Time minimization problem on the group of motions of a plane with admissible control in a half-disk

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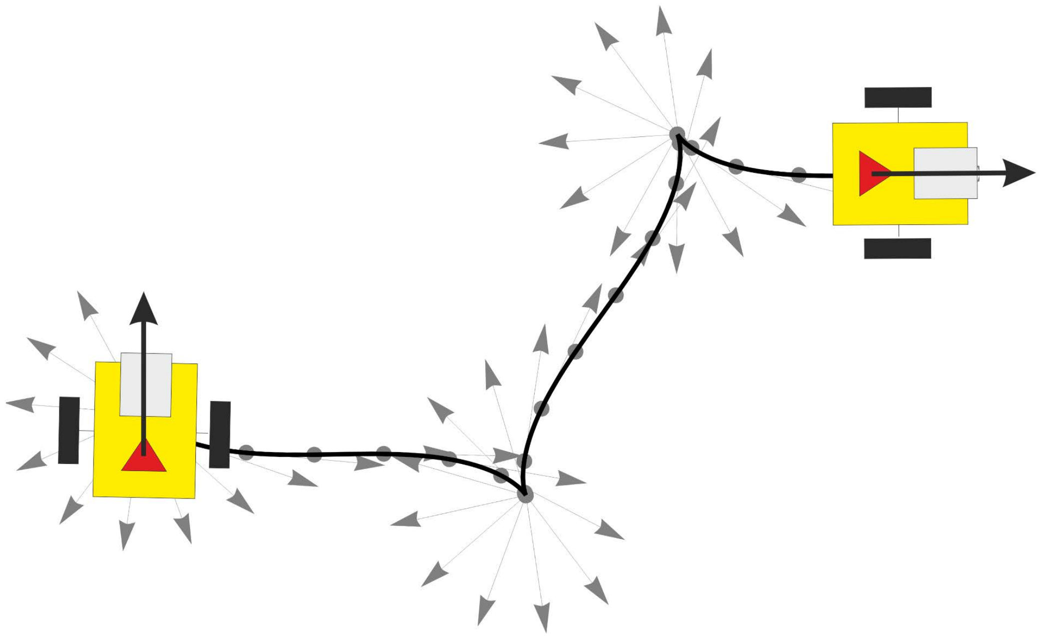
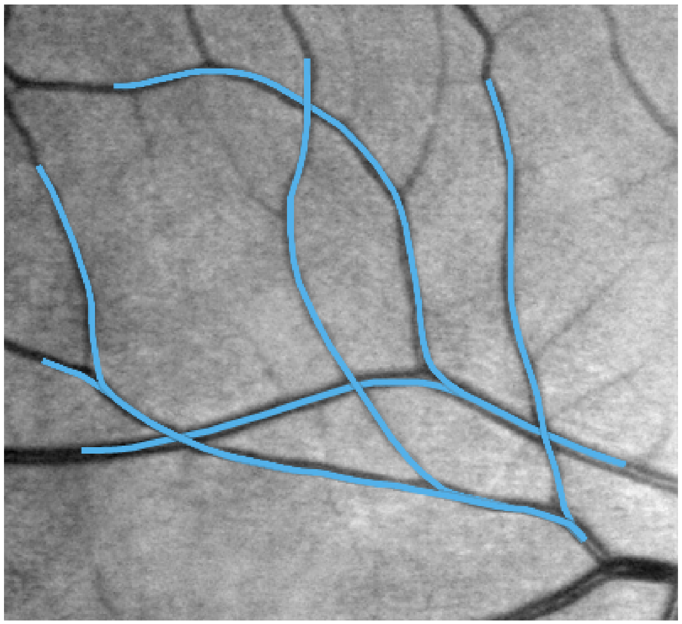
Outline of the Talk

- ◇ Motivation
- ◇ Preliminaries
- ◇ Statement of the problem
- ◇ Controllability
- ◇ Extremal Controls
- ◇ Optimality
- ◇ Conclusion

Motivation: Applications in robotics and image processing

- Motion planning problem for a car-like mobile robot that can move forward and rotate in place
- Extraction of salient curves in images. E.g. vessel tracking on images of human retina.





Preliminaries

• The group of motions of a plane $\mathrm{SE}_2 \equiv M \simeq \mathbb{R}^2_{x,y} \times S^1_\theta \ni q$:

$$qq' = \Big((x,y),\theta\Big)\Big((x',y'),\theta'\Big) = \Big(R_\theta(x',y') + (x,y), \theta+\theta'\Big).$$

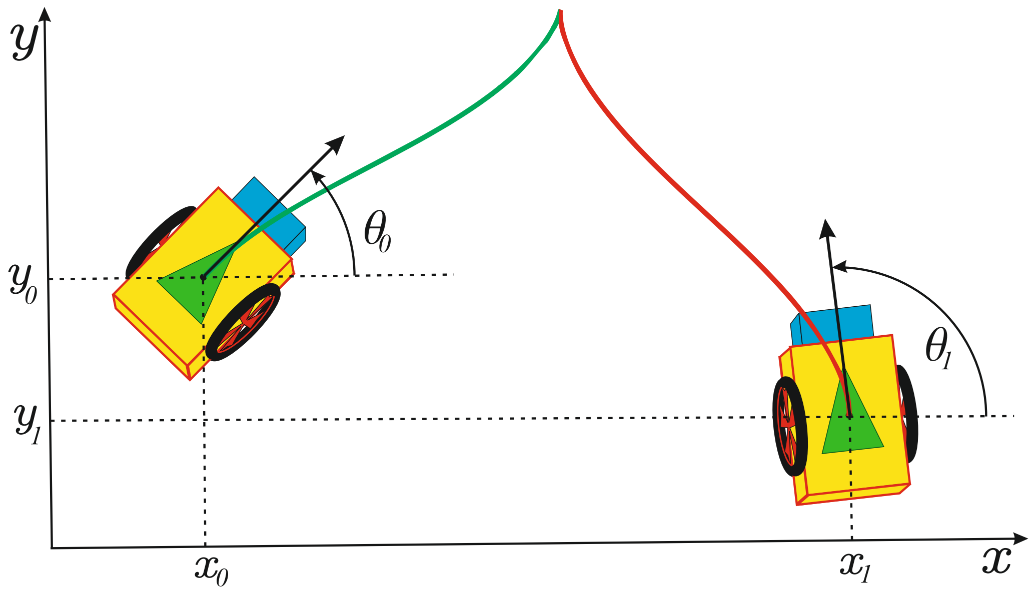
where R_θ is a counter-clockwise planar rotation on angle θ.
The Lie algebra $\mathrm{se}_2 = \mathrm{span}(X_1, X_2, X_3)$, where

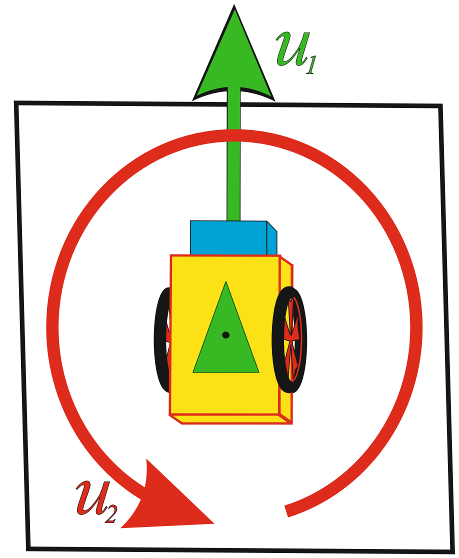
$$X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad X_2 = \partial_\theta, \quad X_3 = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}.$$

• By given a dynamics on M, an extremal trajectory is called a trajectory that satisfies the optimality condition — PMP.

• The wavefront is a set of all points in configuration space M, reachable by all the extremal trajectories in a fixed time T.

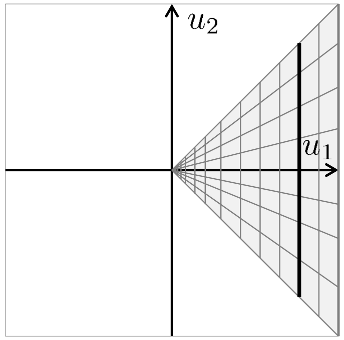
Model of a Car on a Plane



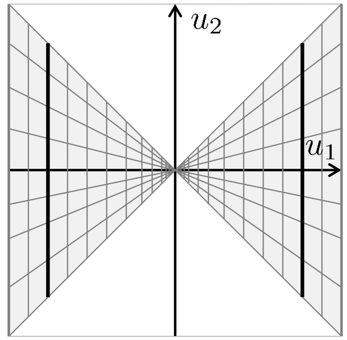


$$\dot{q} = u_1 X_1(q) + u_2 X_2(q),$$

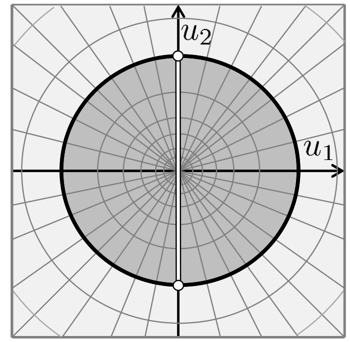
Set of Admissible Controls



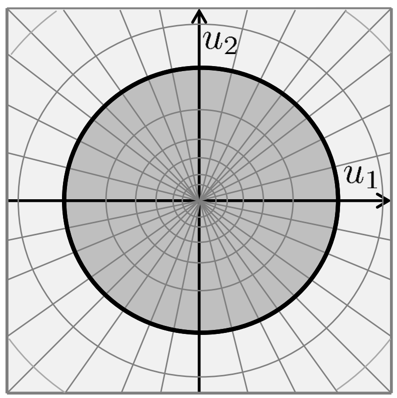
Dubins (1957)



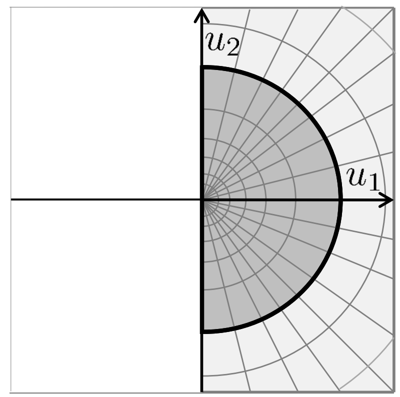
Reeds-Shepp (1990)



Berestovskii (1994)



Sachkov (2010)



Duits (2018)

Statement of the Problem

Consider the following control system (dynamics):

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \begin{array}{l} (x, y, \theta) = q \in \mathrm{SE}_2 = M, \\ u_1^2 + u_2^2 \leq 1,\ u_1 \geq 0. \end{array}$$

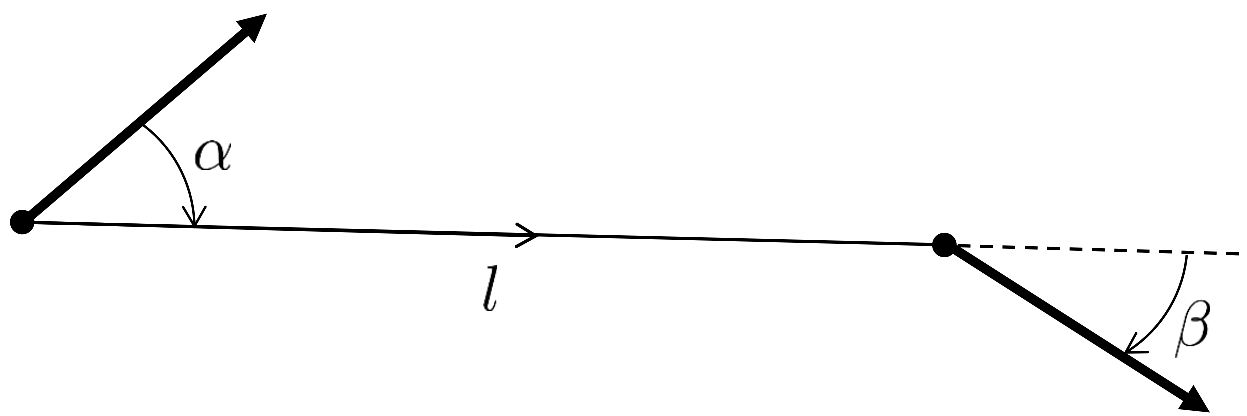
By given q_0, $q_1 \in M$ we aim to find the controls $u_1(t)$, $u_2(t)$ such that the corresponding trajectory $\gamma : [0, T] \to M$ transfers the system from q_0 to q_1 by minimal time

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min\,.$$

Here u_i are $L^\infty([0, T], \mathbb{R})$, and γ is a Lipschitzian curve on M.

Controllability of the System

Theorem. In the time minimization problem for the left-invariant control system on the group of motions of a plane with admissible control in a semicircle, there always exists an optimal trajectory that transfers the system from an arbitrary given initial configuration to an arbitrary given final configuration.



Pontryagin Maximum Principle (PMP)

- A necessary condition of optimality is given by PMP.
- Denote $(p_1, p_2, p_3) \in T^*M$. The Pontryagin function

$$H_u = p_1\sqrt{1-u^2}\cos\theta + p_2\sqrt{1-u^2}\sin\theta + p_3 u.$$

- Let $(u(t), q(t))$, $t \in [0, T]$ be an optimal process. Then

- Hamiltonian system $\dot{p} = -\dfrac{\partial H_u}{\partial q}$, $\dot{q} = \dfrac{\partial H_u}{\partial p}$;

- Maximum condition

$$H = \max_{u\in[-1,1]} H_u(p(t), q(t)) \in \{0, 1\}.$$

- Left-invariant Hamiltonians

$$h_1 = p_1\cos\theta + p_2\sin\theta, \quad h_2 = p_3, \quad h_3 = p_1\sin\theta - p_2\cos\theta.$$

Abnormal Extremal Controls and Trajectories

Theorem. Abnormal extremal control exists when $h_1 \le 0$ and has a form $u_1(t) = 0$, $u_2(t) \in I = [-1, 1]$ — arbitrary $L_\infty([0, T], I)$ function that satisfies the condition

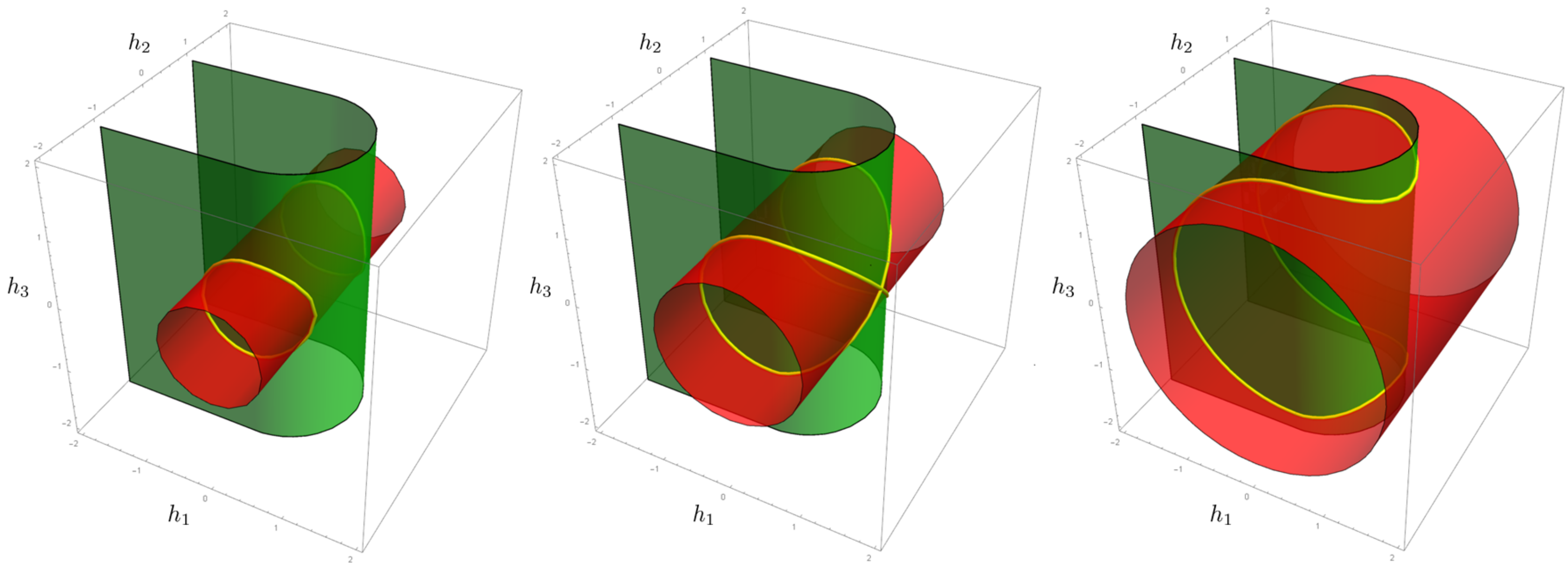
$$h_{10} \cos U_2(t) - h_{30} \sin U_2(t) < 0, \text{ where } U_2(t) = \int_0^t u_2(\tau) d\tau,$$

for all $t \in [0, T]$.

Theorem. Abnormal extremal trajectoriy has a form

$$x(t) = 0, \quad y(t) = 0, \quad \theta(t) = U_2(t).$$

First Integrals of the Hamiltonian System



The Hamiltonian

$$H = \begin{cases} |h_2|, & \text{for } h_1 \leq 0, \\ \sqrt{h_1^2 + h_2^2}, & \text{for } h_1 > 0, \end{cases}$$

The Casimir

$$E = h_1^2 + h_3^2.$$

The Hamiltonian system of PMP

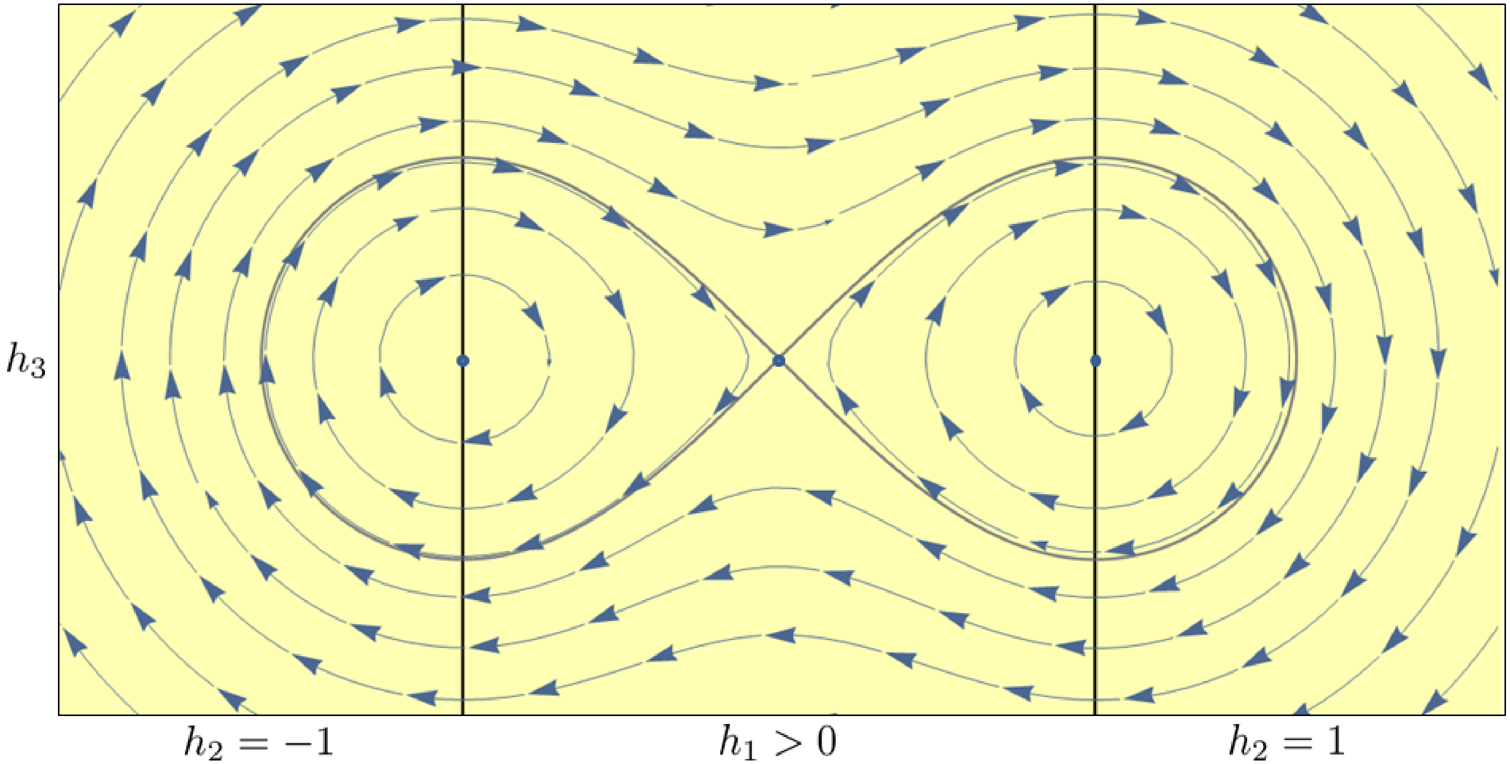
For $h_{10} < 0$

$$\begin{cases} \dot{x} = 0, & x(t_0) = x_0, \\ \dot{y} = 0, & y(t_0) = y_0, \\ \dot{\theta} = s_2, & \theta(t_0) = \theta_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -s_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_3 = s_2 h_1, & h_3(t_0) = h_{30}. \end{cases} \tag{1}$$

For $h_{10} > 0$

$$\begin{cases} \dot{x} = h_1 \cos\theta, & x(t_0) = x_0, \\ \dot{y} = h_1 \sin\theta, & y(t_0) = y_0, \\ \dot{\theta} = h_2, & \theta(t_0) = \theta_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = h_1 h_3, & h_2(t_0) = h_{20}, \\ \dot{h}_3 = h_2 h_1, & h_3(t_0) = h_{30}. \end{cases} \tag{2}$$

Dynamics of Normal Hamiltonian System



Phase portrait on the level surface $H = 1$ of the Hamiltonian.

Normal Extremal Controls

Theorem. A normal extremal control $(u_1(t), u_2(t))$ is uniquely determined by $h_{10}^0 \in (-\infty, 1], \quad s_{20}^0 \in \{-1, 1\}, \quad h_{30}^0 \in \mathbb{R}$.

The function $u_1(t)$ is given by $u_1(t) = \sqrt{1 - u_2^2(t)}$, $t \in [0, T]$.

The function $u_2(t) = h_2(t)$ is defined on time intervals formed by splitting the interval $t \in [0, T]$ by instances

$$t_0 \in \{0 = t_0^0, t_0^1, t_0^2, \dots, T\},$$

where the switching point t_0^i depends on the state

$$h_0^{i-1} = (h_{10}^{i-1}, s_{20}^{i-1}, h_{30}^{i-1}),$$

achieved at t_0^{i-1}, and is determined by the recurrent formula

$$t_0^i(h_0^{i-1}) = \min\{t > t_0^{i-1} \mid h_1(t, h_0^{i-1}) = 0\}, \quad h_0^i = h(t_0^i(h_0^{i-1}), h_0^{i-1}).$$

Here $h(t,h_0^i) = (h_1(t,h_0^i),\, h_2(t,h_0^i),\, h_3(t,h_0^i)) = \mathrm{vert}\left(e^{t\vec{H}} h_0^i\right)$ is the solution of the vertical part of the Hamiltonian system of PMP with the initial value h_0^i for time $t \geq t_0^i$, which has a form

$$\begin{cases} (1), & \text{for } (h_{10}^i < 0) \vee (h_{10}^i = 0 \wedge s_{20}^i s_{30}^i > 0), \\ (2), & \text{for } (h_{10}^i > 0) \vee (h_{10}^i = 0 \wedge s_{20}^i s_{30}^i < 0), \end{cases}$$

$$\text{where } s_{30}^i = \begin{cases} \operatorname{sign} h_{30}^i, & \text{for } h_{30}^i \neq 0, \\ s_{20}^i, & \text{for } h_{30} = 0. \end{cases}$$

Explicit Parametrization of the Extremals

For $h_{10} < 0$ solution to the vertical part is given by

$$h_1(t) = h_{10} \cos(t - t_0) - s_2 h_{30} \sin(t - t_0),$$
$$h_2(t) = h_{20},$$
$$h_3(t) = h_{30} \cos(t - t_0) + s_2 h_{10} \sin(t - t_0).$$

Solution to the horizontal part is given by

$$x(t) = x_0, \quad y(t) = y_0, \quad \theta(t) = \theta_0 + s_2(t - t_0).$$

The instance of switching is determined by

$$t_1 - t_0 = \arg(-s_2 h_{30} - i h_{10}) \in (0, \pi].$$

A corresponding motion of the car is an in-place rotation.

Explicit Parametrization of the Extremals

Let $M = E - 2 = h_1^2 + h_3^2 - 2$.

The vertical part is reduced to the Cauchy problem

$$\ddot{h}_2 + M h_2 + 2h_2^3 = 0,$$

with the initial conditions

$$h_2(t_0) = h_{20}, \quad \dot{h}_2(t_0) = h_{10}\, h_{30} = \sqrt{1 - h_{20}^2}\, h_{30}.$$

Let $\xi(t) = \begin{cases} \frac{t-t_0}{k} - s_3 F(\alpha, k), & \text{for } E > 1, \\ -s_2 s_3 \frac{t-t_0}{k} + s_2 F(\alpha, k), & \text{for } E < 1, \end{cases}$

where $s_2 = \operatorname{sign} h_{20},\ s_3 = \operatorname{sign} h_{30},\ F(\alpha, k) = \int\limits_0^\alpha \frac{\mathrm{d}\, a}{\sqrt{1 - k^2 \sin^2 a}}$.

An explicit solution to the Cauchy problem is given by

$$h_2(t) = -s\,\mathrm{cn}\,(\xi(t), k)\,,$$

where

$$k = \frac{1}{\sqrt{E}} = \frac{1}{\sqrt{1 - h_{20}^2 + h_{30}^2}}, \qquad s = \begin{cases} s_3, & \text{for } E > 1, \\ -s_2, & \text{for } E < 1. \end{cases}$$

The remaining components of the momentum covector are

$$h_1(t) = \mathrm{sn}\,(\xi(t), k)\,, \quad h_3(t) = h_{30} + \frac{s}{k}\,(\mathrm{dn}\,(\xi(t), k) - \mathrm{dn}\,(\xi(t_0), k))\,.$$

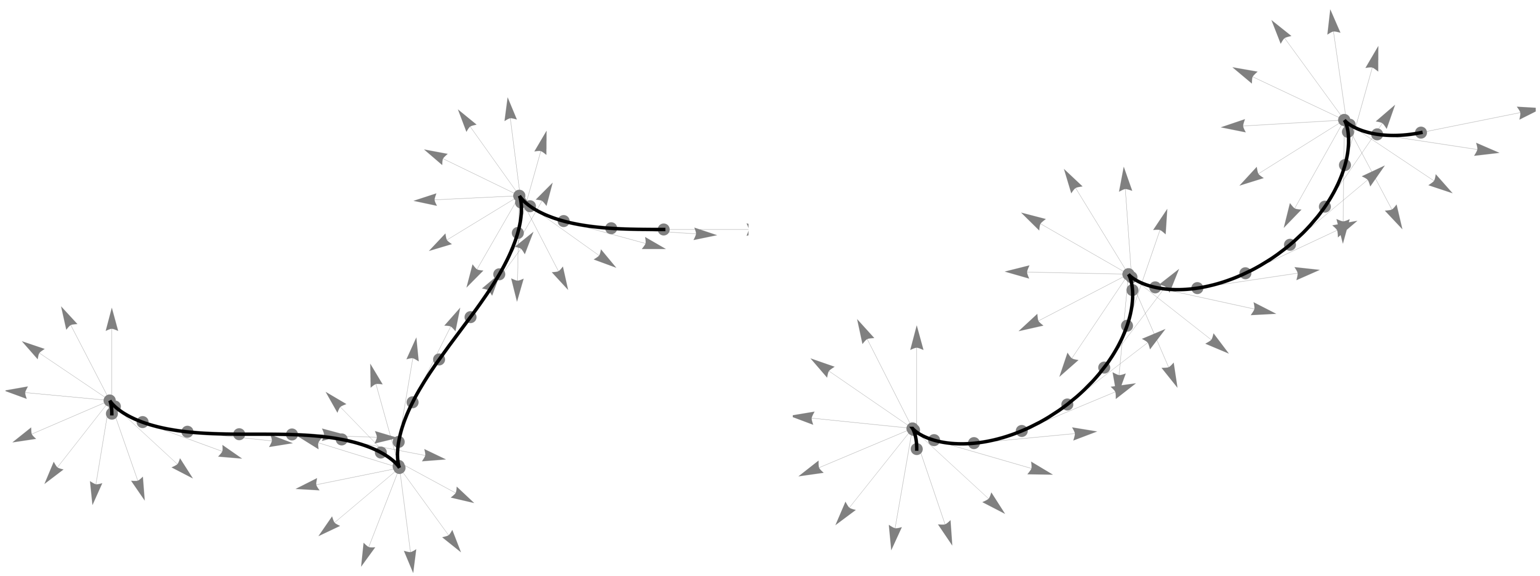
The car is moving along the sub-Riemannian geodesics in SE_2 whose planar projections do not have cusps.

Extremal Trajectories

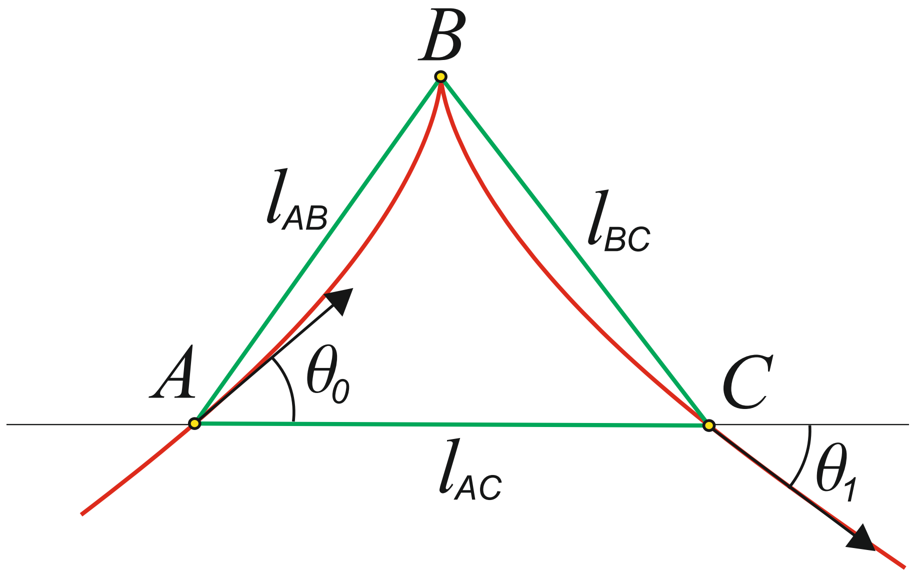
- The extremal trajectories are obtained by integration

$$x(t) = \int_0^t u_1(\tau) \cos\theta(\tau)\, \mathrm{d}\,\tau,\ y(t) = \int_0^t u_1(\tau) \sin\theta(\tau)\, \mathrm{d}\,\tau,\ \theta(t) = \int_0^t u_2(\tau)\, \mathrm{d}\,\tau.$$

- Explicit parametrization by Jacobi elliptic functions



Optimality of Extremal Trajectories



An optimal trajectory does not have internal turn points.

Proof by contradiction. For $\gamma : [0, T] \to \mathrm{SE}_2$ with an internal turn point there exists a shortcut $\gamma_0 : [0, T_0] \to \mathrm{SE}_2$.

$$T_0 = |\theta_0| + l_{AC} + |\theta_1| < T.$$

Structure of Optimal Synthesis

Theorem. Any optimal trajectory has a form

$t \in$	$[0, t_0^1)$	$[t_0^1, t_0^2)$	$[t_0^2, T]$
$x(t)$	0	$x_s(t)$	x_1
$y(t)$	0	$y_s(t)$	y_1
$\theta(t)$	$s_1 t$	$\theta_s(t)$	$\theta_s(t_0^2) + s_2(t - t_0^2)$,

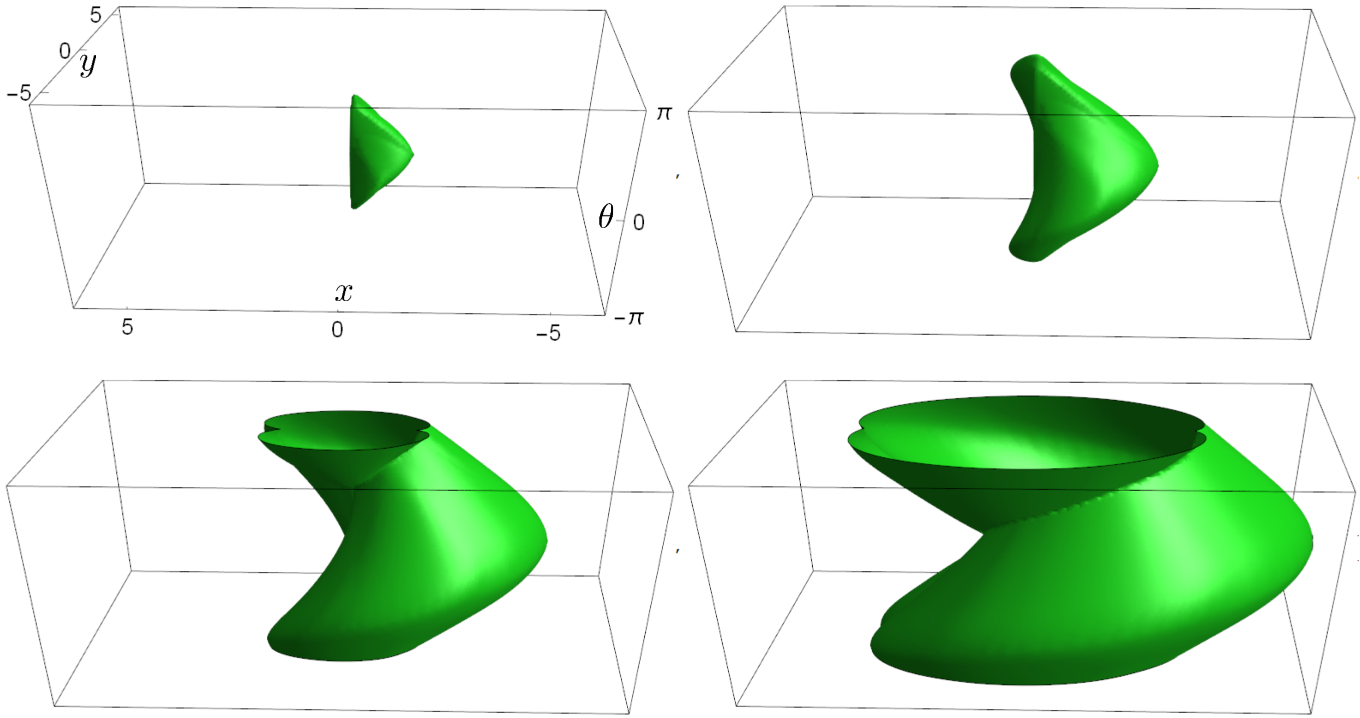
where $0 \leq t_0^1 \leq t_0^2 \leq T$ — control switching points, the signs $s_i = \pm 1$ are determined by initial values, the trajectory

$$(x_s(t), y_s(t), \theta_s(t)) =: q_s(t),$$

$$q_s(t_0^1) = (0, 0, \theta_0^1), \quad q_s(t_0^2) = (x_1, y_1, \theta_0^2)$$

is a sub-Riemannian length minimizer in SE_2 that does not have internal cusps in its planar projection (i.e. for any $t \in (t_0^1, t_0^2)$ the inequality $\dot{x}_s(t)^2 + \dot{y}_s(t)^2 > 0$ holds).

Wavefront along Optimal Trajectories



Wavefronts along optimal trajectories for $T = \frac{\pi}{2}, \pi, \frac{7\pi}{5}, 2\pi$.

Duits et.al. Optimal Paths for Variants of the 2D and 3D Reeds–Shepp Car with Applications in Image Analysis, JMIV, 2018.

Conclusion

- Solution to the left-invariant control problem, with the set of admissible controls containing zero on the boundary.

- Proof of existence of optimal control

- Explicit formulas for extremal controls and trajectories

- Partial analysis of optimality

- Structure of optimal synthesis

Thank you for your attention!