Geodesic Flow of the Sub-Riemannian Structure of Engel Type with Strictly Abnormal Extremals

Alexey Mashtakov and **Alexey Podobryaev**

A.K. Ailamazyan Program Systems Institute of RAS

XVI International Conference «Stability and Oscillations of Nonlinear Control Systems» (Pyatnitskiy's Conference), Moscow, 3 Jun 2022

Outline of the Talk

1. Motivation
2. Preliminaries
3. Statement of the problem
4. Existence of the solution
5. Pontryagin maximum principle
6. Qualitative analysis in a general case
7. Explicit integration in the special case

Motivation

- Model example in sub-Riemannian geometry.
- Left invariant structure of low dimension.
- Liouville integrability of normal geodesic flow.
- Existence of strictly abnormal geodesic.
- The most complicated singularities of sub-Riemannian metric.

Preliminaries

- *A sub-Riemannian manifold* is a triple (M, Δ, g), where M is a connected smooth manifold, $\Delta \subset TM$ is a subbundle of the tangent bundle (called *a distribution*) and g is a scalar product on Δ.
- A Lipschitzian curve $\gamma : [0, T] \to M$ is called *admissible* if it is tangent to Δ a.e.
- *A sub-Riemannian length* of an admissible curve $\gamma(\,\cdot\,)$ is

$$l(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, \mathrm{d}t.$$

- Δ satisfies the Hörmander condition, i.e., the Lie brackets of the vector fields tangent to Δ span the whole tangent bundle TM.
- Complete controllability by Rashevskii-Chow theorem.
- The sub-Riemannian distance between $x, y \in M$ is defined as infimum for the lengths of the admissible curves connecting x and y.

Preliminaries

- Let $\Delta^1 = \Delta$, $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. Due to the Hörmander condition, there exists $s \in \mathbb{N}$ such that $\Delta^s = TM$. The tuple of numbers $(\dim \Delta^i)_{i=1}^s$ is called *the growth vector* of the distribution Δ.
- For a left-invariant sub-Riemannian structure M is a Lie group, Δ is generated by left-invariant vector fields, g is left-invariant.
- The growth vector of left-invariant sub-Riemannian structures is constant.
- A left-invariant sub-Riemannian structure with the growth vector $(2, 3, 4)$ is called *an Engel type sub-Riemannian structure.*
- Such structures were classified by Almeida (1998). Then Beschastnyi and Medvedev (2018) found an Engel type structure such that the geodesic flow is Liouville integrable and there exists a strict abnormal geodesic.

Preliminaries

- Let G be a connected Lie group, and $\mathfrak{L}$ be the Lie algebra of left-invariant vector fields on G. A left-invariant sub-Riemannian (SR) structure can be defined via an orthonormal frame $X_1,\ X_2 \in \mathfrak{L}$ as

$$\Delta_q = \mathrm{span}(X_1(q), X_2(q)), \quad \langle X_i, X_j \rangle = \delta_{ij},\ i,j = 1,2,$$

  where $q \in G$ and δ_{ij} is the Kronecker delta.
- A length minimizer can be found as a solution to the optimal control problem

$$\dot\gamma(t) = u_1(t)\,X_1(\gamma(t)) + u_2(t)\,X_2(\gamma(t)), \quad \gamma(0) = q^0, \quad \gamma(T) = q^1, \quad l(\gamma) \to \min,$$

  where $X_1,\ X_2$ are orthonormal w.r.t. the scalar product $g(\,\cdot\,,\,\cdot\,)$, the controls $u_1,\ u_2$ are real-valued L_∞-functions. and the length $l(\gamma)$ is equal to

$$l(\gamma) = \int\limits_0^T \sqrt{u_1^2(t) + u_2^2(t)}\,\mathrm{d}t.$$

Preliminaries

- Without loss of generality one can chose q^0 as identity of the group.
- Due to the Cauchy-Schwarz inequality, minimization of $l(\gamma)$ is equivalent to minimization of the action

$$E(\gamma) = \frac{1}{2} \int_0^T (u_1^2(t) + u_2^2(t))\, \mathrm{d}t.$$

- An *extremal trajectory* is called a trajectory that satisfies the necessary optimality condition — Pontryagin maximum principle (PMP). If sufficiently small arcs of an extremal trajectory are optimal, then it is called *a geodesic*.
- In PMP there are two cases $\nu \in \{0, 1\}$. When $\nu = 0$, the extremal and geodesic are called *abnormal*. When $\nu = 1$, the extremal and geodesic are called *normal*.
- An abnormal geodesic that is not equal to the projection of any normal extremal is called *a strict abnormal geodesic*.

Problem formulation

We study the problem of finding length minimizers on on the Lie group $\overline{\mathrm{SL}}_2$ that is the central extension of the special linear group SL_2 over real numbers $\mathbb{R}$:

$$q \in \overline{\mathrm{SL}}_2 = \left\{ \left(\begin{array}{ccc} x & y & 0 \\ z & w & 0 \\ 0 & 0 & C \end{array} \right) \middle| \begin{array}{l} x, y, z, w, C \in \mathbb{R}, \\ xw - yz = 1 \end{array} \right\}.$$

We consider the following basis left-invariant vector fields:

$$X_1(q) = (L_q)_* \left(\partial_y - \frac{1}{2} \partial_C \right) \Big|_e, \quad X_2(q) = (L_q)_* \left(\frac{T_4}{2} \partial_z \right) \Big|_e,$$

where $(L_q)_*$ is push–forward under left translation $L_q h = qh$, e is identity of the group, and $T_4 > 0$ is a constant parameter.
In coordinates $(x, y, z, w, C) \in \mathbb{R}^5$ we have

$$X_1(q) = x\partial_y + z\partial_w + \frac{C}{2}\partial_C, \quad X_2(q) = \frac{T_4}{2} y\partial_x + \frac{T_4}{2} w\partial_z.$$

Optimal control problem

- Control system

$$\dot{x} = \frac{T_4}{2} y u_2, \quad \dot{y} = x u_1, \quad \dot{z} = \frac{T_4}{2} w u_2, \quad \dot{w} = z u_1, \quad \dot{C} = \frac{1}{2} C u_1.$$

- Boundary conditions

$$\begin{aligned} &x(0) = w(0) = C(0) = 1,\ y(0) = z(0) = 0, \\ &x(T) = x^1,\ y(T) = y^1,\ z(T) = z^1,\ w(T) = w^1,\ C(T) = C^1. \end{aligned}$$

- Minimizing functional

$$\frac{1}{2} \int_0^T (u_1^2(t) + u_2^2(t))\, \mathrm{d}t \to \min .$$

Existence of the solution

Theorem. For any point $q^1 = (x^1, y^1, z^1, w^1, C^1)$, satisfying $x^1 w^1 - y^1 z^1 = 1$, there exists an optimal trajectory that departs from q_0 and arrives to q^1.

Complete controllability is guaranteed by the Rashevskii–Chow theorem. Indeed, the vector fields X_1, X_2, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3] + T_4 X_1$ *form a basis of the Lie algebra* $\mathfrak{L}$.

Existence of optimal trajectories is guaranteed by the Filippov theorem.

Pontryagin maximum principle

Denote $h_i = \langle \cdot, X_i \rangle$, $i = 1, \dots, 4$. The Pontryagin function is given by

$$H_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \frac{\nu}{2}(u_1^2 + u_2^2), \qquad \lambda \in T^*G.$$

PMP: If $\tilde{q}(\cdot)$ and $\tilde{u}(\cdot)$ are an optimal process, then there exist a curve $\lambda \in \mathrm{Lip}\,([0, T], T^*G)$, $\pi(\lambda(t)) = \tilde{q}(t)$ and a number $\nu \in \{0, 1\}$ such that for a.e. $t \in [0, T]$

$$\begin{aligned}
&\dot{\lambda}(t) = \vec{H}_{\tilde{u}(t)}(\lambda(t)), \\
&H^\nu_{\tilde{u}(t)}(\lambda(t)) = \max_{u \in \mathbb{R}^2} H_u^\nu(\lambda(t)), \\
&(\lambda(t), \nu) \neq 0,
\end{aligned}$$

where $\pi : T^*G \to G$ is the natural projection and $\vec{H}$ is the Hamiltonian vector field.

Pontryagin maximum principle

The vertical (adjoint) subsystem of the Hamiltonian system of the PMP reads as $\dot{h}_i = \{H^\nu_u, h_i\}$ for $i = 1, \dots, 4$, where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket on T^*G. Computation of the Poisson brackets gives

$$\begin{aligned} \dot{h}_1 &= -u_2 h_3, & \dot{h}_3 &= u_1 h_4 - T_4(u_1 h_1 - u_2 h_2), \\ \dot{h}_2 &= u_1 h_3, & \dot{h}_4 &= 0. \end{aligned}$$

From the results of Beschasnyi and Medvedev it follows that the abnormal extremal trajectory with $u_1 = 0$ and $u_2 = 1$ is strict abnormal geodesic. It is given by

$$x = 1, \quad y = 0, \quad z = e^{\frac{T_4}{2}t} - 1, \quad w = 1, \quad C = 1.$$

Normal Hamiltonian system in the general case

It follows from the maximum condition that in the normal case $u_1 = h_1$, $u_2 = h_2$. The vertical subsystem reads as

$$\begin{aligned} \dot h_1 &= -h_2 h_3, & \dot h_3 &= h_1 h_4 - T_4(h_1^2 - h_2^2), \\ \dot h_2 &= h_1 h_3, & \dot h_4 &= 0. \end{aligned}$$

This system has two obvious first integrals: $h_4 = \text{const}$ and $H = \frac{1}{2}(h_1^2 + h_2^2) = \frac{1}{2}$.

Introduce the polar angle $\theta \in S^1$ as $h_1 = \cos\theta$, $h_2 = \sin\theta$.

The vertical subsystem reduces to the system of a skewed pendulum

$$\begin{cases} \dot\theta = h_3, & \theta(0) = \theta^0 \\ \dot h_3 = h_4 \cos\theta - T_4 \cos 2\theta, & h_3(0) = h_3^0. \end{cases}$$

Normal Hamiltonian system in the general case

The system of a skewed pendulum is a conservative system of one degree of freedom. Such systems have the first integral – the total energy

$$G = \frac{1}{2}h_3^2 - (h_4 - T_4\cos\theta)\sin\theta = \frac{1}{2}h_3^2 - h_4h_2 + T_4h_1h_2.$$

Theorem. The trajectories of the vertical subsystem are the curves of intersection of the cylinder $H = \frac{1}{2}$ and the surface $G = \text{const}$, which is a two-sheeted hyperboloid for $G > 0$, a one-sheeted hyperboloid for $G < 0$ and a cone for $G = 0$. The apex of the cone is at the point $h_1 = \frac{h_4}{T_4}$, $h_2 = h_3 = 0$ and the generatrix is parallel to the axis h_2.

Proof. The quadratic form $\frac{1}{2}h_3^2 - h_4h_2 + T_4h_1h_2$ *is reduced to canonical form* $\frac{1}{2}h_3^2 + T_4(h_1' - \frac{h_4}{2T_4})^2 - T_4(h_2' - \frac{h_4}{2T_4})^2$*, where* $h_1' = \frac{1}{2}(h_1 + h_2)$ *and* $h_2' = \frac{1}{2}(h_1 - h_2)$*.*

Fixed points of the vertical subsystem

Fixed points are determined by $\dot{h}_3 = \dot{\theta} = 0$ and have the form

$$h_3 = 0, \quad \theta = \arctan\left(\frac{s_1\sqrt{B}}{h_4 + s_2 A}\right),$$

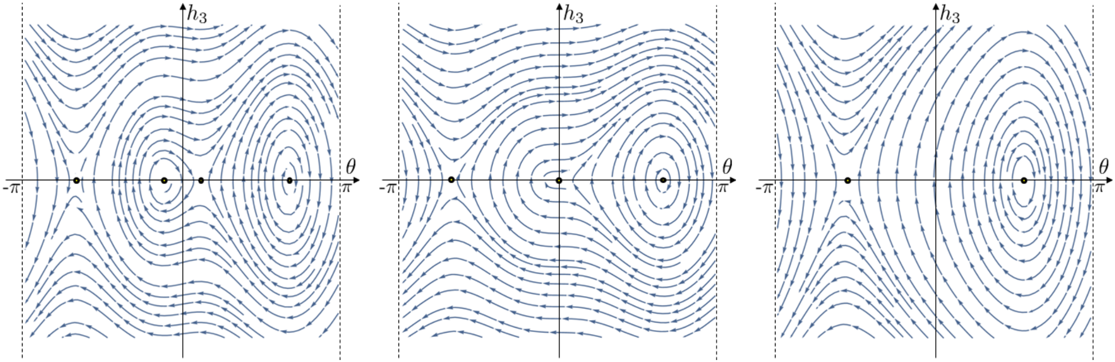
where $A = \sqrt{h_4^2 + 8T_4^2} > 0$, $B = 8T_4^2 + 2s_1h_4(A - s_1h_4) \geq 0$ and s_1, s_2 take values ± 1.

Note that $B = 0$ iff $|h_4| = T_4$.

Qualitative analysis of the vertical subsystem

By analyzing the condition $B \geq 0$ and the type of extremum at the fixed points we obtain

Theorem The fixed points of the vertical subsystem: (1) Two centers and two saddles for $|h_4| < T_4$; (2) Center, saddle and cusp for $|h_4| = T_4$; (3) A center and a saddle for $|h_4| > T_4$.



Special case $h_4 = 0$

In this case, the vertical subsystem is reduced to the system of mathematical pendulum

$$\begin{cases} \dot\theta = h_3, & \theta(0) = \theta^0 = \arg(h_1^0 + \boldsymbol{i} h_2^0) \\ \dot h_3 = -T_4 \cos 2\theta, & h_3(0) = h_3^0. \end{cases}$$

The function $h_3(t)$ can be found as as a solution to the Cauchy problem

$$\ddot h_3 - 2G h_3 + h_3^3 = 0, \quad h_3(0) = h_3^0,\ \dot h_3(0) = -T_4 \cos 2\theta^0.$$

An explicit solution is given by $h_3(t) = A\,\mathrm{cn}(\Omega(t - t_0), k)$, where the constant parameters A, Ω, t_0 and k are determined by the initial conditions and the constant T_4.
The polar angle $\theta(t)$ is obtained as the integral

$$\theta(t) = \theta^0 + \int_0^t h_3(\tau)\mathrm{d}\tau.$$

This integral admits expression in Jacobi elliptic functions.
The extremal controls are given by $u_1(t) = \cos\theta(t)$, $u_2(t) = \sin\theta(t)$.

Conclusion

Summary:

- Left-invariant sub-Riemannian structure of Engel type with strict abnormal geodesic.
- Model example in sub-Riemannian geometry.
- Proof of existence of optimal control.
- Necessary optimality condition — PMP.
- Qualitative analysis of dynamics in general case.
- Explicit formulas for extremal controls in the special case.

Plans:

- Study of singularity of the wavefront near the abnormal geodesic.

Thank you for your attention!