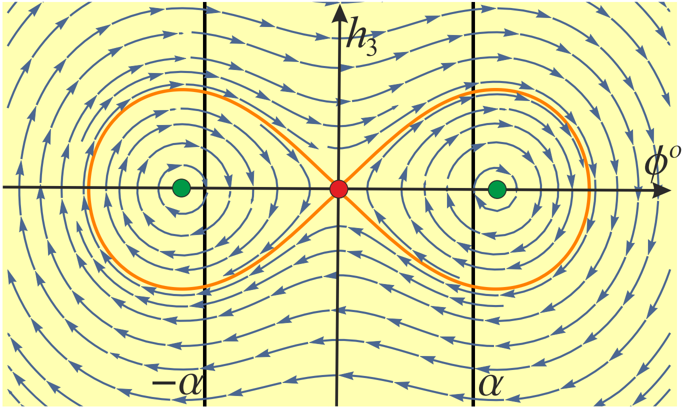
$$\begin{array}{l} \text{for } |\phi^o|\le\alpha :\ \cos_{U^o}\phi^o=\cos\phi^o,\ \sin_{U^o}\phi^o=\sin_{U^o}\phi^o; \\ \text{for } |\phi^o|>\alpha :\ \cos_{U^o}\phi^o=\cos\alpha-\sin\alpha(\phi^o-\alpha),\ \sin_{U^o}\phi^o=\operatorname{sign}(\phi^o)\,(\sin\alpha+\cos\alpha(\phi^o-\alpha)) \end{array}$$

Along the extremal trajectories $u_1=\cos\phi$, $u_2=\sin\phi, h_1=\cos_{U^o}\phi^o$, $h_2=\sin_{U^o}\phi^o$. Denote $K(\phi^o)=\frac{1}{2}\cos^2_{U^o}\phi^o$. The Casimir $E=\frac{h_1^2}{2}+\frac{h_3^2}{2}=\frac{h_3^2}{2}+K(\phi^o)$ can be seen as a total energy integral of conservative system with one degree of freedom

$$\dot{\phi}^o=h_3, \quad \dot{h}_3=K'(\phi^o).$$

Phase Portrait on the Level Surface of the Hamiltonian

- $E = 0 \Rightarrow (\phi^o, h_3) \equiv (\pm(\alpha + \cot\alpha), 0)$ is stable equilibrium;
- $E \in (0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty) \Rightarrow$ the trajectory $(\phi^o, h_3)(t)$ is periodic;
- $E = \frac{1}{2} \Rightarrow$ either $(\phi^o, h_3) \equiv (0, 0)$ is unstable equilibrium or $(\phi^o, h_3)(t)$ is a separatrix.



The Hamiltonian System in s–parameterization

Since $u_1 > 0$ the trajectories can be parametrized by the arc-length on the plane Oxy:

$$s(t) = \int_0^t \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}\, \mathrm{d}\,\tau = \int_0^t u_1(\tau)\, \mathrm{d}\,\tau.$$

The Hamiltonian in s-parameterization is given by

$$\begin{cases} x' = \cos\theta, \\ y' = \sin\theta, \\ \theta' = u, \end{cases} \qquad \begin{cases} h_1' = -uh_3, \\ h_2' = h_3, \\ h_3' = uh_1, \end{cases} \qquad \text{where } {}' = \frac{d}{ds},\ u = \frac{u_2}{u_1}.$$

Explicit Expression for the Extremals

Denote $M = 1 + p_2^2 - p_{30}^2$, $P(s) = p_{30}\sinh(s) - p_2\cosh(s)$, $f(s) = p_{30}\cosh s - p_2 \sinh s - p_{30}$.
Case $|\phi^o| > \alpha$. The extremal control is $u \equiv \pm\sin\alpha$, and the corresponding extremal trajectories are circular arcs on the plane Oxy:

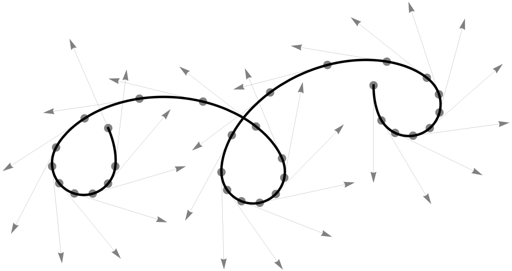
$$x(s) = \frac{1}{u}\sin(us), \quad y(s) = \frac{1}{u}(\cos(us) - 1), \quad \theta(s) = us.$$

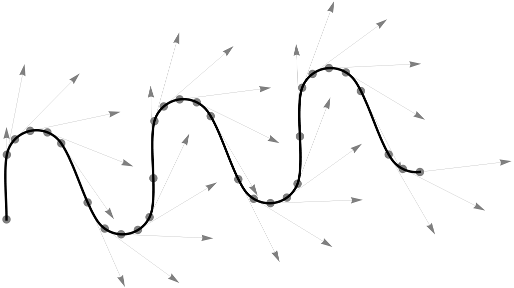
Case $|\phi^o| < \alpha$. The extremal trajectories are the arcs of sub-Riemannian geodesics in SE_2:

$$x(s) = \frac{1}{M}\big(p_1\, g(s) - p_2\, f(s)\big),\ y(s) = \frac{1}{M}\big(p_1\, f(s) + p_2\, g(s)\big),\ \theta(s) = \arcsin\frac{P(s)}{\sqrt{M}} - \arcsin\frac{P(0)}{\sqrt{M}},$$

where $g(s) = \int\limits_0^s \sqrt{M - P^2(\sigma)}\,\mathrm{d}\,\sigma$ is expressed in Jacobi elliptic functions.

Examples of the Extremal Trajectories





Conclusion

Summary:

- Left-invariant time minimization problem in SE_2 with admissible controls in a sector.
- Applications in robotics and image processing.
- Proof of existence of optimal control.
- Necessary optimality condition — PMP.
- Qualitative analysis of dynamics.
- Explicit formulas for extremal controls and trajectories.

Plans:

- Optimality of extremals.
- Optimal synthesis.

Thank you for your attention!