On the hyperbolic model of visual color perception

17-11-01387-P

Alexey Mashtakov

Ailamazyan Program Systems Institute of RAS

Совещание “Математика в эпоху суперкомпьютеров”

В рамках Национального суперкомпьютерного форума

Pereslavl-Zalesskiy, 29.11.2021

Structure of the Talk

- Historical Remarks
- Physiology of Human Vision
- Functional Architecture of Primary Visual Cortex V1
  - Mathematical Model of Contour Completion
- Color Space Metrics
- Color Vision Metrics
- Applications

  Conclusion

Historical Remarks

Historical Remarks

In [1] it was shown that there are only two possible Riemannian manifolds that model the perceived color space, compatible with the set of Schrödinger axioms supplemented by the homogeneity hypothesis.
One possible case is given by the well-studied Helmholtz-Styles metric on the space (RGB)

$$P_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \quad ds^2 = a_1 \left(\frac{dx_1}{x_1}\right)^2 + a_2 \left(\frac{dx_2}{x_2}\right)^2 + a_3 \left(\frac{dx_3}{x_3}\right)^2.$$

The second case is given by the invariant metric g on the homogeneous space of hyperbolic type

$$P_2 = \mathbb{R}^+ \times SL_2/SO_2, \quad ds^2 = \frac{1}{2} Tr((x^{-1}dx)^2).$$

Advanced results on the study of this model were obtained in a series of papers [2]—[5].

1 Resnikoff, H.L. Differential geometry and color perception. JMB (1974).

2 Provenzi, E. Geometry of color perception. J. Math. Neurosc (2020).

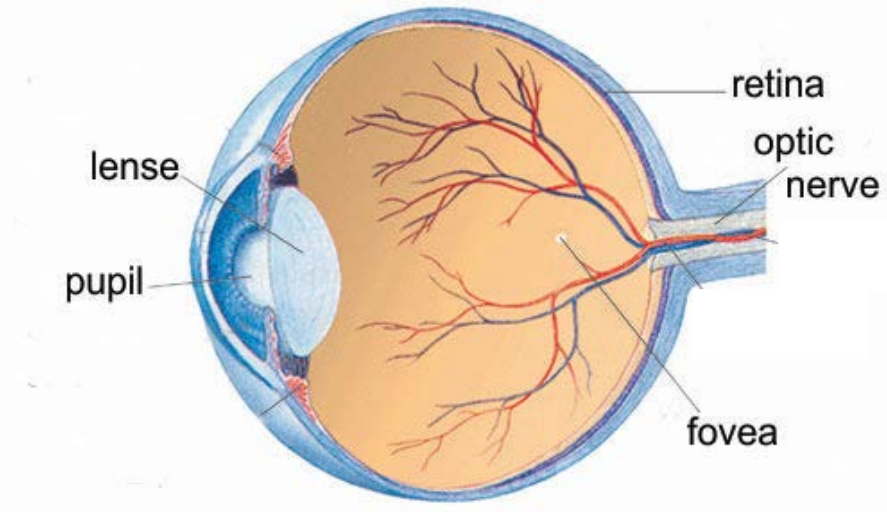
3 M. Berthier, E. Provenzi, The quantum nature of color perception: Uncertainty relations for chromatic opposition, J. Imaging, (2021).

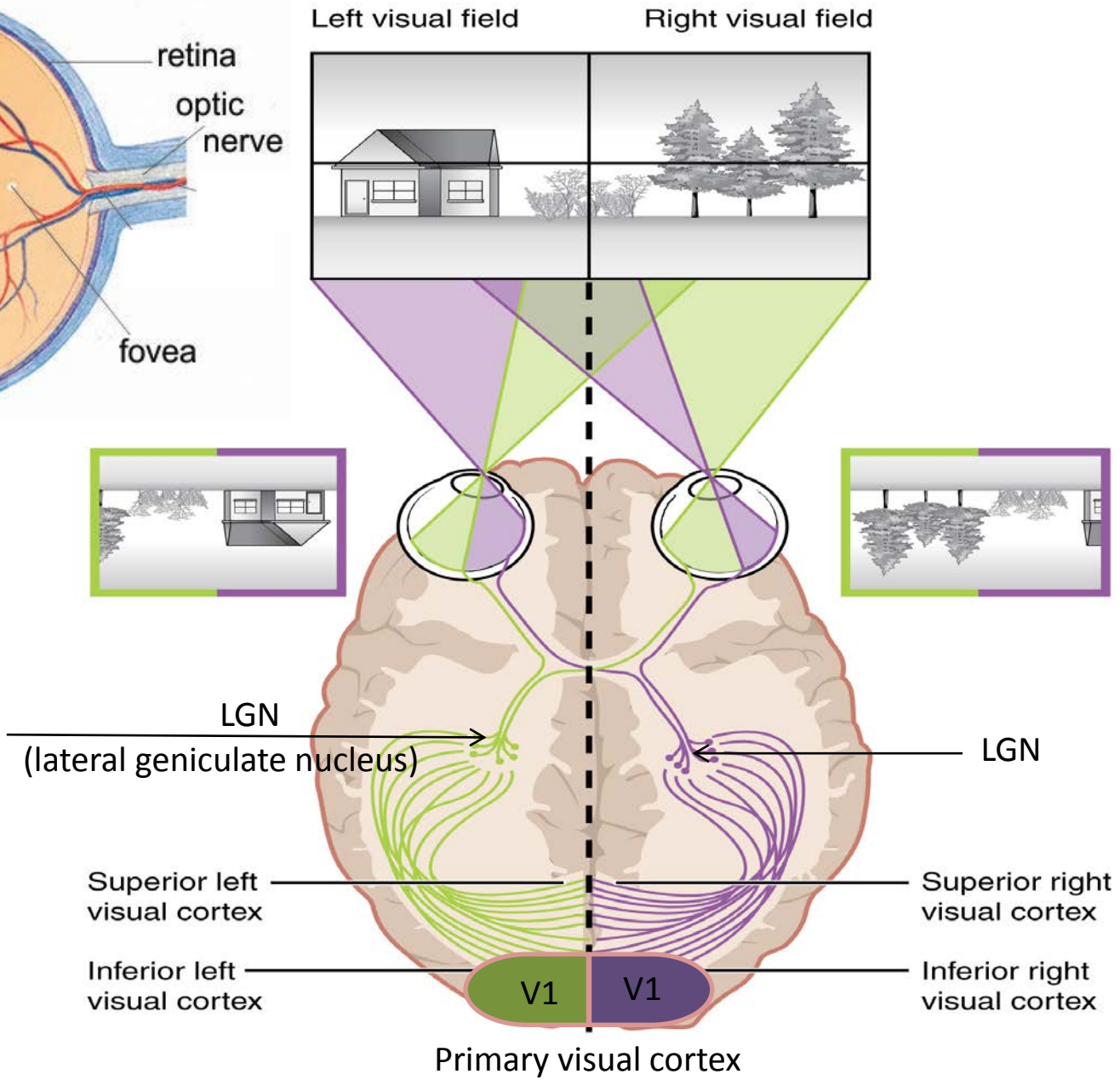
4 M. Berthier, E. Provenzi, From Riemannian trichromacy to quantum color opponency via hyperbolicity, Jour of Math Imag and Vision, (2021)

5 M. Berthier, V. Garcin, N. Prencipe, E. Provenzi, The relativity of color perception, Jour of Math Psychology, (2021)

Brief Tour in Physiology of Human Vision

Perception of Visual Information by Human Brain





Physiology of Human Eye

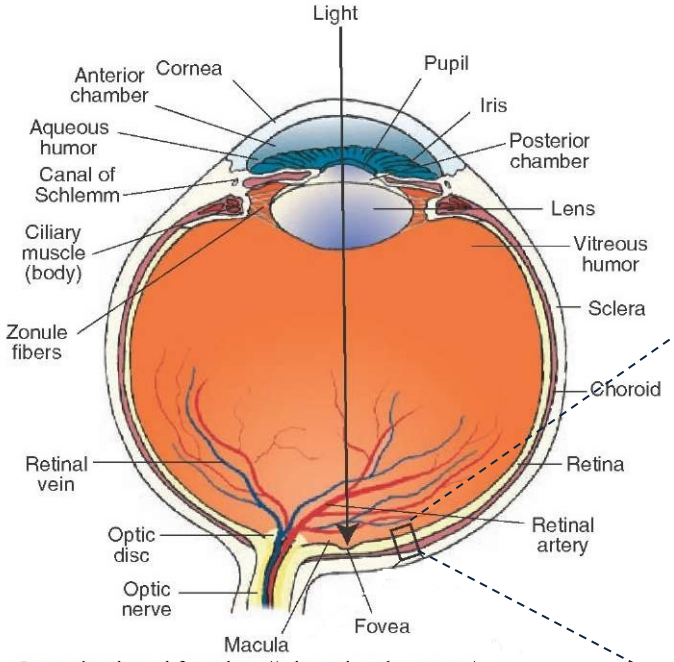
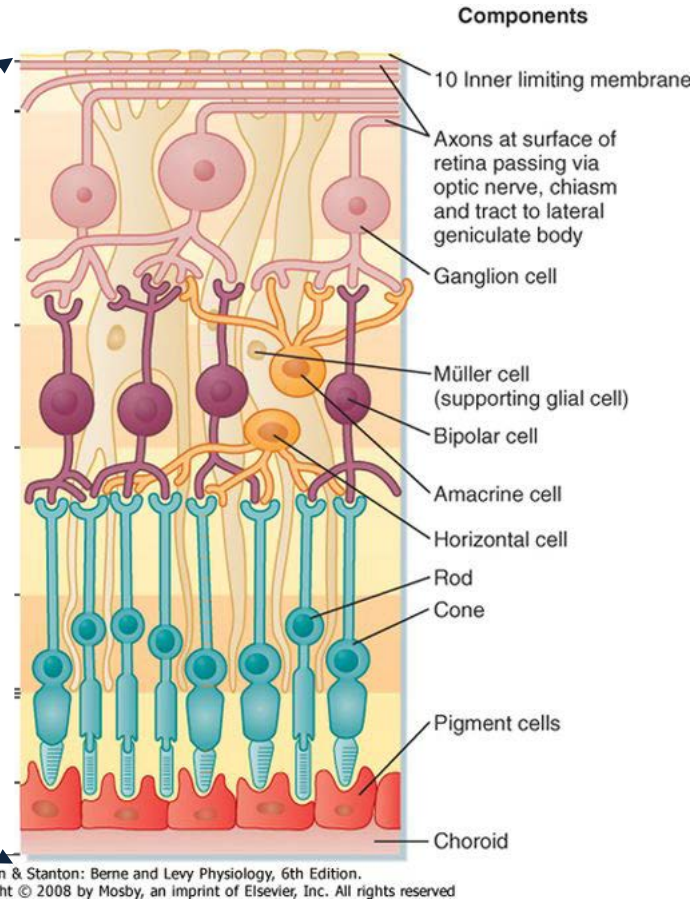
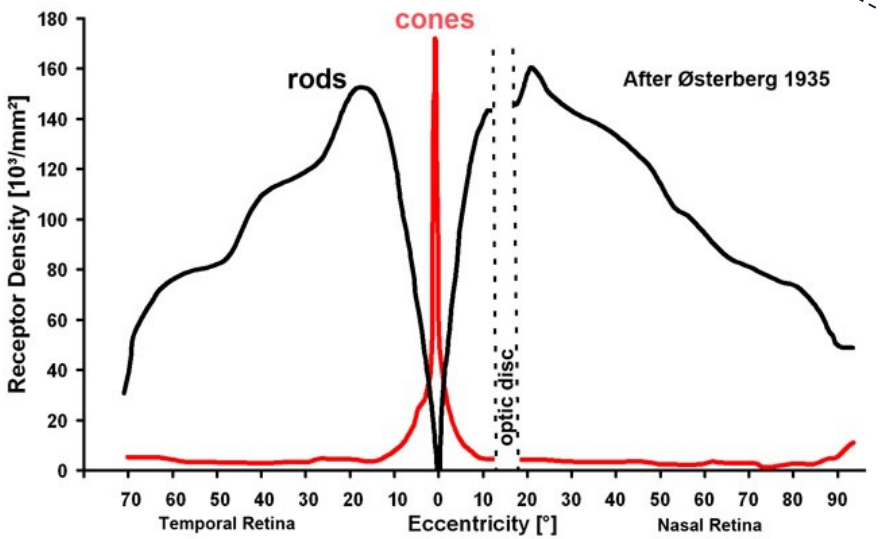


Image is adapted from http://what-when-how.com/

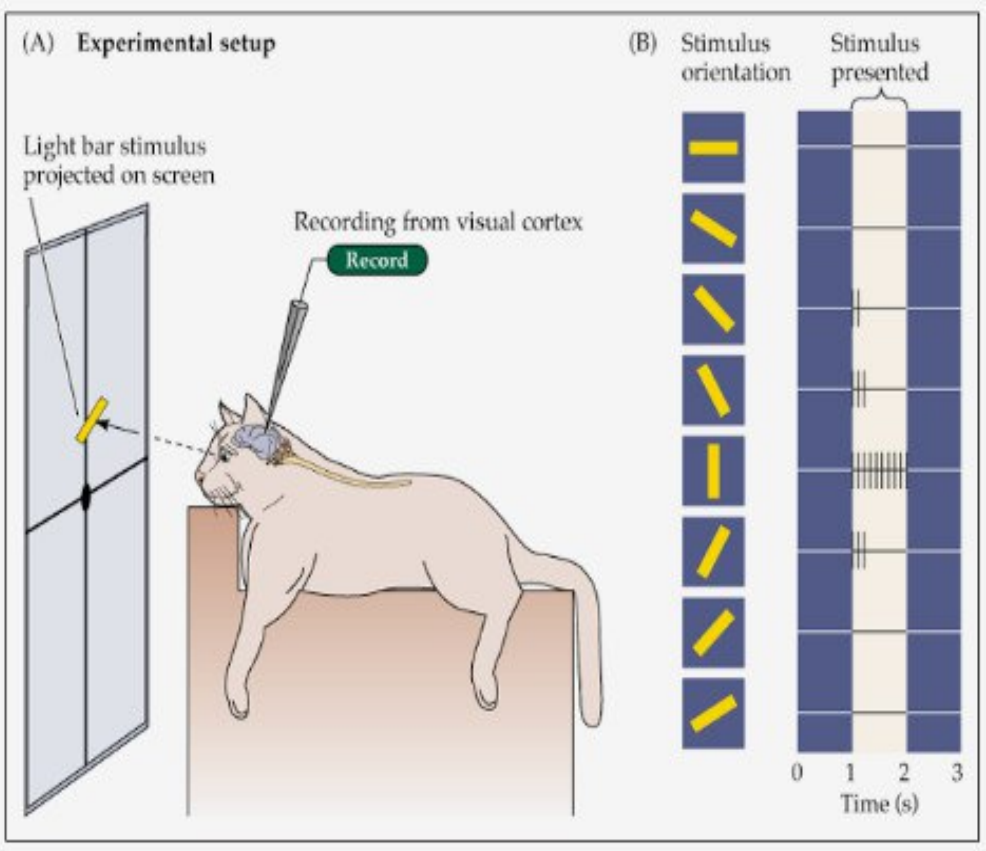


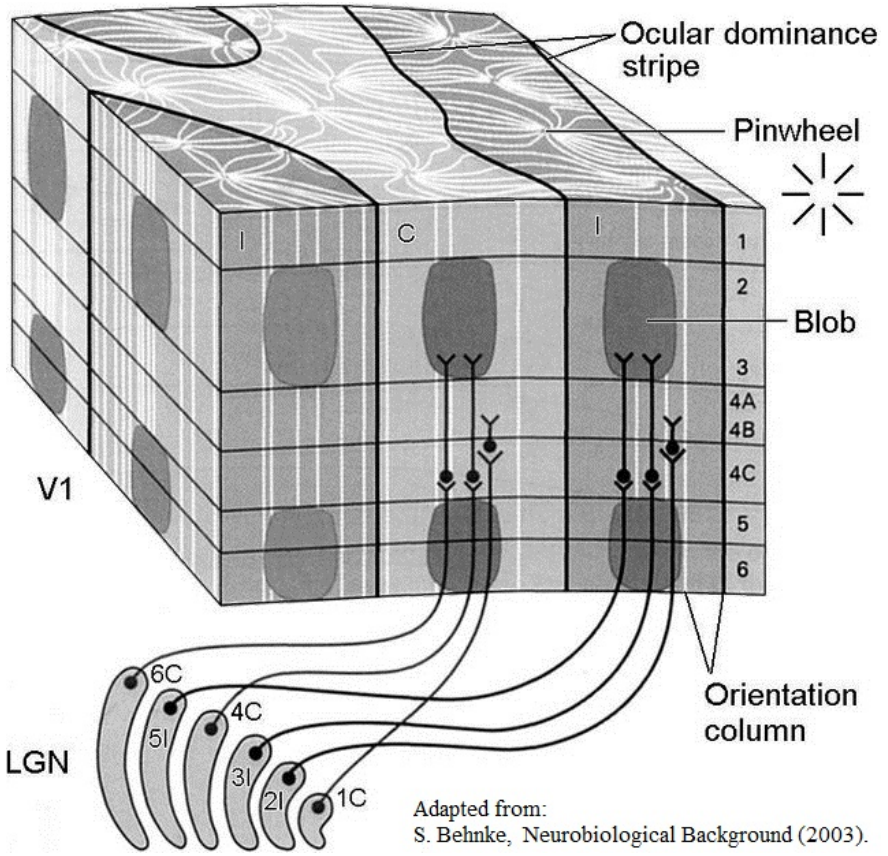


Opsins:

- three types in cones,
- one type in rods,
- one type in ganglion cells.

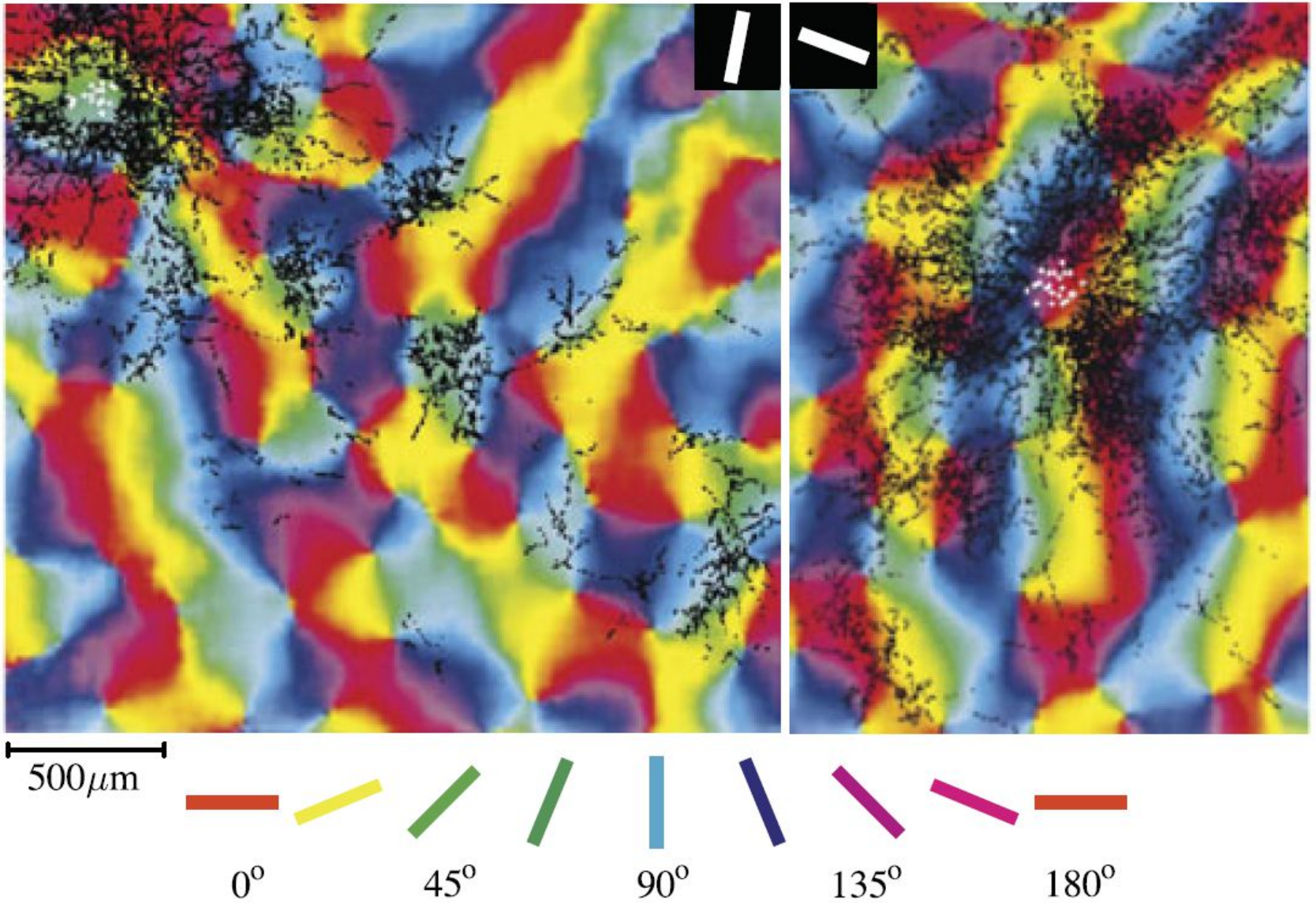
Primary Visual Cortex V1 of Mammals





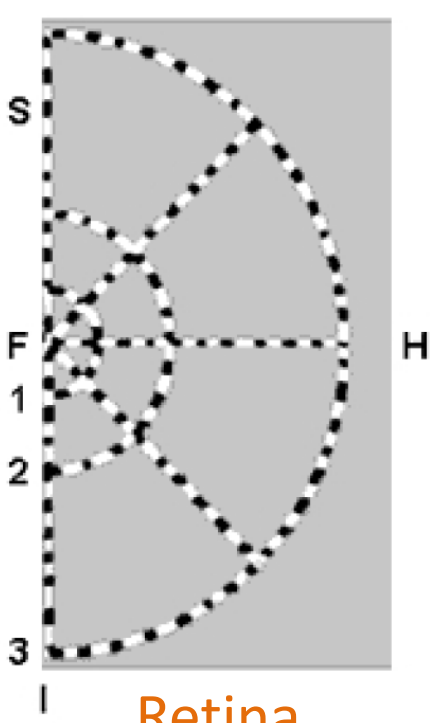
D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.

Primary Visual Cortex V1

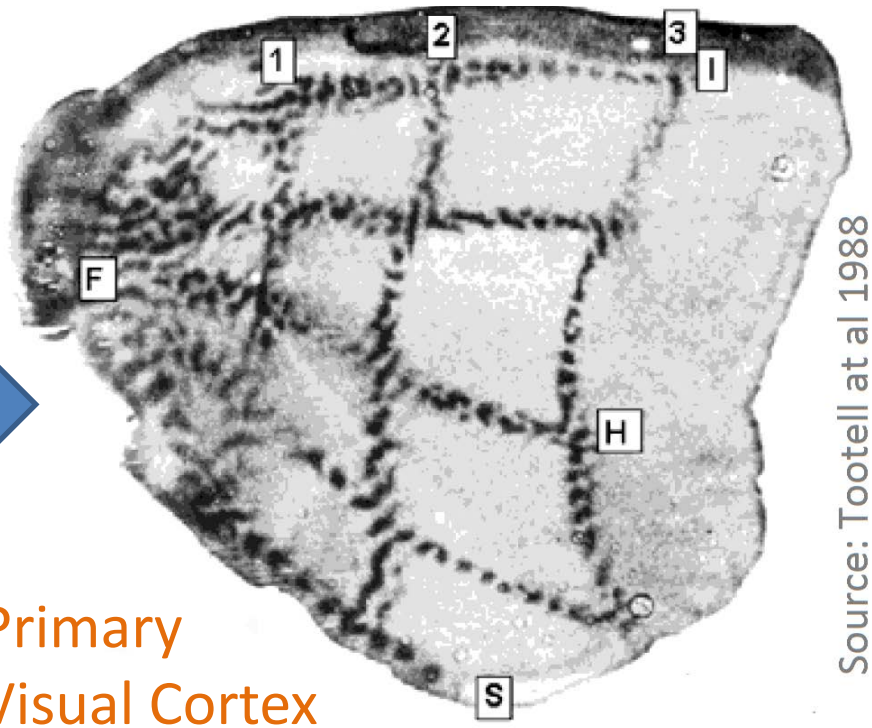


Replicated from Bosking, W.H., et al., Orientation selectivity and the arrangement of horizontal connections in tree shrew striate cortex. J. Neuroscience, 1997

Cortical Magnification



Retina



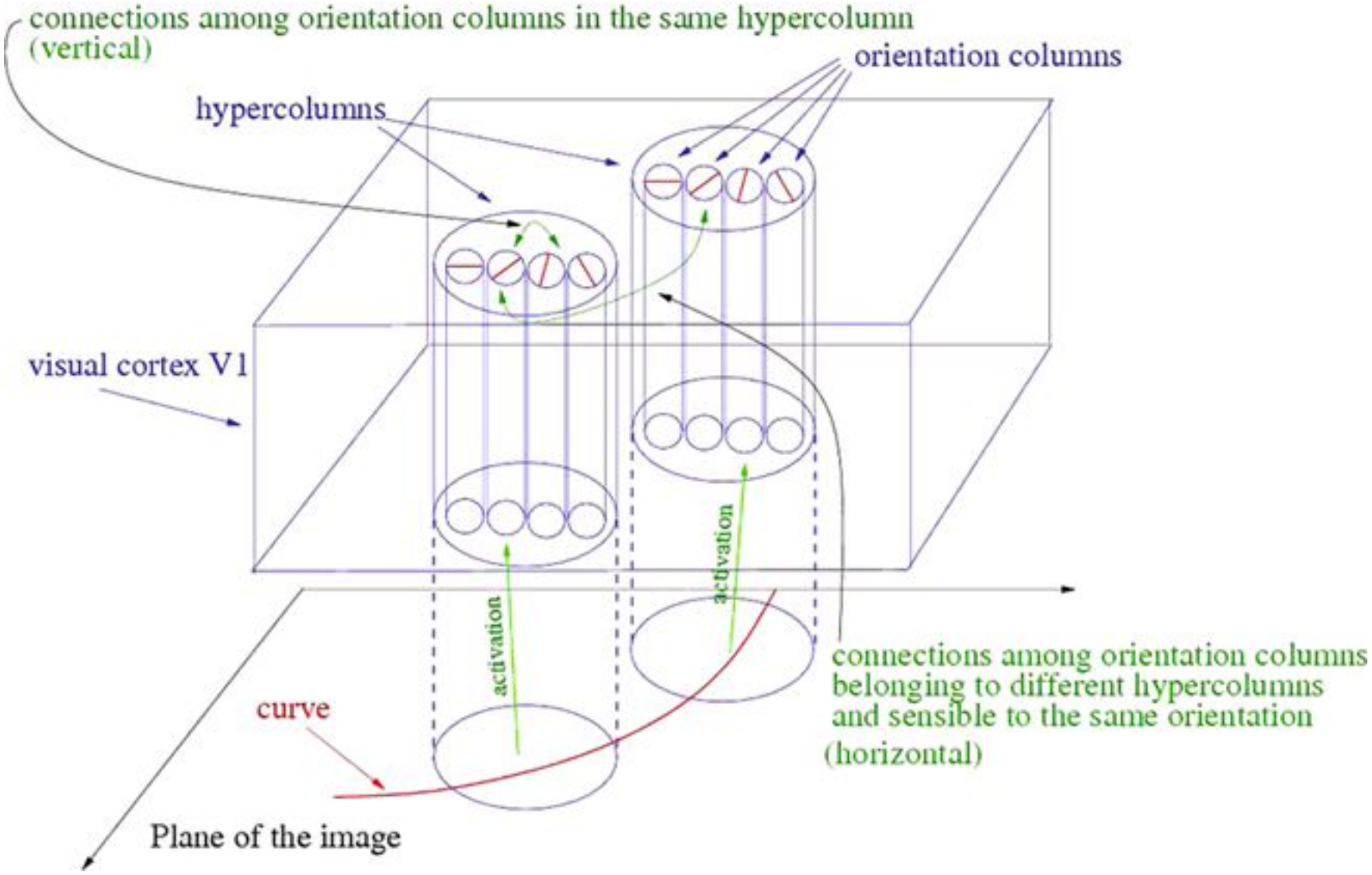
Primary
Visual Cortex

Source: Tootell at al 1988

R.B.H. Tootell, E. Switkes, M.S. Silverman, S.J. Hamilton. Functional anatomy of macaque striate cortex. II. Retinotopic organization. Journal of Neuroscience, 1988.

Mathematical Modelling of Functional Architecture of the Primary Visual Cortex V1

A Model of the Primary Visual Cortex V1

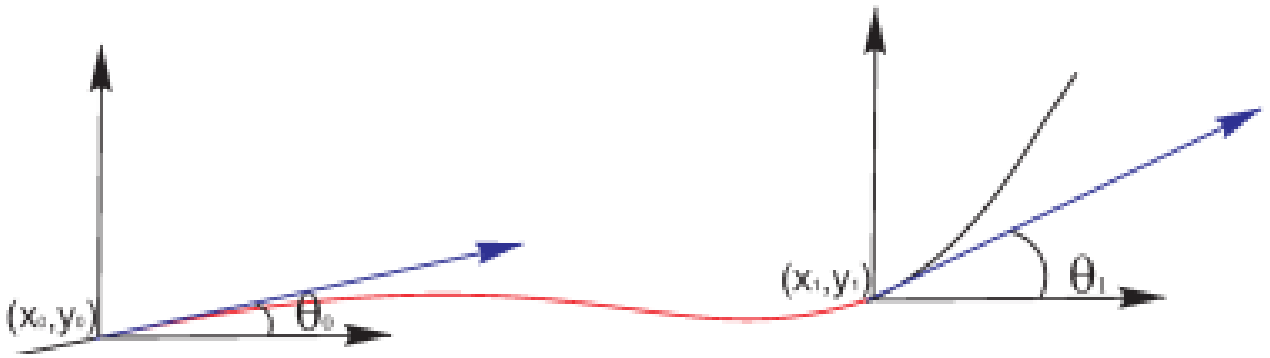


Replicated from R. Duits, U. Boscain, F. Rossi, Y. Sachkov,
Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2), JMIV, 2013.

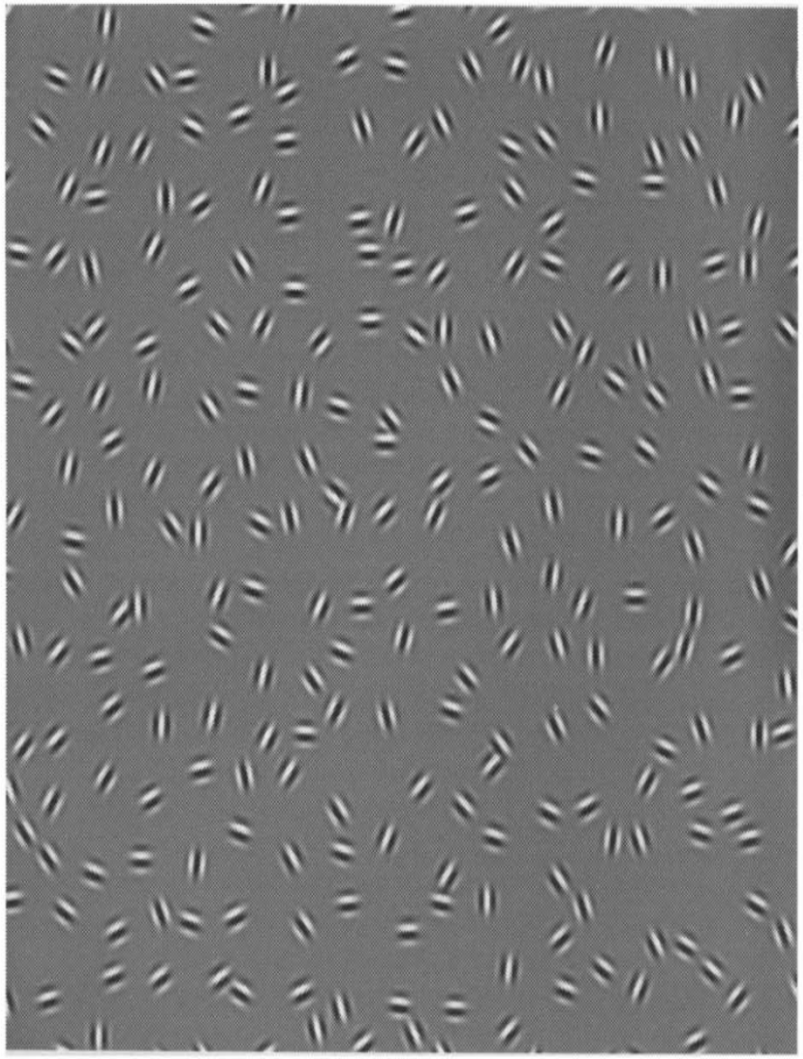
Cortical Based Model of Perceptual Completion

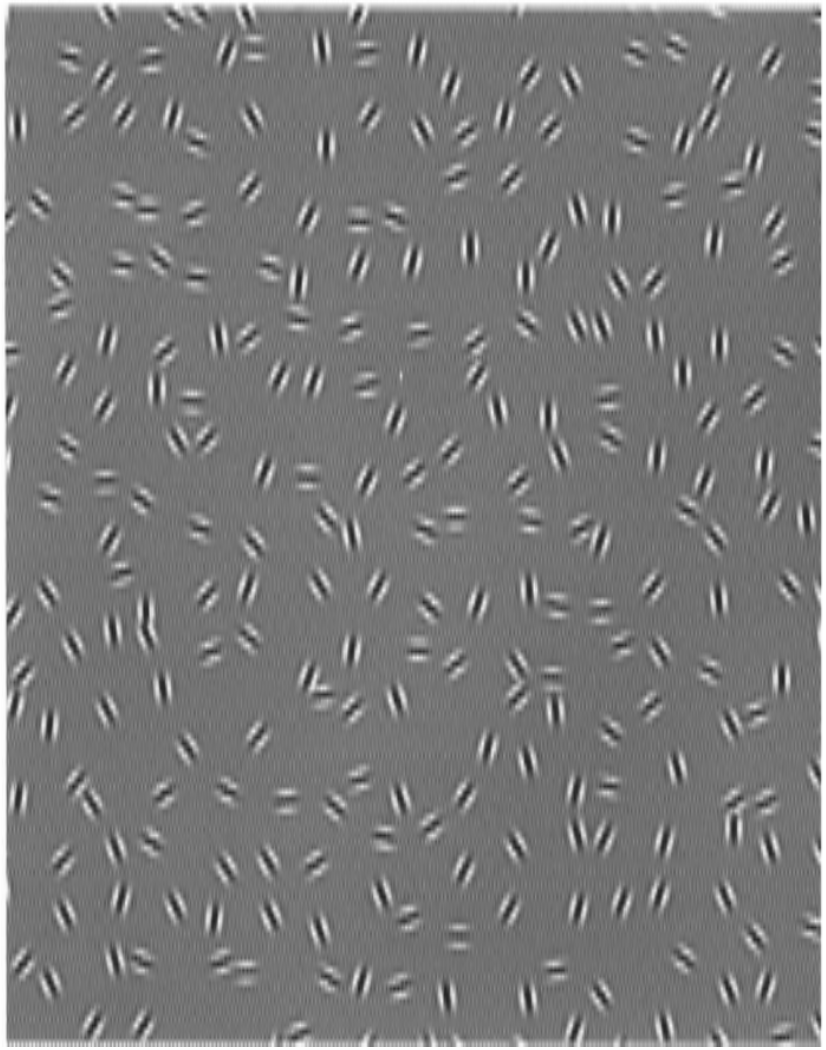
- D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.
- Sub-Riemanian structures in neurogeometry of the vision:
  - J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003. (Heisenberg group.)
  - G. Citti and A. Sarti, A Cortical Based Model of Perceptual Completion in the Roto-Translation Space, 2006. ($SE(2)$ group.)
- Variational principle: recovered arc has minimal length in the space (x, y, θ):

$$\int \sqrt{\xi^2 \left(\dot{x}^2 + \dot{y}^2\right) + \dot{\theta}^2} dt \rightarrow \min, \text{ under constraint } \dot{\theta} = \arg\left(\dot{x} + \mathrm{i}\, \dot{y}\right)$$



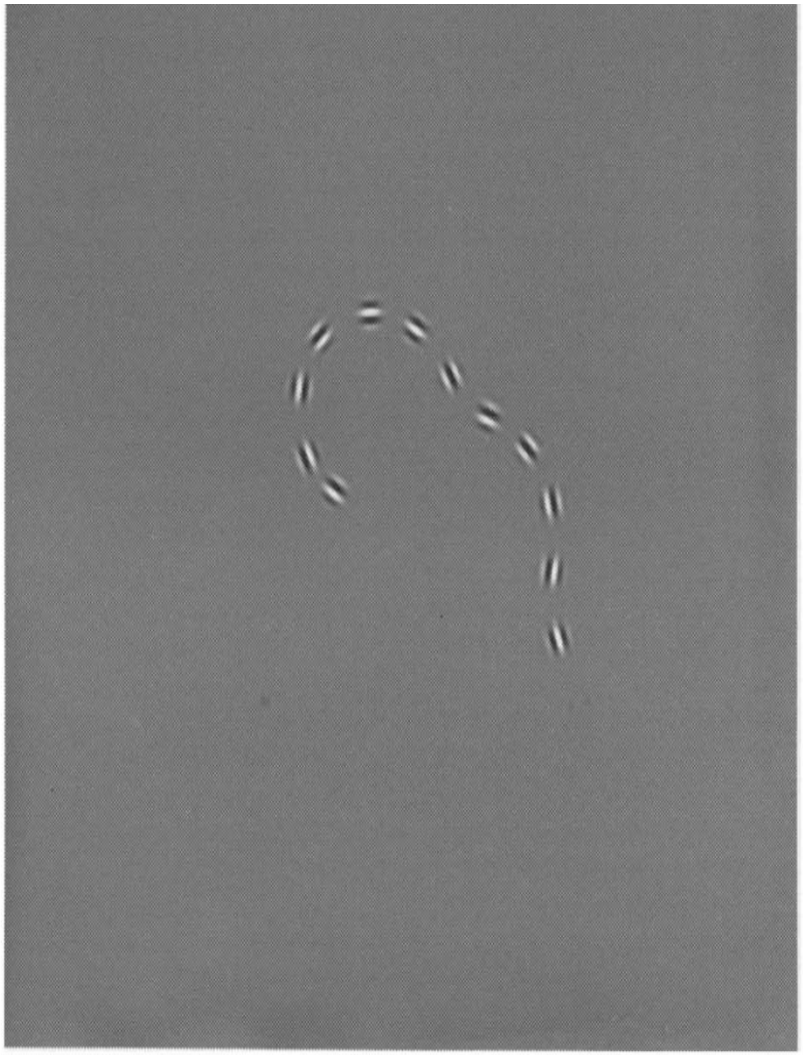
Association fields: conditions for hidden contour completion

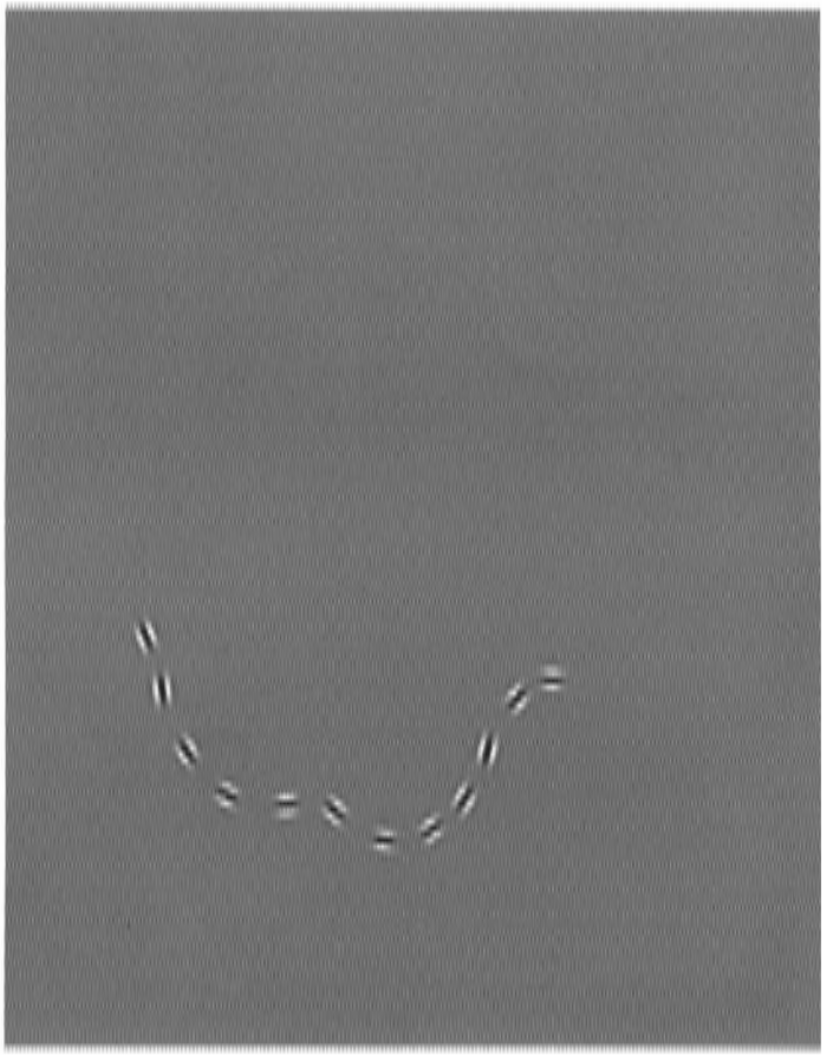




- D.J. Field, A. Hayes, R. Hess. Contour integration by the human visual system: Evidence for a local "association field", Vision Research, 1993.

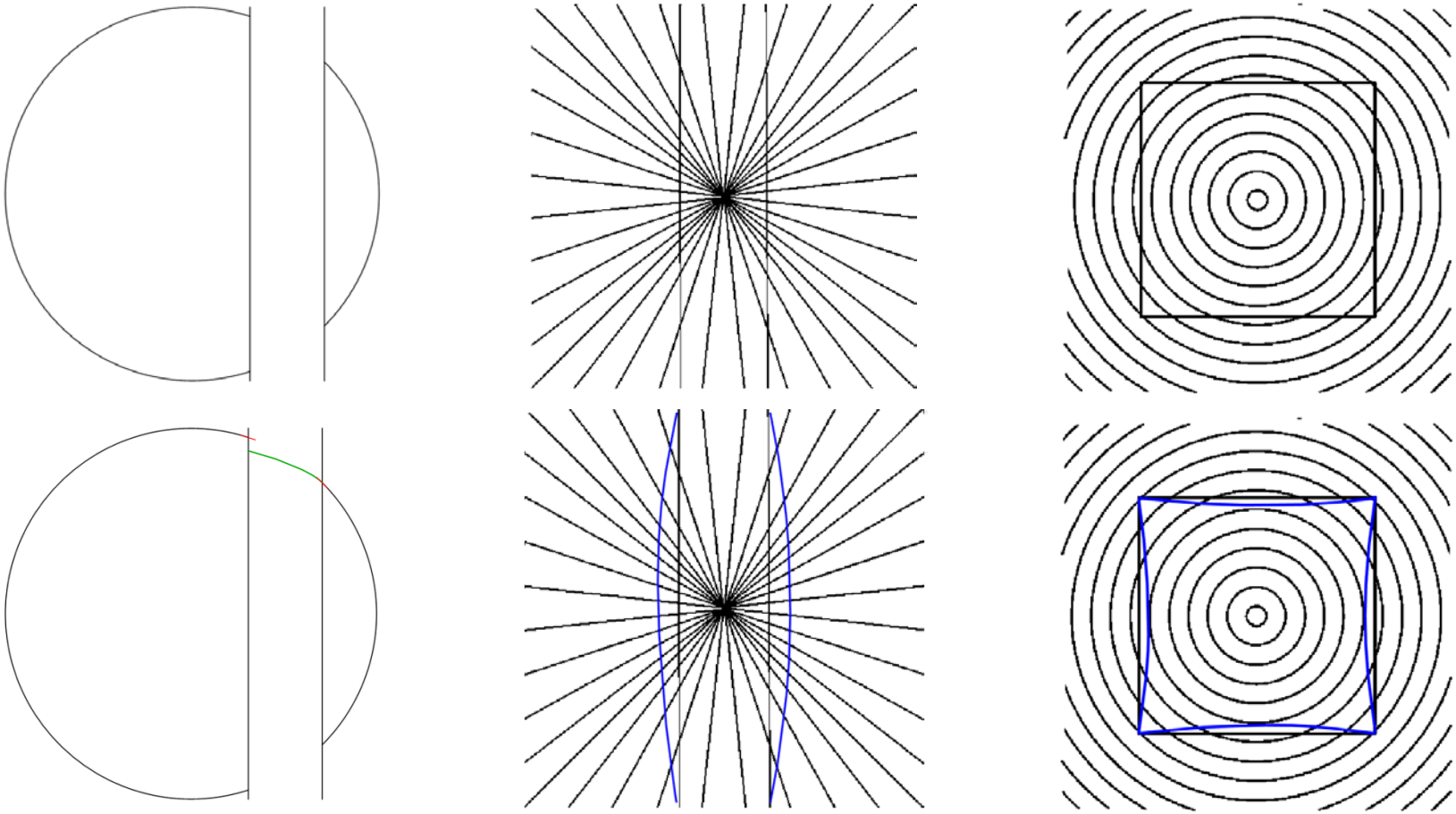
Association fields: conditions for hidden contour completion





- D.J. Field, A. Hayes, R. Hess. Contour integration by the human visual system: Evidence for a local "association field", Vision Research, 1993.

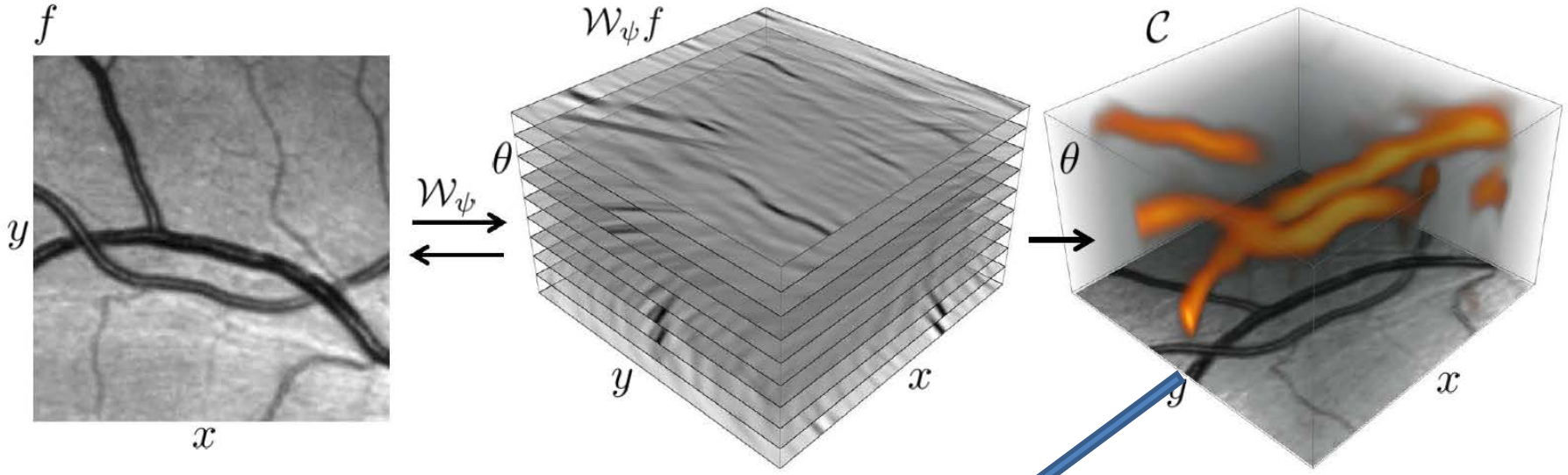
Geometrical Optical Illusions



[**1**] B. Franceschiello, A. Mashtakov, G. Citti, A. Sarti, *Modelling of the Poggendorff Illusion via Sub-Riemannian Geodesics in the Roto-Translation Group*, LNCS, 2017.

[**2**] B. Franceschiello, A. Mashtakov, G. Citti, A. Sarti, *Geometrical optical illusion via sub-Riemannian geodesics in the roto-translation Group*, DGA, 2019.

Adaptive Model of Vision via Data-driven Sub-Riemannian Geodesics



The primary visual cortex lifts the image from the two-dimensional surface of the retina into the extended space of positions and orientations, and sets a sub-Riemannian metric adapted to the image. The hidden contour is reconstructed by the SR geodesics with the specified boundary conditions.

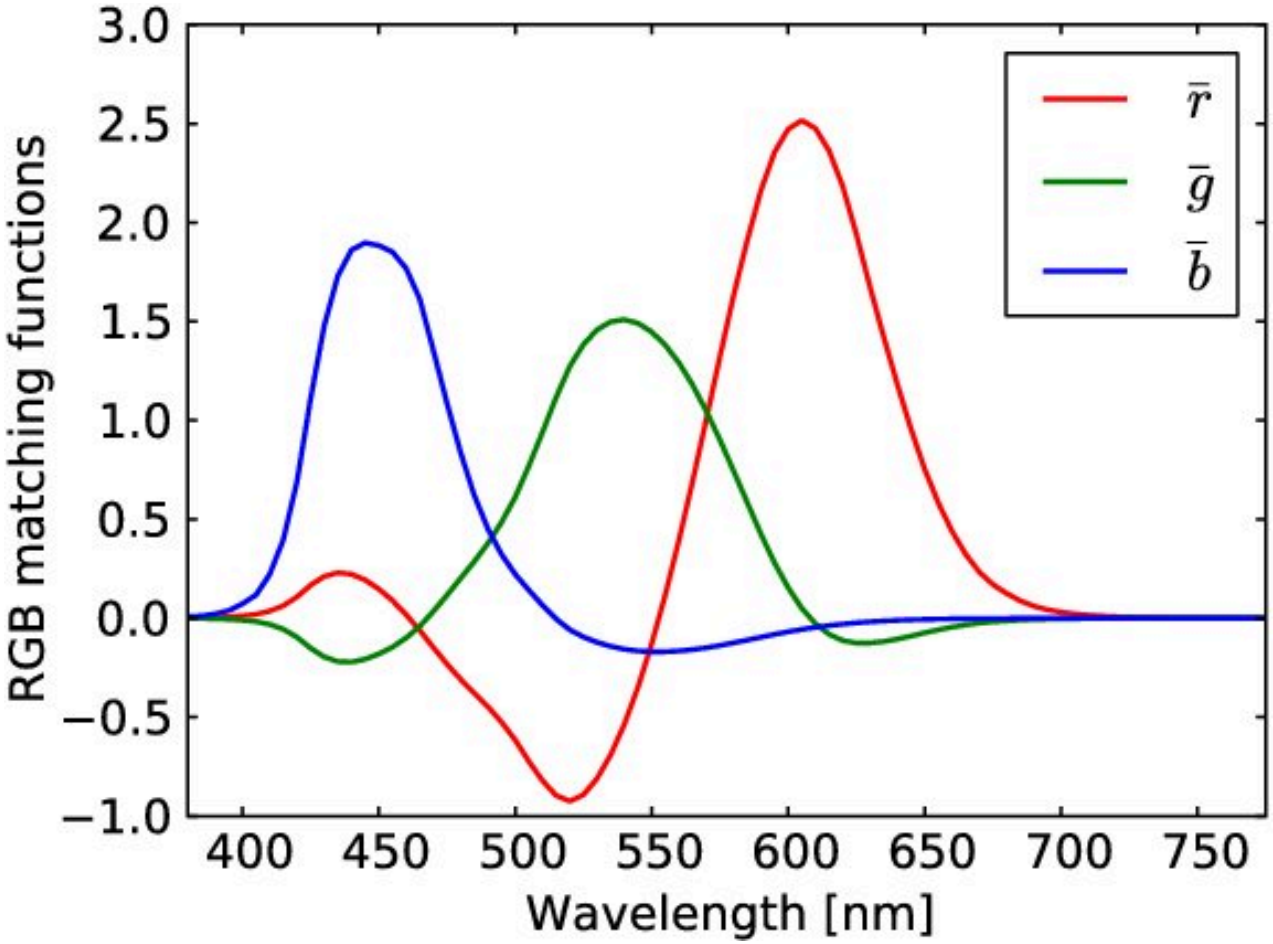
$$l = \int_0^T \mathcal{C}(\gamma(t))\sqrt{u_1^2(t) + u_2^2(t)}\, dt \to \min,$$

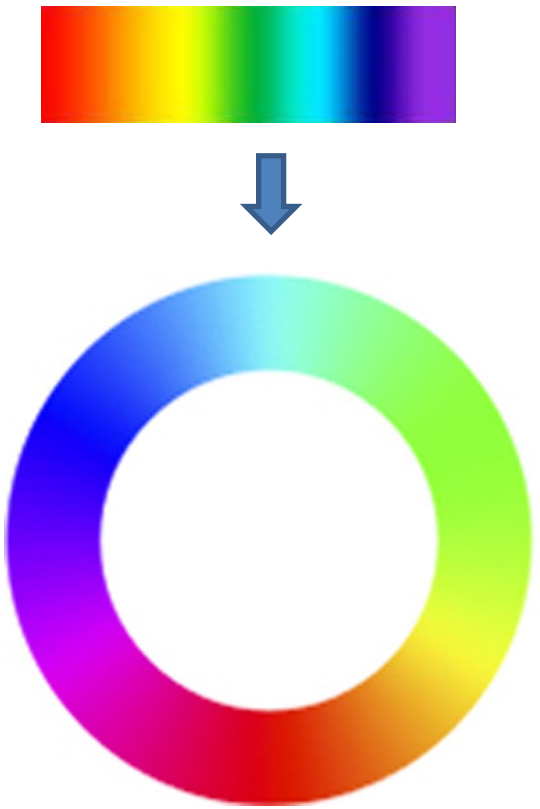
$$\dot{\gamma} = u_1\mathcal{A}_1 + u_2\mathcal{A}_2, \quad \gamma(0) = e,\ \gamma(T) = g$$

$\mathcal{A}_i$ — left invriant vector fields on the rototranslations Lie group.

Color Space Metrics

Photoreceptor Responses

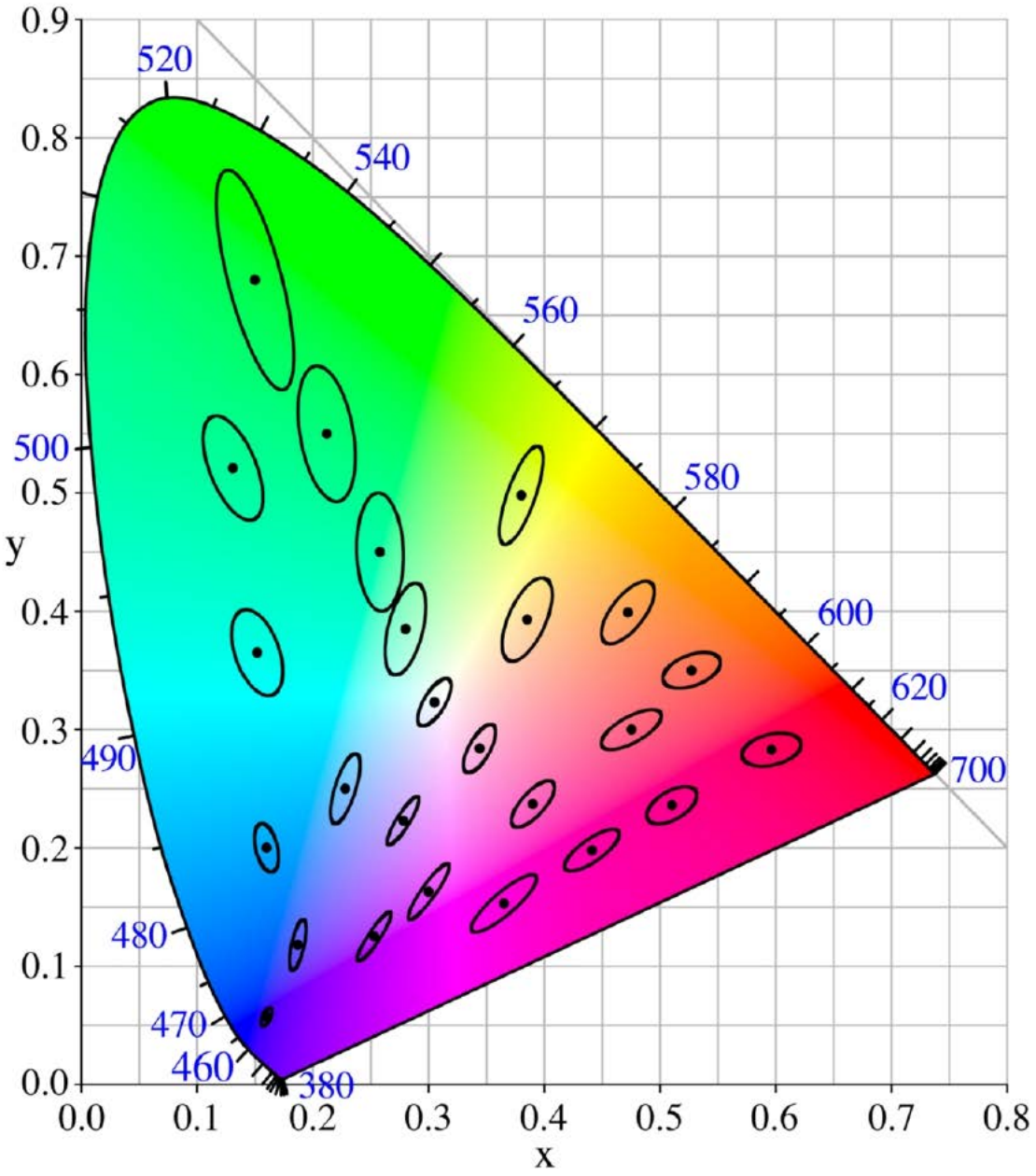




Perceptual color space model should be based on the input data — reaction of activation of photoreceptors under the light source.

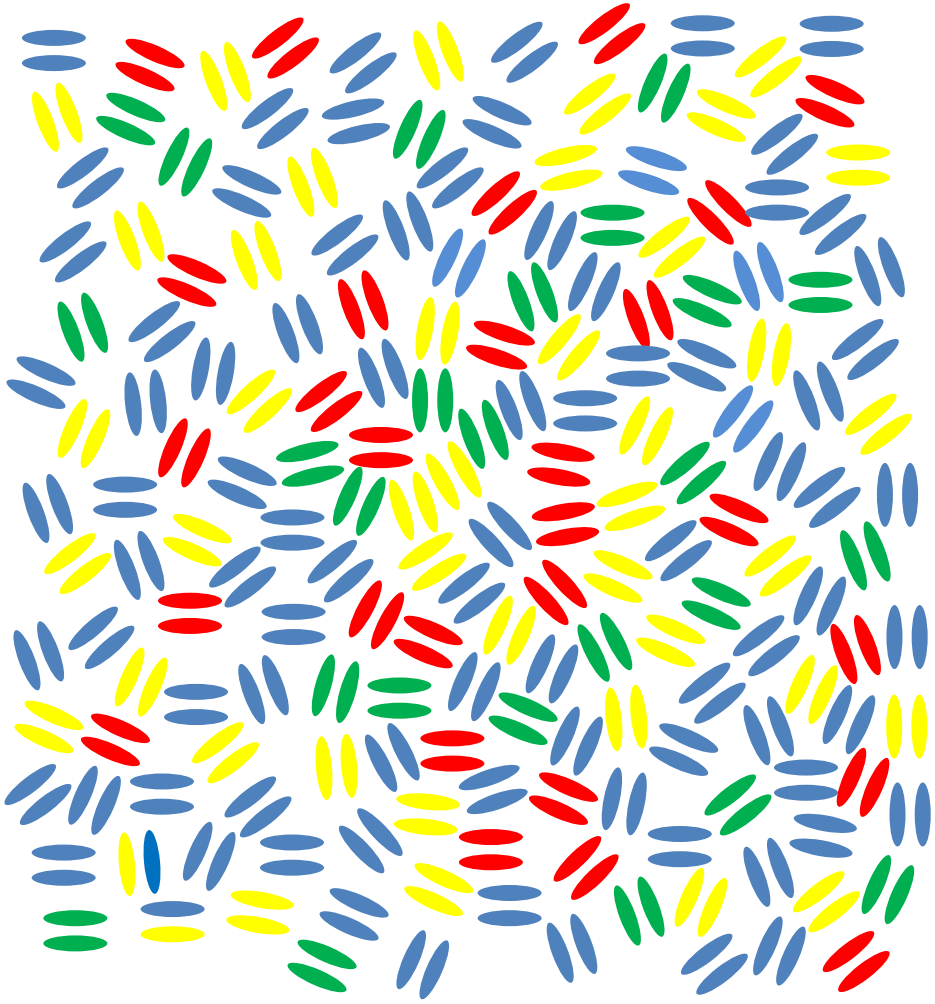
Noticeable Difference Ellipses in Chromaticity Diagram

- D.L. MacAdam, Visual sensitivities to color differences in day light, Josa, 1942.
- Stiles W., The basic data of colour-matching, Phys. Soc. Year Book, 1955.
- D.L. MacAdam, "Uniform color scales", Josa, 1974.
- and many other.

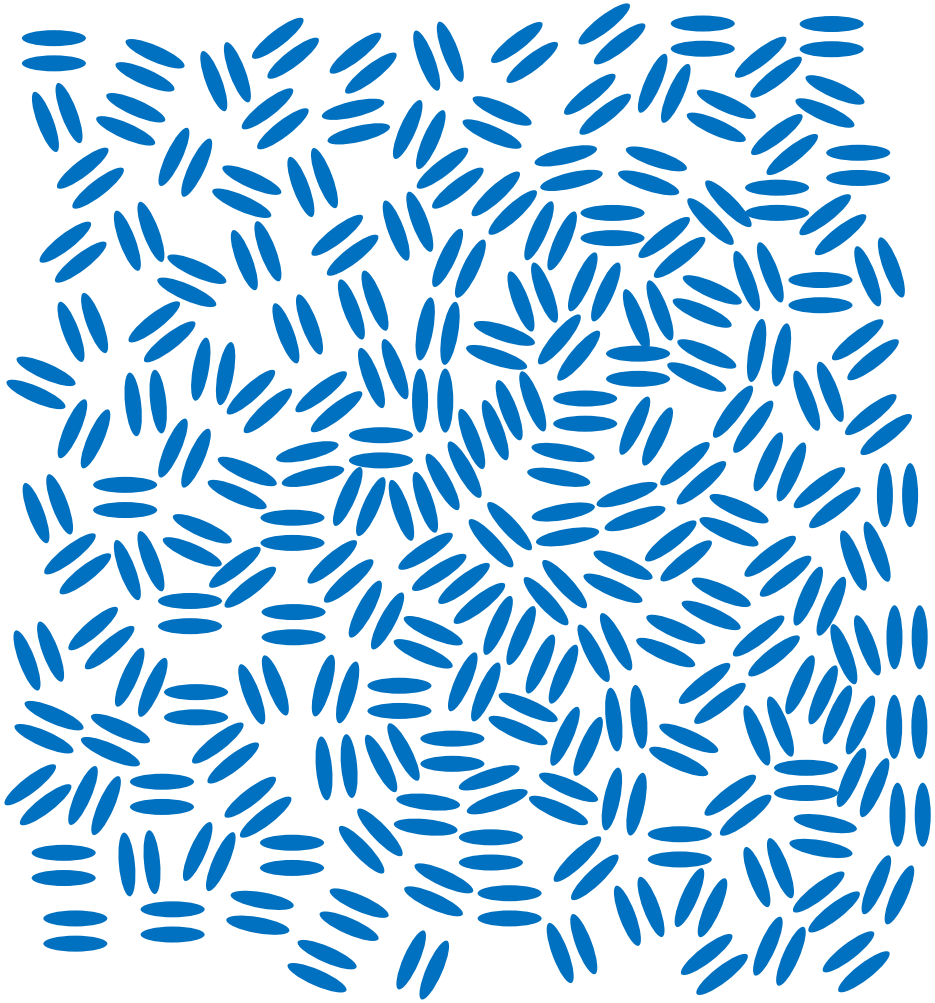


Color Vision Metrics

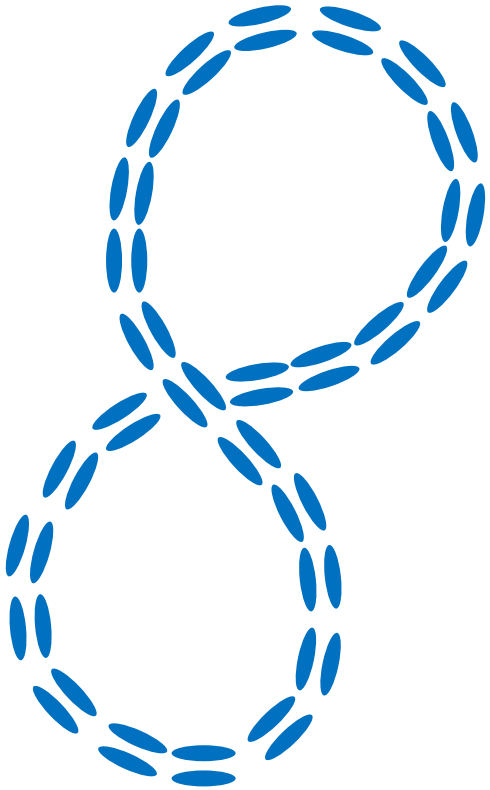
Association Fields in Color Images



Association Fields in Monochromatic Images



Association Fields in Monochromatic Images



Adaptive Color Vision Metric

We propose a model of contour completion in color vision that takes into account both internal geometry of human visual cortex and external information induced by observed image.

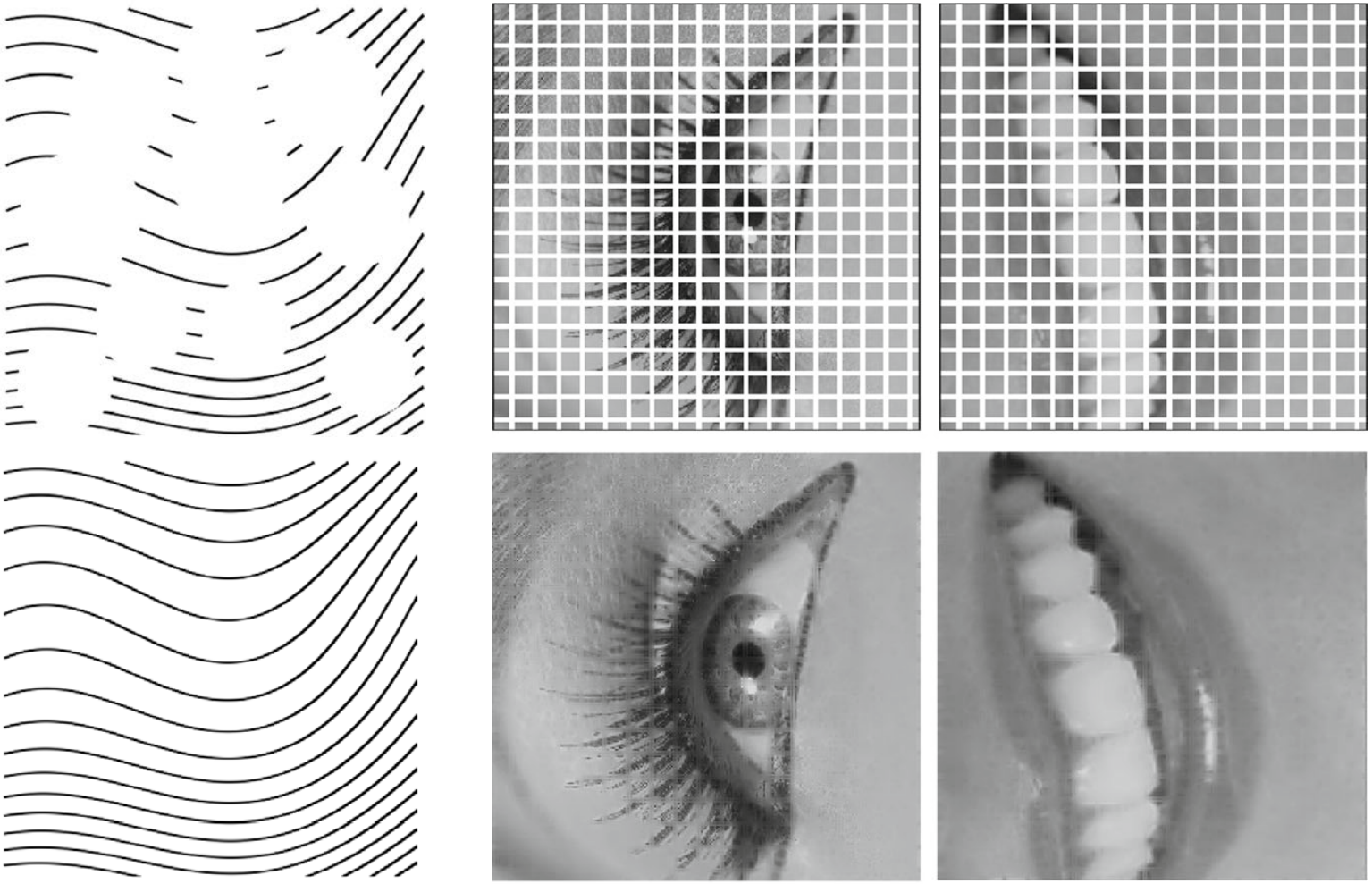
$$\dot{\gamma} = \sum_{i=1}^{2} u_i \mathcal{A}_i, \quad \gamma(0) = e,\ \gamma(T) = g \in SE_2,$$

$$l(\gamma) = \int_0^T \mathcal{C}(e, \gamma(t)) \sqrt{\sum_{i=1}^{2} u_i^2(t)}\, dt \to \min,$$

where the function $\mathcal{C}(e, g) \in \mathbb{R}^+$ is a distance in color space between colors p_0 and p_1, assigned to elements e and g.

Applications

Anthropomorphic Image Reconstruction



[**1**] A. Mashtakov, A. Ardentov, Yu. Sachkov, *Parallel Algorithm and Software for Image Inpainting via Sub-Riemannian Minimizers on the Group of Roto-translations*, NMTMA, 2013.

[**2**] Boscain, U.V., Chertovskih, R., Gauthier, JP. et al. Highly Corrupted Image Inpainting Through Hypoelliptic Diffusion, JMIV, 2018.

Analysis of Images of the Retina

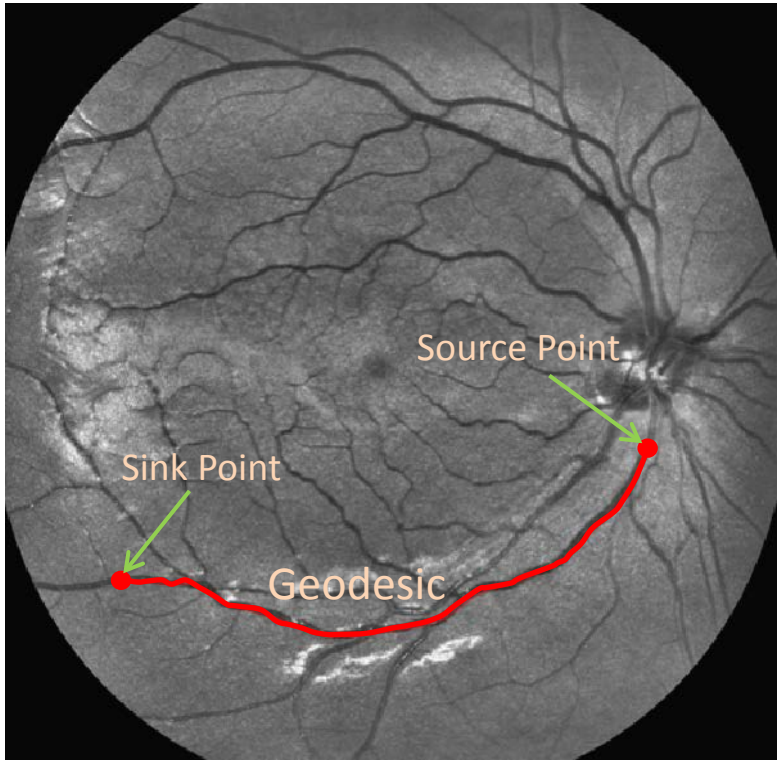
Diabetic retinopathy --- one of the main causes of blindness.
Patients are found early --> treatment is well possible.
Early warning --- leakage and malformation of blood vessels.
The retina --- excellent view on the microvasculature of the brain.

Healthy retina

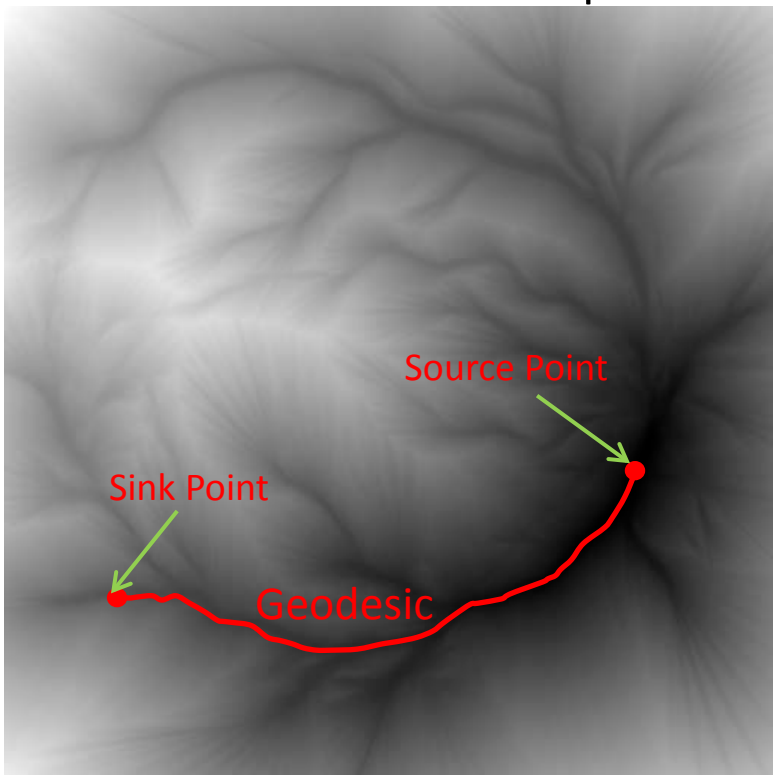
Diabetes Retinopathy with tortuous vessels

Geodesic Methods in Computer Vision

Image



Distance from source point



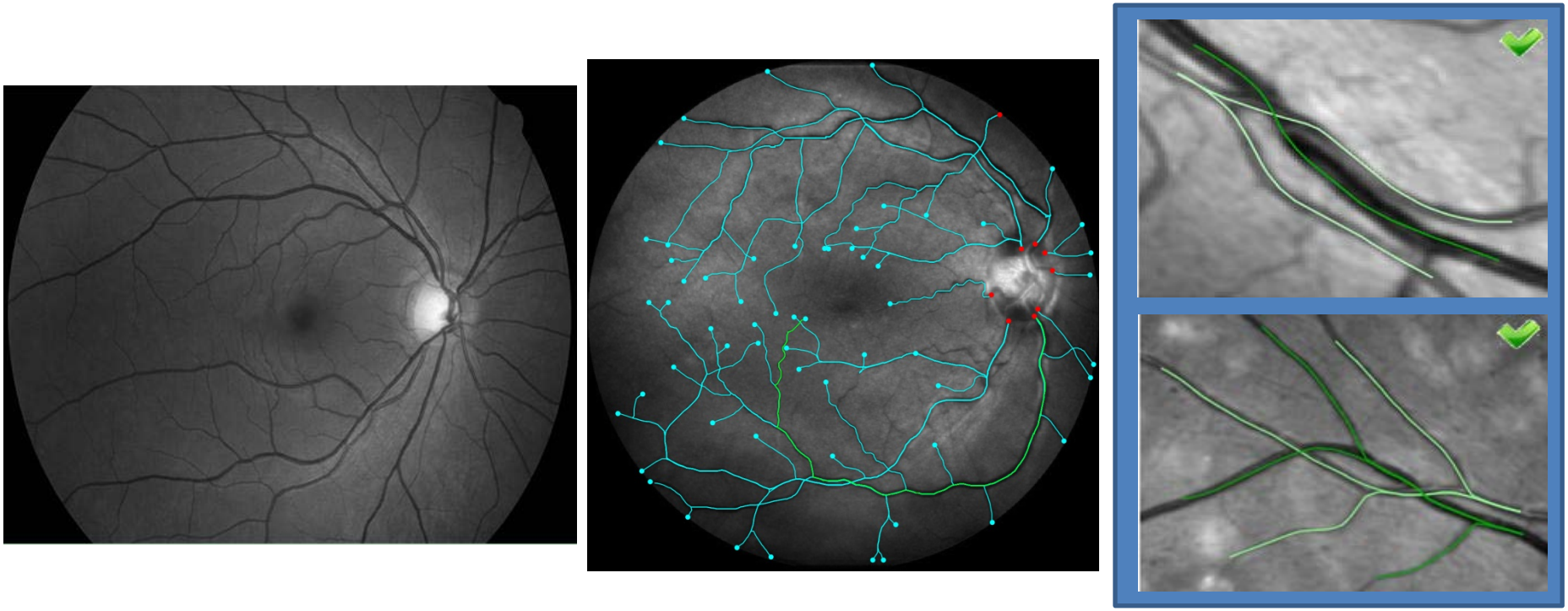
Tracking of salient lines via data-driven minimal paths (or geodesics).

Data-driven Geodesic – curve that minimizes length functional weighted by external cost (function with high values at image locations with high curve saliency).

Fast Marching method to compute geodesics:

1) Computation of distance map from source point,
2) Geodesic via steepest decent on distance map.

Tracking of Lines in Flat Images via Sub-Riemannian Geodesics in SE(2)



[1] E.J. Bekkers, R. Duits, A. Mashtakov and G.R. Sanguinetti, *Data-driven Sub-Riemannian Geodesics in* SE(2), Proc. SSVM, 2015.

[2] E.J. Bekkers, R. Duits, A. Mashtakov and G.R. Sanguinetti, *A PDE Approach to Data-driven Sub-Riemannian Geodesics in* SE(2), SIIMS, 2015.

[3] G. Sanguinetti, R. Duits, E. Bekkers, M. Janssen, A. Mashtakov, J-M. Mirebeau, *Sub-Riemannian Fast Marching in* SE(2), Proc. CIARP, 2015.

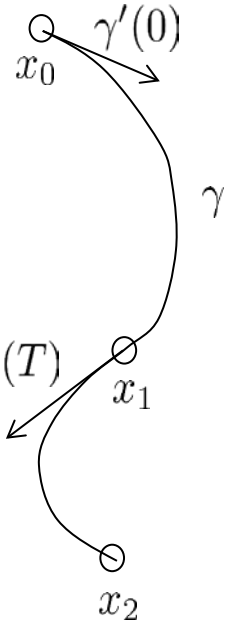
Contrast enhancement of color images

Initial low-contrast image

Metric coefficients – the parameters that specify the color perception by an individual observer

Extrapolation by extending the geodesic that connects the base colorless image with the given low-contrast image in a metric in color space defined by the individual observer

Enhanced high-contrast image adapted for the selected individual observer



x_0

Colorless image

geodesic

x_1

Initial image

extrapolation

x_2

Resulting image

Conclusion

- Hyperbolicity naturally appears in modelling of perceptual color space.
- Combination of hyperbolic color space model with sub-Riemannian model of contour completion provides an interesting novel model of color vision.
- The model has applications in color image processing.
- Computation of the distance map is a nontrivial problem.

Thank you for your attention!