Extremal Trajectories of a Spherical Robot on Inhomogeneous Surfaces

Alexey Mashtakov
Ailamazyan Program Systems Institute of RAS

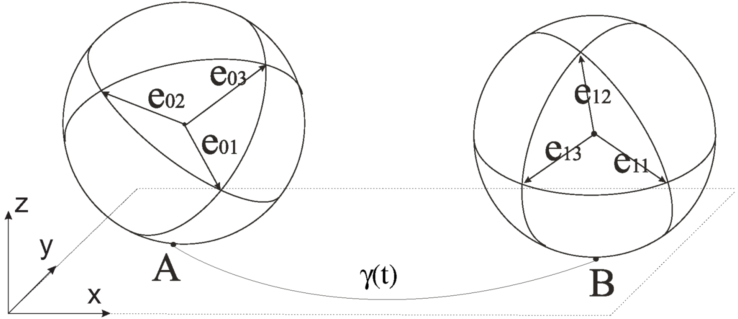
Nonlinearity, information and robotics

the Second International Conference
«Nonlinearity, Information and Robotics»
Innopolis, Russia, 27.08.2021

RSF

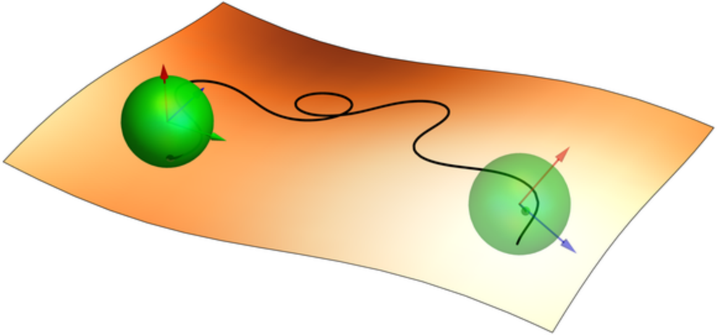
Problem Formulation

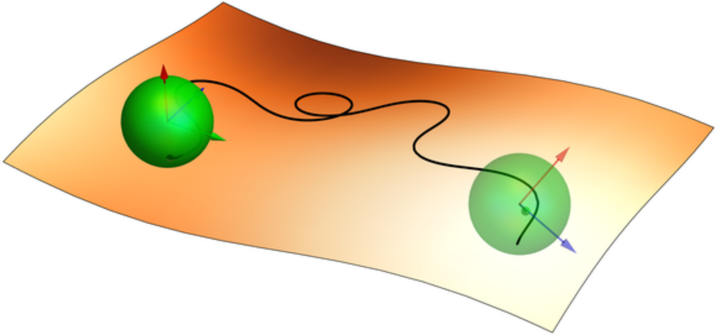
Given: $A, B \in \mathbb{R}^2$, frames (e_{01}, e_{02}, e_{03}), (e_{11}, e_{12}, e_{13}) in $\mathbb{R}^3$,
external cost $\mathscr{C} : \mathbb{R}^2 \to [\varepsilon, +\infty) : (x, y) \mapsto \mathscr{C}(x, y)$, $\varepsilon > 0$.



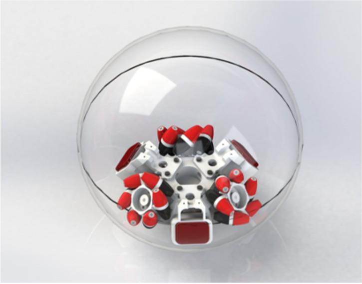
Find: $\gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$, $t \in [0, T]$, — the shortest (w.r.t. $\mathscr{C}$) curve, s.t.:
$\gamma(0) = A$, $\gamma(T) = B$, orientation transfers from (e_{01}, e_{02}, e_{03}) to (e_{11}, e_{12}, e_{13}),

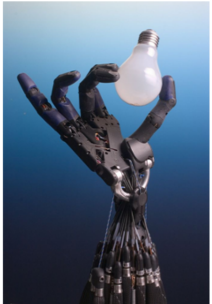
$$\int_0^T \mathscr{C}(x(t), y(t)) \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, \mathrm{d}t \to \min.$$





Motivation





History of the problem

1983 J.M. Hammersley: statement of the plate-ball problem.

1986 A.M. Arthur, G.R.Walsh: integrability of Hamiltonian system of PMP in quadratures.

1990 Z.Li, J. Canny: controllability of the system.

1993 V. Jurdjevic:

- projections of extremal curves $(x(t), y(t))$ — Euler elasticae,
- description of different qualitative types of extremal trajectories,
- differential equations for evolution of Euler angles along extremal trajectories.

History of the problem

2010 Yu. L. Sachkov:

- continuous and discrete symmetries;
- fixed points of symmetry (Maxwell points);
- a necessary optimality condition: an upper bound for the cut time.

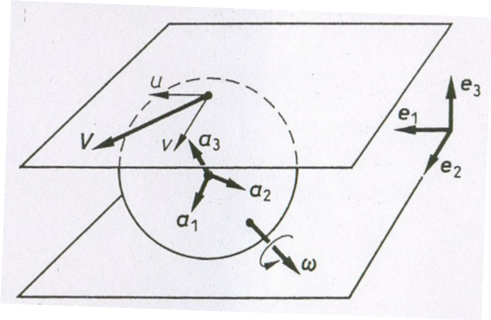
2011 A.P. Mashtakov, Yu.L. Sachkov:

- explicit parameterization of extremal trajectories;
- asymptotics of the extreme trajectories of Maxwell's time when the sphere rolls along sinusoids of small amplitude.

2021 Current work: Generalization to the external cost case

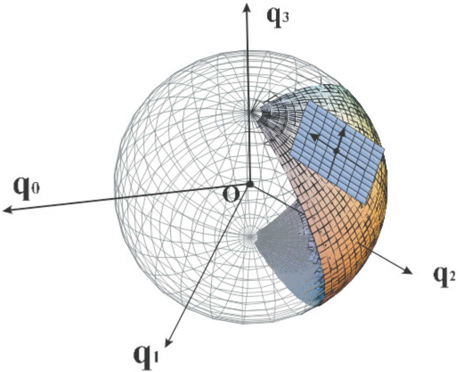
State and control variables

- Contact point $(x, y) \in \mathbb{R}^2$
- Orientation of sphere $R : a_i \mapsto e_i,\ i = 1, 2, 3,\ R \in SO(3)$
- State of the system $Q = (x, y, R) \in \mathbb{R}^2 \times SO(3) = M$
- Controls $u_1 = u/2,\ u_2 = v/2,\ (u_1, u_2) \in \mathbb{R}^2$



Representation of rotations in $\mathbb{R}^3$ by quaternions

- $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 | q_0, \ldots, q_3 \in \mathbb{R}\}$
- $S^3 = \{q \in \mathbb{H} | |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$
- $I = \{q \in \mathbb{H} | \mathrm{Re}\, q = q_0 = 0\}$
- $q \in S^3 \Rightarrow R_q(a) = qaq^{-1}, \quad a \in I, R_q \in \mathrm{SO}(3) \cong \mathrm{SO}(I)$
- $\dot{q} = \frac{1}{2}\vec{\omega} q$



Optimal control problem

- Control System

$$\begin{cases} \dot{x} = u_1, \qquad \dot{y} = u_2, \\ \dot{q}_0 = \frac{1}{2}(q_2 u_1 - q_1 u_2), \\ \dot{q}_1 = \frac{1}{2}(q_3 u_1 + q_0 u_2), \\ \dot{q}_2 = \frac{1}{2}(-q_0 u_1 + q_3 u_2), \\ \dot{q}_3 = \frac{1}{2}(-q_1 u_1 - q_2 u_2), \end{cases} \qquad q \in S^3, \ (u_1, u_2) \in \mathbb{R}^2,$$

$$x(0) = y(0) = 0, \ q(0) = 1.$$

- Boundary conditions $Q(0) = Q_0, \ Q(T) = Q_1$
- Cost functional $I(Q) = \int_0^T \mathscr{C}(Q)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^T \mathscr{C}(Q)\sqrt{u_1^2 + u_2^2}\, dt \to \min$

Existence of solutions

- Sub-Riemannian problem:

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \qquad (u_1, u_2) \in \mathbb{R}^2,$$
$$Q(0) = Q_0, \qquad Q(T) = Q_1, \qquad Q \in M = \mathbb{R}^2 \times \mathrm{SO}(3),$$
$$l = \int_0^T \mathscr{C}(Q)\sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

- Complete controllability by Rashevskii-Chow theorem:

$$\mathrm{span}(X_1(Q), X_2(Q), X_3(Q), X_4(Q), X_5(Q)) = T_q M \quad \forall\, Q \in M,$$
$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, [X_1, X_2]], \qquad X_5 = [X_2, [X_1, X_2]].$$

- Filippov's theorem: $\forall Q_0, Q_1 \in M$ optimal trajectory exists.
- $Q_0 = (0, 0, \mathrm{Id}) \in \mathbb{R}^2 \times \mathrm{SO}(3)$.

Minimization functional

The Cauchy-Schwarz inequality ensures that the original problem

$$I = \int_0^T \mathscr{C}(Q)\sqrt{u_1^2 + u_2^2}\,dt \to \min\,.$$

is equivalent to the problem of minimizing the action:

$$J = \frac{1}{2}\int_0^T \mathscr{C}^2(Q)(u_1^2 + u_2^2)\,dt \to \min\,.$$

Pontryagin Maximum Principle (PMP)

Left-invariant hamiltonians:

$$h_i = \langle \lambda, X_i \rangle \,, \quad \lambda \in T^*(\mathbb{R}^2 \times S^3),\ i = 1, \dots, 5$$

The Pontryagin function:

$$h_u^\nu = u_1 h_1 + u_2 h_2 + \frac{\nu}{2} \mathscr{C}^2(Q)(u_1^2 + u_2^2)$$

Maximum Condition:

$$H_{\bar{u}}^\nu = \max_{u \in \mathbb{R}^2} h_u^\nu$$

The Hamiltonian system of PMP:

$$\dot{Q} = \bar{u}_1 X_1 + \bar{u}_2 X_2, \quad \dot{h}_i = \{H_{\bar{u}}^\nu, h_i\} \,.$$

Abnormal Case $\nu = 0$

Rolling of the sphere along straight lines.

Normal Case $\nu = 1$

Maximum Condition: $u_1 = \frac{h_1}{\mathscr{C}^2(Q)}, u_2 = \frac{h_2}{\mathscr{C}^2(Q)}$

The Hamiltonian System of PMP:

$$
\begin{cases}
\dot{x}(t) = \dfrac{h_1(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{y}(t) = \dfrac{h_2(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{q}_0(t) = \dfrac{-h_2(t)q_1(t) + h_1(t)q_2(t)}{2\mathscr{C}^2(Q(t))}, \\
\dot{q}_1(t) = \dfrac{h_2(t)q_0(t) + h_1(t)q_3(t)}{2\mathscr{C}^2(Q(t))}, \\
\dot{q}_2(t) = \dfrac{-h_1(t)q_0(t) + h_2(t)q_3(t)}{2\mathscr{C}^2(Q(t))}, \\
\dot{q}_3(t) = -\dfrac{h_1(t)q_1(t) + h_2(t)q_2(t)}{2\mathscr{C}^2(Q(t))}.
\end{cases}
\qquad
\begin{cases}
\dot{h}_1(t) = \dfrac{1}{\mathscr{C}^3(Q(t))}\partial_x \mathscr{C}(Q(t)) + \dfrac{h_2(t)h_3(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{h}_2(t) = \dfrac{1}{\mathscr{C}^3(Q(t))}\partial_y \mathscr{C}(Q(t)) + \dfrac{h_1(t)h_3(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{h}_3(t) = \dfrac{h_1(t)h_4(t) + h_2(t)h_5(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{h}_4(t) = -\dfrac{h_1(t)h_3(t)}{\mathscr{C}^2(Q(t))}, \\
\dot{h}_5(t) = -\dfrac{h_2(t)h_3(t)}{\mathscr{C}^2(Q(t))},
\end{cases}
$$

with the initial condition $h_1(0) = \pm\sqrt{2\mathscr{C}^2(Q(0)) - (h_2^0)^2}$, $h_i(0) = h_i^0$, $i = 2, \ldots, 5$.

Conclusion

1. Model of a spherical robot rolling on an inhomogeneous surface
2. The external cost encodes the inhomogeneity of the surface
3. Optimal control formulation
4. A necessary optimality condition — PMP
5. Hamiltonian system of PMP that determines the extremals
6. Numerical simulation of robot motion

Thank you for your attention!