Modelling of optimal parking for a wheeled robot

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***Abstract*—We consider a kinematic model for a differential wheeled robot. The corresponding optimal control problem is stated in the general form with an arbitrary domain for the controls and an arbitrary minimized functional. The known specifications of the problem lead to the classical models given by Markov-Dubins problem, Reeds-Shepp problem, Euler's elastic problem and the sub-Riemannian problem on the rototranslation group. We describe the algorithms for constructing optimal trajectories of the robot for the given boundary conditions. These algorithms are implemented in an interface software developed in Wolfram Mathematica system. Additionally, we allow to equip the robot with a trailer. Trailer trajectories are computed for each model. We assume that the robot stays at the initial and final positions with zero velocities and accelerates to the given maximum linear velocity along the way. The interface program animates the movement of the robot (with a trailer) along the chosen optimal paths.**

***Index Terms*—optimal control, rototranslation group, kinematic model, wheeled robot, Markov-Dubins path, Reeds-Shepp car, Euler elastica, sub-Riemannian geometry, trailer, Wolfram Mathematica.**

I. Introduction

The motion planning problem for mobile robots often leads to the models given by optimal control problems. Since for arbitrary boundary configurations we have a continuum number of trajectories connecting the initial and final configuration of the robot, one should choose a cost criterion that is important depending on the motion planning situation. How to choose such a criterion? An important step in the direction of resolving this question is to provide a computer simulation of the motion planning with different criteria. This work aims to collect the known criteria provided by important four optimal control problems for a mobile robot, to describe the developed motion planning interface as a navigation tool for these problems, and to describe the situations, where a certain criterion is preferable. We use Wolfram Mathematica environment for our interface. So far, only the simplest problem (Markov-Dubins problem [1]) is presented on Wolfram Demonstration Project site where such interfaces are collected.

This work continues the investigation of the known models for a wheeled robot (or a car) started in [2]. The robot has two driving coaxial wheels and its shape is given by convex hexagon (see Fig. 1). Configuration space of the wheeled robot moving on a plane is given by the rototranslation group

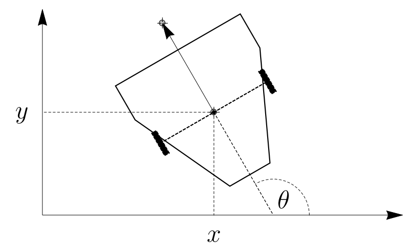

Fig. 1. Geometric model of the robot with two locators for setting its position

$$\mathrm{SE}(2) = \{q = (x, y, \theta) \mid (x, y) \in \mathbb{R}^2, \theta \in S^1\},$$

where (x, y) is a position of the drive axle center and θ is an angle of its orientation in the plane.

A parking problem is a problem of finding a path starting from the given initial position q_0 and arriving to the final one q_1. We assume that driving wheels are rolling with no slipping and no skidding, therefore the kinematic motion of the robot is expressed by nonholonomic system [3]. Also, angular and linear velocities of the robot u_1, u_2 are understood as controls, since there is a one-to-one correspondence with the actual controls of the real robot — the velocities of two driving wheels [4]. A domain for possible controls and a cost functional to be minimized define the optimal control problem.

In the case when the robot is equipped with a trailer, the choice of the initial position $\varphi_0 \in S^1$ for the trailer determines a unique trajectory of the trailer $\varphi(t)$ for the given controls u_1, u_2.

II. Optimal control problem

A. General problem statement

The optimal control problem on the rototranslation group SE(2) is given by linear in controls system

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \tag{1}$$

boundary conditions

$$q(0) = (x_0, y_0, \theta_0), \qquad q(T) = (x_1, y_1, \theta_1), \tag{2}$$

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minimized cost functional

$$J = \int_0^T f(u_1, u_2)\, dt \to \min, \tag{3}$$

and restriction on the controls

$$(u_1, u_2) \in U \subseteq \mathbb{R}^2, \tag{4}$$

where the terminal time T can be fixed or free.

Further, we describe four models which can be obtained from probelm (1)–(4) by fixing the function f and the shape of the domain U.

B. Models

The following specifications of problem (1)–(4) are considered:

1) Markov-Dubins problem [5], [6]:

$$U = U_{MD} := \{(u_1, u_2) \mid u_1 = 1, |u_2| \leq \kappa_{\max}\}, \tag{5}$$

$$f(u_1, u_2) = f_{MD} \equiv 1. \tag{6}$$

where $\kappa_{\max} = 1/R > 0$ is the maximum curvature allowed for the trajectory, i.e., the robot can turn along a circle with the minimal radius R.

2) Reeds-Shepp problem [7]:

$$U = U_{RS} := \{(u_1, u_2) \mid |u_1| = 1, |u_2| \leq \kappa_{\max}\}, \tag{7}$$

$$f(u_1, u_2) = f_{RS} \equiv 1. \tag{8}$$

3) Euler's elastic problem [8], [9]:

$$U = U_E := \{(u_1, u_2) \mid u_1 = 1\}, \tag{9}$$

$$f(u_1, u_2) = f_e(u_2) := \frac{u_2^2}{2}. \tag{10}$$

4) Sub-Riemannian problem on SE(2) [10]:

$$U = \mathbb{R}^2, \tag{11}$$

$$f(u_1, u_2) = f_{SR}(u_1, u_2) := \sqrt{u_1^2 + u_2^2}. \tag{12}$$

Problems 1), 2) with $f \equiv 1$ are called time-optimal problems. Such a problem with an arbitrary compact symmetric control domain $U = -U$ containing the origin with a neighborhood is called a sub-Finsler problem, e.g., convexified problem 2) is equivalent to a sub-Finsler problem. Note that problem 4) is equivalent to the sub-Finsler problem with

$$U = U_{SR} := \{(u_1, u_2) \mid u_1^2 + u_2^2 \leq 1\}.$$

Problems 1), 3) correspond to the models of the robot which is moving only forward with unit velocity, so terminal time T is equal to the length of the curve on the plane (x, y).

Optimal synthesis algorithms for problems 1), 2) were implemented in the first version of the interface [2]. Optimal synthesis for problems 3), 4) was reduced to solving certain systems of algebraic equations [9], [10]. It is known that such equations have a unique solution on the described domains for the unknown variables which determine the optimal path. Algorithms for searching the roots of the equations can be found in [11]. We refine all these algorithms and join them in one interface.

C. Trailer

The described models of parking problem were overviewed in [12], where it was proposed to use them for controlling the robot-trailer system. We extend system (1) by additional linear in controls equation for the trailer motion:

$$\dot{\varphi} = -u_1 \frac{\sin \varphi}{l_t} - u_2, \tag{13}$$

here the trailer is attached at the center of the robot (x, y) and l_t corresponds to the distance between the centers of the robot and the trailer. We implicitly determined the function $\varphi(t)$ depending on an initial condition $\varphi(0) = \varphi_0$ and the optimal controls u_1, u_2 for Markov-Dubins and Reeds-Shepp problems in [2]. For the other two problems (Euler's and Sub-Riemannian) we calculate the corresponding trailer trajectory $\varphi(t)$ numerically.

III. Motion planning interface

The interface program is developed in the Wolfram Mathematica. It is divided into two parts: the control panel and the active part of the plane (x, y). Control elements of the interface set the parameters for the general problem (1)–(4).

A. Description of control elements

Control panel has two tabs. The main **Menu** tab is provided with the following elements (see top of Fig. 2):

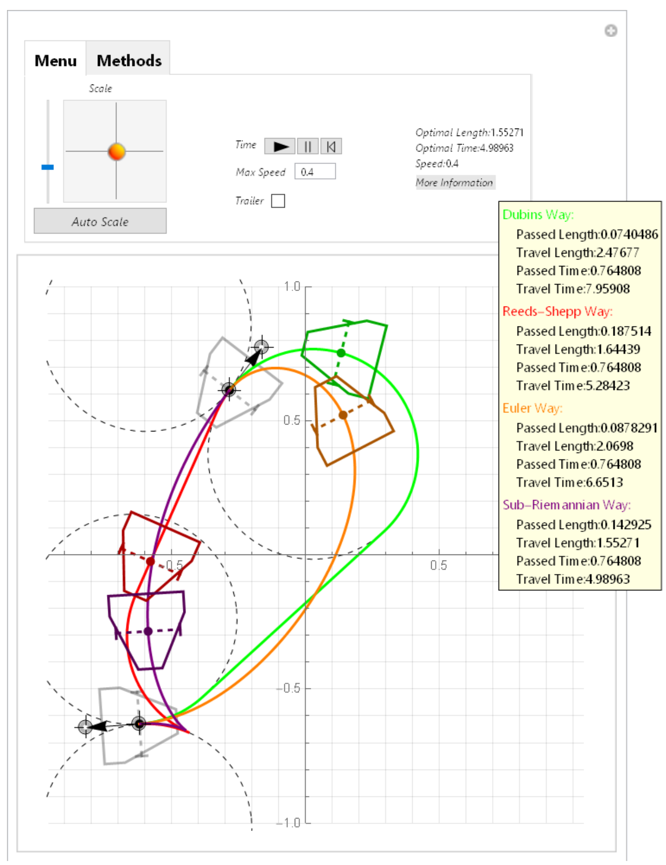

Fig. 2. Screenshot of the interface with *More Information* tooltip

- *Time* timer allows to observe the movement of the robot (with a trailer) along the optimal path.
- *Max Speed* window sets the maximum speed of the robot.
- *Trailer* checkbox adds a trailer to the robot.
- *Scale* options allow to choose the active part of the plane (x, y) (details see in [2]).

The main tab also contains the basic information about the optimal path: *Optimal Length*, *Optimal Time*, *Speed*. Also, it is possible to get more details on the applied methods by mouseover on *More Information* (see Fig. 2).

The additional tab called **Methods** allows one to choose models for the parking problem and to display the chosen trajectories (see the top of Fig. 3).

We implement the *All Ways* option to display also all candidate optimal paths for a single chosen method (*Dubins*, *Reeds-Shepp* or *Euler*). For the chosen *All Ways* option, it is possible to switch between paths using *Way* popup menu (see Fig. 3). Moreover, a priori non-optimal paths are marked with red for the Reeds-Shepp method (see Fig. 3, details can be found in [2]).

Radius slider allows one to change the radius of maximum curvature $R = 1/\kappa_{\max}$ for both Markov-Dubins and Reeds-Shepp methods.

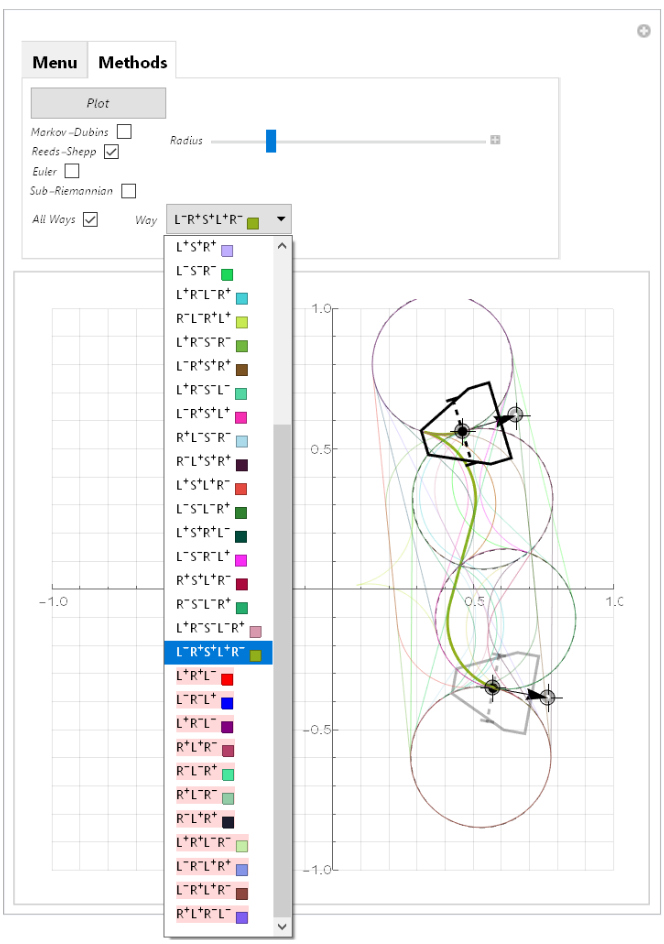

Fig. 3. Screenshot of the interface with open **Methods** tab and list of possible ways for the Reeds-Shepp problem

Similar *Length* slider determines the length of an optimal elastica for Euler's problem.

Locators on the rendered plane (x, y) set boundary positions of the robot q_0, q_1 (as well as the initial trailer orientation φ_0 when *Trailer* option is chosen). *Plot* button starts the calculation of the desired solutions for the given boundary conditions with the appropriate configuration of the solution method and eventually draws the corresponding trajectories in the active part of the plane (x, y).

B. *Plotting of optimal curves*

There are 6 possible motion trajectories for the Markov-Dubins problem [6] and 48 possible variants for the Reeds-Shepp problem [7]. Each variant consists of a combination of straight lines segments and arcs of circles. We inherit the function for plotting such kinds of curves from the first version of the interface [2].

Both the Euler elastic problem and the sub-Riemannian problem on $\mathrm{SE}(2)$ were reduced to solving systems of three equations depending on elliptic integrals F, E and Jacobi elliptic functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ [13]. Explicit formulas for these systems are obtained from the expressions for the exponential mapping given in [15], [14]. Description of the domains for the roots corresponding to optimal solutions is given in [9], [16].

Euler's elastic problem has two candidates for the solution; we plot the one with minimal energy using a fast function for the construction of Euler elastica. If *AllWays* checkbox is chosen, then the other candidate will be displayed as well but in a semi-transparent color (the robot won't move along this route).

However, a situation, when both solutions are optimal, is possible as well, see Fig. 4 illustrating the so-called hidden Maxwell point [17].

The sub-Riemannian problem for the general boundary conditions has only one candidate for optimality, since the problem is studied completely and the cut time is known for this problem [16].

The computation of optimal solutions starts by *Plot* button and the corresponding function returns the following data for each of the selected methods:

- the number of the optimal path or the path selected by the user in *Ways* popup menu,
- length,
- list of all possible trajectories,
- functions for constructing optimal ones $x(t), y(t), \theta(t)$ defined for $t \in [0, T]$ (if *Trailer* checkbox is chosen, then $\varphi(t)$ is also included).

These data allow one to draw the corresponding paths on the active part of (x, y) and to animate the movement of the robot (with a trailer) along the paths.

The trajectories of the center of the trailer

$$\big(x(t) - l_t \cos(\theta + \varphi), y(t) - l_t \sin(\theta + \varphi)\big)$$

are displaied by dotted lines with the same color as the robot trajectory (see Fig. 4).

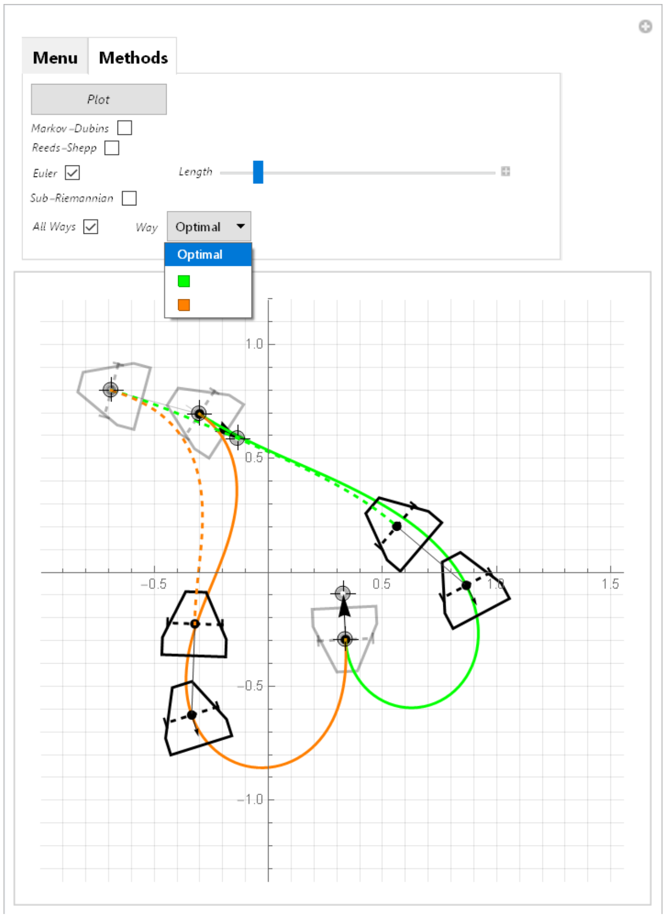

Fig. 4. Screenshot of the interface with two non-symmetric Euler elasticae as optimal paths: movement of the robot with a trailer

TABLE I
LENGTH OF THE PATH AND TIME OF CALCULATIONS

Method	Length,m	Time,sec
Markov-Dubins problem	1.714	0.015625
Reeds-Shepp problem	1.233	0.109375
Euler's elastic problem	1.714	0.640625
Sub-Riemannian problem on SE(2)	1.26	2.45313

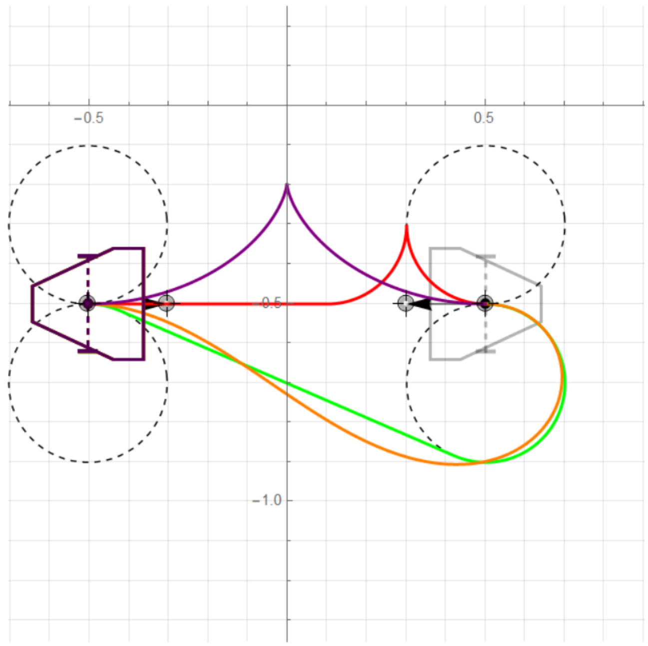

Fig. 5. Comparison of the models

C. Animation

The resulting trajectories admit arbitrary time (or equivalently length) reparametrization $t = f(\hat{t})$ with condition $f'(\hat{t}) > 1$ to define the inverse function $\hat{t} = f^{-1}(t)$. Assuming $\hat{u}_1(0) = \hat{u}_1(\hat{T}) = 0$ we rescale the solutions of the parking problem by the method of acceleration and deceleration along paths used for full-scale robots to move along optimal Euler elasticae [18]. We generalize this method to the considered 4 models and implement the corresponding algorithm in the animation tool of the interface.

In the situation when several methods are chosen, the animation of the robot movement will continue until the robot reaches the final position of the time-optimal path.

IV. Comparison of the Models

All methods work with the same accuracy 10^{-3} specified in the program since there is no need to display trajectories of higher accuracy. The tasks will be compared by the length of the path and the time of calculations for each of the methods (see TABLE I).

The reversal problem was analyzed using the following configuration: the initial position of the robot is $(-0.5, -0.5.0)$ and the final position is $(0.5, -0.5, \pi)$, see Fig. 5.

In the case of Euler's problem, any length of the path can be set, therefore, it is chosen to be the same as for the Markov-Dubins problem.

According to TABLE I, the Reeds-Shepp method is the shortest for this configuration. The robot reaches the endpoint in 3.96 seconds, on condition the acceleration and deceleration function is used.

The fastest solution method is the Markov-Dubins method since it requires less amount of calculation than the other methods. It can be used for motion planning problems when it requires getting a solution on the fly. The method based on Euler's problem gives more elegant method and is preferable when we have enough time to plan the motion on the spot.

The method based on the sub-Riemannian problem requires an additional parameter for the fair comparison with the other methods. The parameter analogous to *Radius* (for Markov-Dubins and Reeds-Shepp) and *Length* (for Euler's problem) is a compromise parameter $\mu > 0$ generalizing function (12) to $\sqrt{u_1^2 + \mu^2 u_2^2}$.

V. Future Plans

We aim to obtain a greater instrument designed both for specialists in optimal control theory and for engineers in robotics.

We will add buttons for exporting the data for performing the corresponding experiment with the real robot (with a trailer), e.g., the file with discrete velocity data for the driving wheels along the desired way.

The algorithm for reparking the trailer along sub-optimal path [19] will be implemented considering the given final configuration of the trailer position $\varphi(T) = \varphi_1$.

We are already implementing quite new models of the parking problem arising from sub-Finsler geometry and convex trigonometry [20].

Also, we are going to consider the parking problem with holonomic constraints in the plane (x, y). We plan to implement an option for adding a circle obstacle in the active part of the plane (x, y), the resulting trajectories should not pass over this obstacle.

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