Liouville integrability in a four-dimensional model of the visual cortex

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The mathematics of vision: from cortical-inspired modelling to perception

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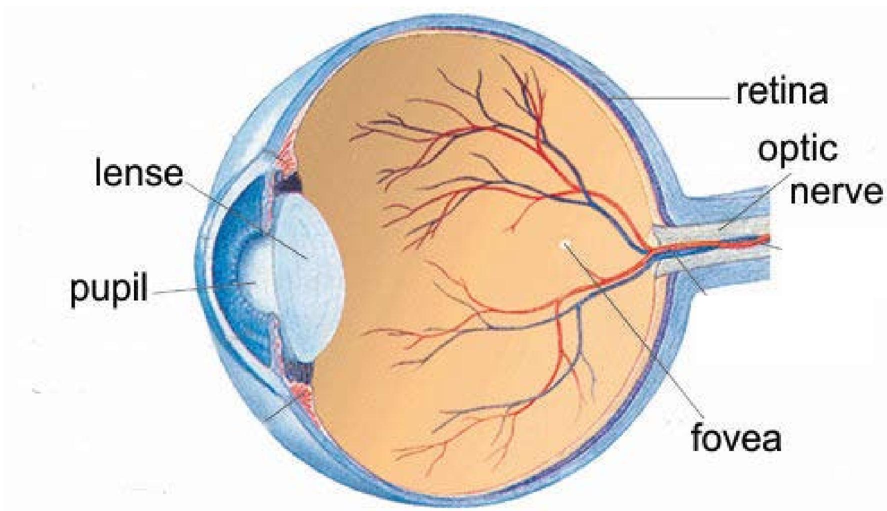
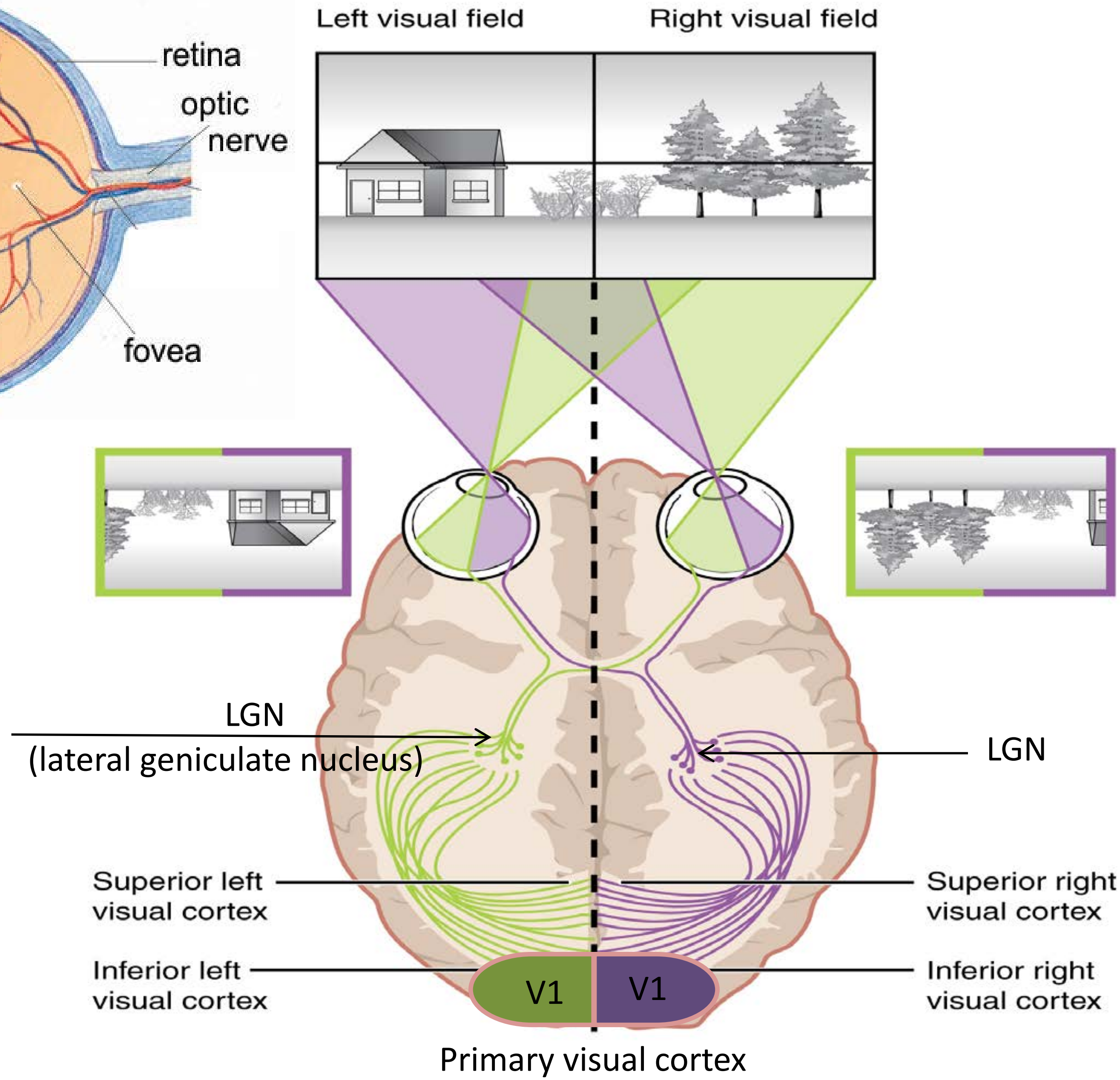
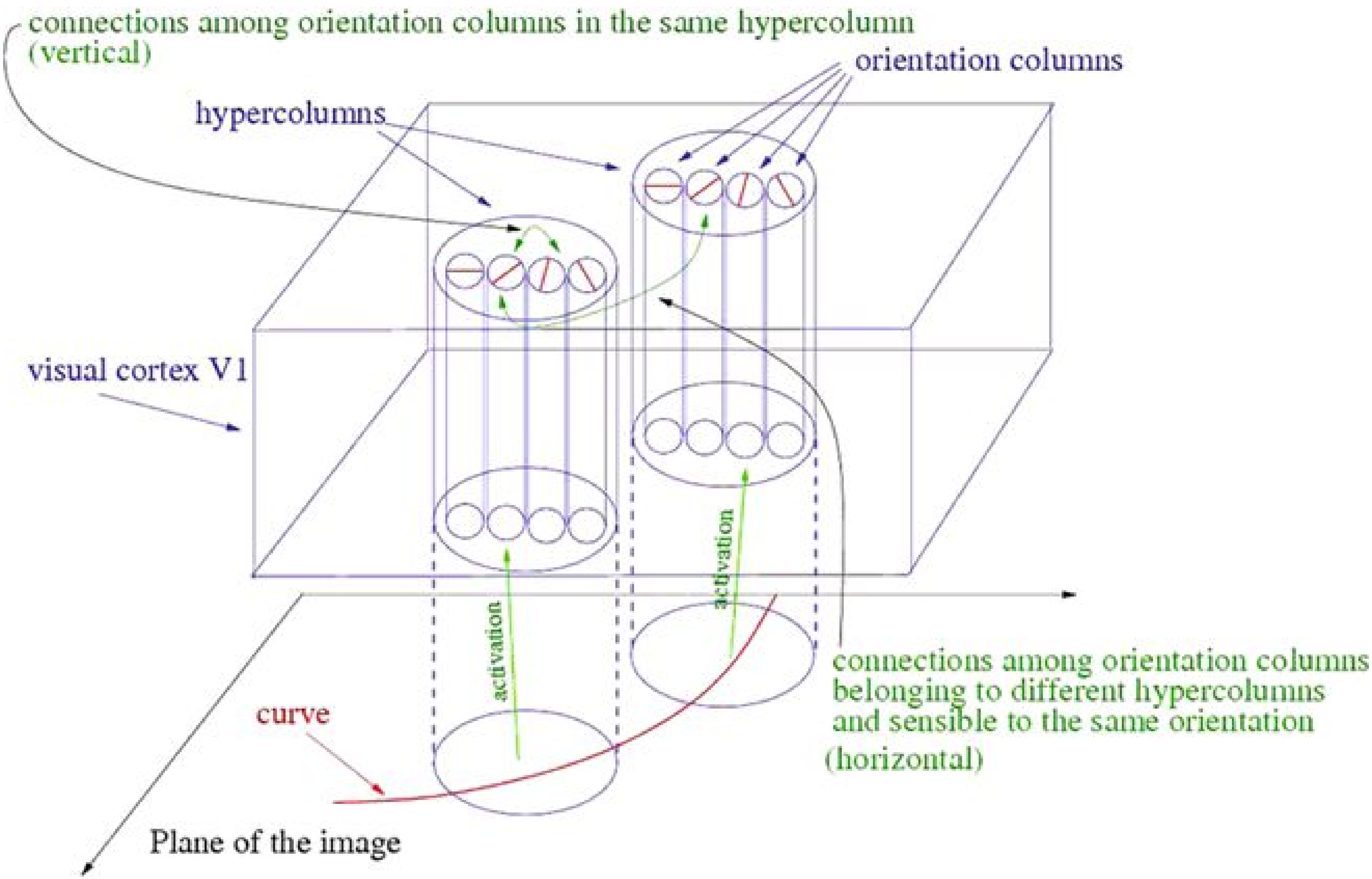
Perception of Visual Information by Human Brain





3D Model of the Primary Visual Cortex V1



Replicated from R. Duits, U. Boscain, F. Rossi, Y. Sachkov,
Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2), JMIV, 2013.

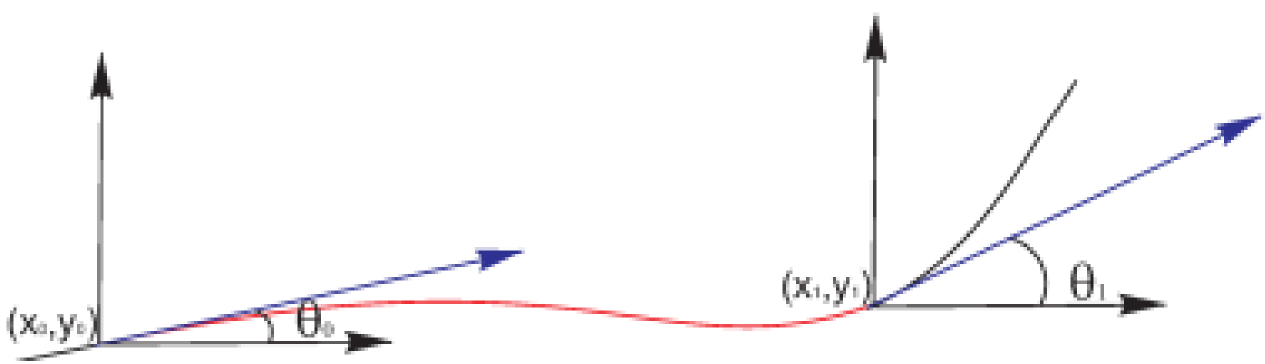
3D Model of the Primary Visual Cortex

- D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.

- Sub-Riemanian structures in neurogeometry of the vision:
  - J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003. (Heisenberg group.)
  - G. Citti and A. Sarti, A Cortical Based Model of Perceptual Completion in the Roto-Translation Space, 2006. ($SE(2)$ group.)

- Variational principle: recovered arc has minimal length in the space (x, y, θ):

$$\int \sqrt{\xi^2\left(\dot{x}^2+\dot{y}^2\right)+\dot{\theta}^2}dt \rightarrow \min, \text{ under constraint } \theta = \arg\left(\dot{x} + \mathrm{i}\ \dot{y}\right)$$



SE(2): Group of Roto-translations of a Plane

The group of Euclidean motions (rototranslations) of the plane:

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in S^1,\ x, y \in \mathbb{R} \right\} \cong \mathbb{R}^2_{x,y} \rtimes S^1_\theta .$$

Associated Lie algebra $\quad \mathrm{se}(2) = T_{\mathrm{Id}}\,\mathrm{SE}(2) = \mathrm{span}(A_1, A_2, A_3),$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lie algebra of left-invariant vector fields $\mathrm{L} = \mathrm{span}(X_1, X_2, X_3)$

$$X_1(q) = qA_1, \quad X_2(q) = qA_2, \quad X_3(q) = qA_3, \quad q \in \mathrm{SE}(2)\,.$$

Via the isomorphism $\mathrm{SE}(2) \cong \mathbb{R}^2_{x,y} \rtimes S^1_\theta$

$$X_1 \sim \mathcal{A}_1 = \cos\theta\partial_x + \sin\theta\partial_y, \quad X_2 \sim \mathcal{A}_2 = \partial_\theta, \quad X_3 \sim \mathcal{A}_3 = -\sin\theta\partial_x + \cos_\theta \partial_y.$$

Left-invariant Sub-Riemannian Problem on SE(2)

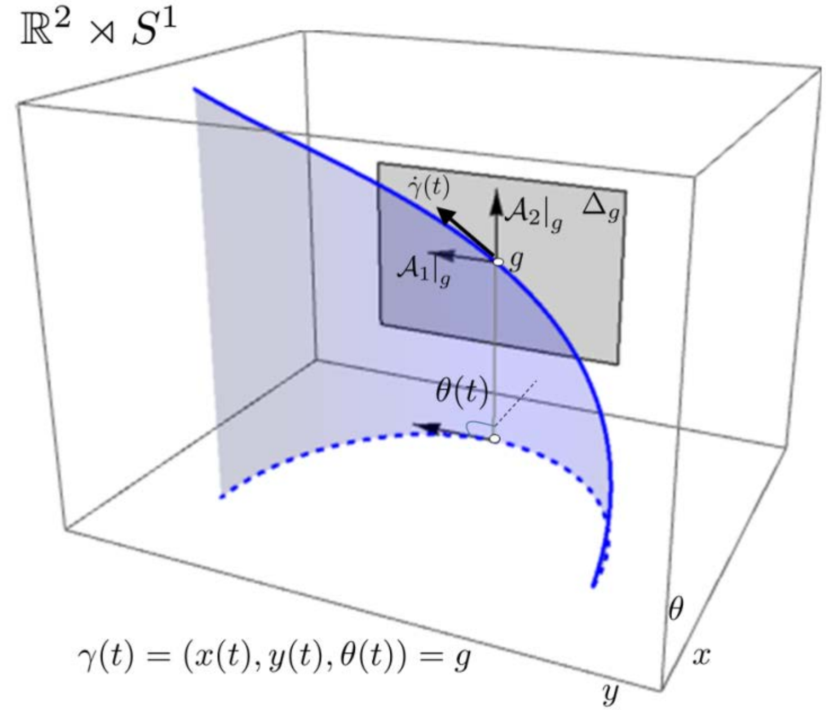
$$\dot\gamma(t) = u_1(t)\,\mathcal{A}_1|_{\gamma(t)} + u_2(t)\,\mathcal{A}_2|_{\gamma(t)}\,,$$

$$\mathcal{A}_1 = \cos\theta\partial_x + \sin\theta\partial_y, \quad \mathcal{A}_2 = \partial_\theta,$$

$$\gamma(0) = \mathrm{Id}, \qquad \gamma(T) = g,$$

$$l(\gamma) = \int_0^T \sqrt{u_1^2(t) + u_2^2(t)}\,\mathrm{d}t \to \min,$$

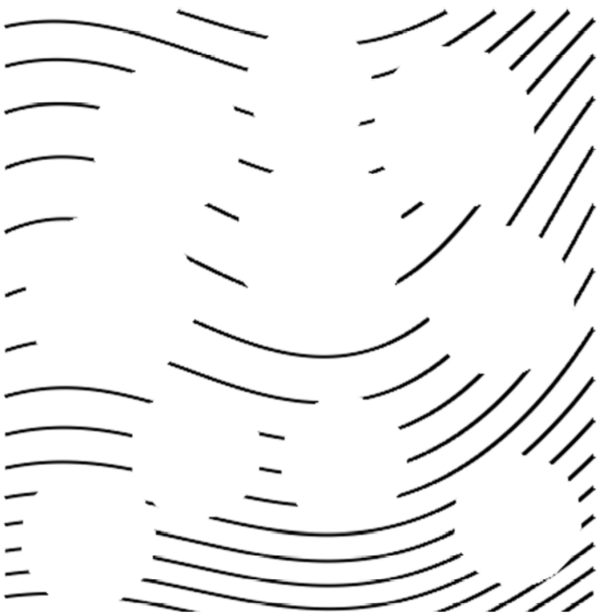
$$\gamma(t) \in \mathrm{SE}(2), \quad (u_1(t), u_2(t)) \in \mathbb{R}^2$$



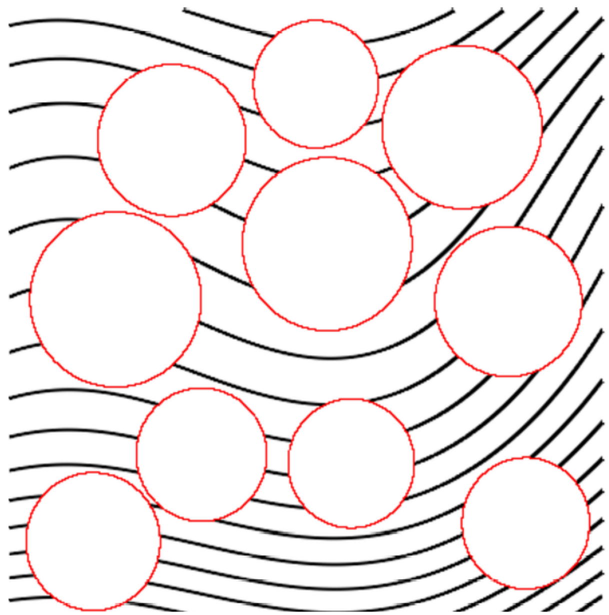
- I. Moiseev, Yu. L. Sachkov, Maxwell strata in sub-Riemannian problem on the group of motions of a plane, (2010)
- Yu. L. Sachkov, Conjugate and cut time in sub-Riemannian problem on the group of motions of a plane, (2010)
- Yu. L. Sachkov, Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane, (2011)

Anthropomorphic Image Reconstruction

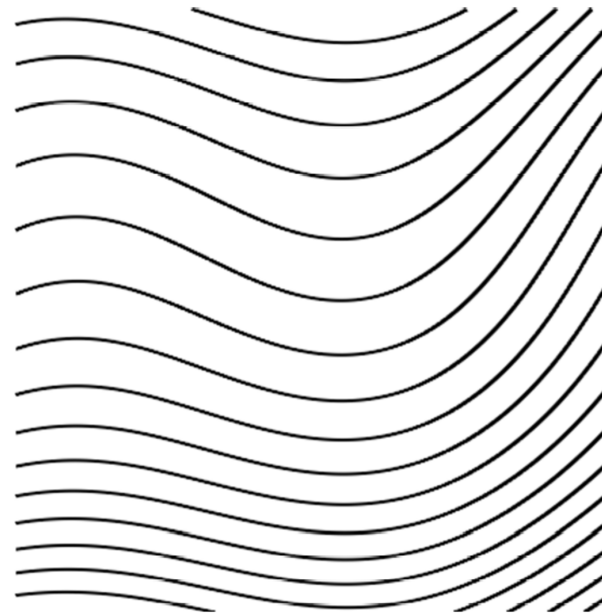
By given binary or grayscale image represented as series of isophotes (level lines of brightness) with some corrupted regions to restore the image in a natural (for human eye) way.



Input (Corrupted) Image



Input Image with Detected Corrupted Regions

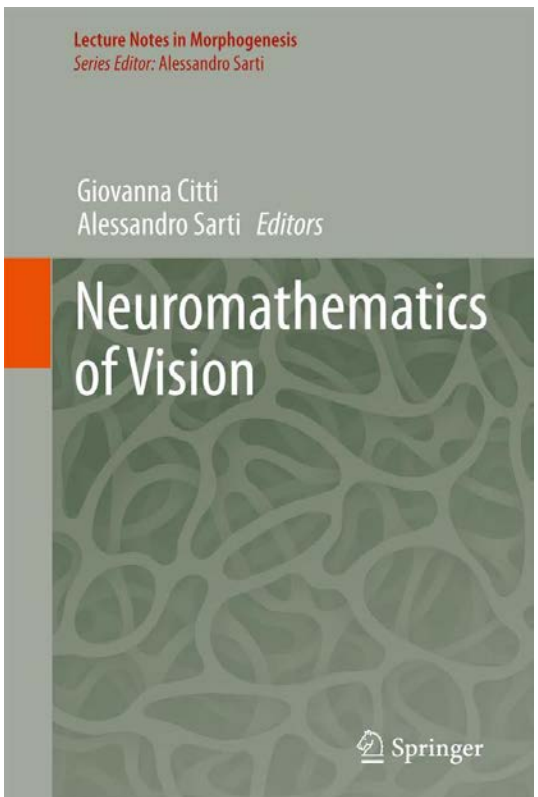


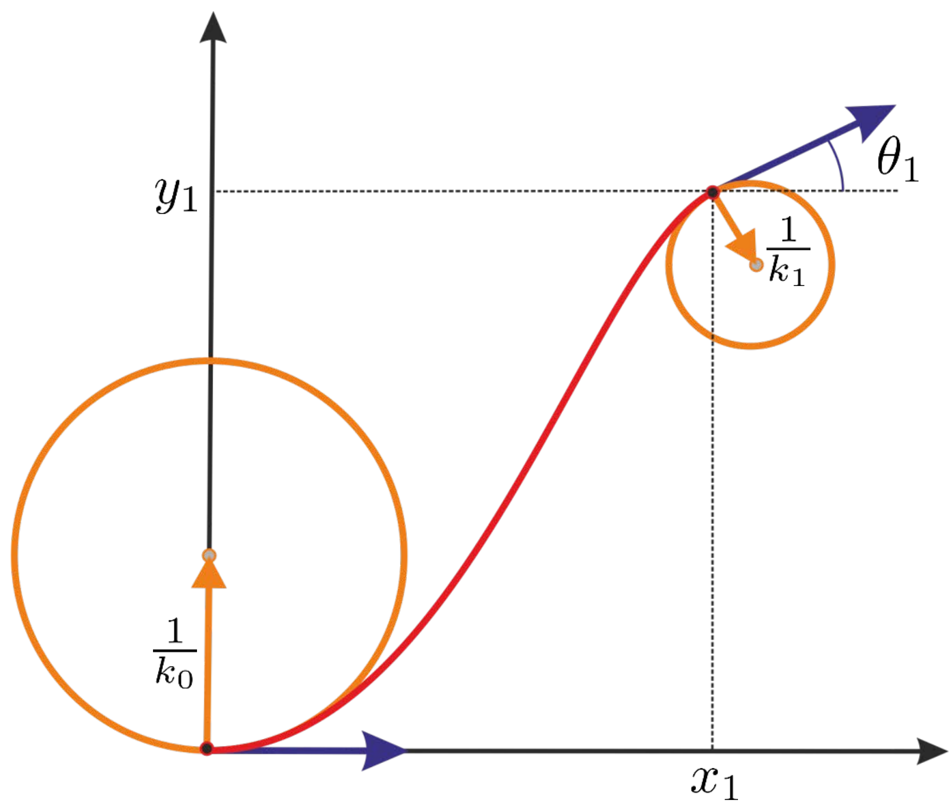
Restored Image

[**1**] A. Mashtakov, A. Ardentov, Yu. Sachkov, *Parallel Algorithm and Software for Image Inpainting via Sub-Riemannian Minimizers on the Group of Roto-translations*, NMTMA, 2013.

4D Model of the Primary Visual Cortex

- S. Zucker, The computational connection in vision: Early orientation selection, 1986. (Hypothesis of detectors of curvatre in V1).
- J. Petitot, Landmarks for Neurogeometry. In: Citti G., Sarti A. (eds) Neuromathematics of Vision, 2014. (Sub-Riemannian model in space of positions, orienttions and curvatures).





4D Model of the Primary Visual Cortex

- Configuration of a neuron — position, orientation and curvature

  $M = \mathrm{SE}(2) \times \mathbb{R} = \mathbb{R}^2_{x,y} \times S^1_\theta \times \mathbb{R}_k$

- Sub-Riemannian (SR) structure $(M, \Delta, \mathcal{G})$:

  distribution $\Delta = \mathrm{span}(\mathcal{A}_1, \mathcal{A}_2)$,

$$\mathcal{A}_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y + k\partial_\theta, \quad \mathcal{A}_2 = \partial_k$$

  inner product $\mathcal{G} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2, \quad \langle \omega^i, \mathcal{A}_j \rangle = \delta^j_i.$

- $\gamma : [0,T] \to M$ – horizontal (i.e. admissible) curve if

$$\dot\gamma(t) \in \Delta_{\gamma(t)} \text{ for a.e. } t \in [0,T].$$

  SR minimizers are horizontal curves γ that have minimum length

$$l(\gamma) = \int_0^T \sqrt{\mathcal{G}|_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt \to \min.$$

- SR geodesics are curves whose short arcs are SR minimizers.

Optimal Control Formulation

$$\dot{\gamma}(t) = u_1(t)\ \mathcal{A}_1|_{\gamma(t)} + u_2(t)\ \mathcal{A}_2|_{\gamma(t)}\ ,$$
$$\mathcal{A}_1 = \cos\theta\partial_x + \sin\theta\partial_y + k\partial_\theta, \quad \mathcal{A}_2 = \partial_k,$$
$$\gamma(0) = q_0, \qquad \gamma(T) = q_1,$$
$$l(\gamma) = \int_0^T \sqrt{u_1^2(t) + u_2^2(t)}\,\mathrm{d}t \to \min,$$
$$\gamma(t) = (x(t), y(t), \theta(t), k(t)) \in M = \mathrm{SE}(2) \times \mathbb{R}, \quad (u_1(t), u_2(t)) \in \mathbb{R}^2$$

Remark 1. Due to invariance under parallel translations and rotations in the (x, y) plane, without loss of generality, we can reduce the problem for an arbitrary $q_0 = (x_0, y_0, \theta_0, k_0)$ to the case $q_0 = (0, 0, 0, k_0)$.

Remark 2. By virtue of Cauchy-Schwarz inequality the problem of minimizing the length is equivalent to the problem of minimizing the action

$$J = \int\limits_0^T \frac{u_1^2(t) + u_2^2(t)}{2}\,dt \to \min\,.$$

Existence of Optimal Control

- For any q_0, $q_1 \in M$ there exists a horizontal curve γ, connecting q_0 to q_1. The system is completely controllable by Chow-Rashevsky theorem.

$$\mathcal{A}_3 = [\mathcal{A}_1, \mathcal{A}_2] = -\partial_\theta, \quad \mathcal{A}_4 = [\mathcal{A}_1, \mathcal{A}_3] = -\sin\theta\partial_x + \cos\theta\partial_y$$

$$\det(\mathcal{A}_1, \dots, \mathcal{A}_4) = 1$$

- Existence of optimal control by Filippov theorem.

Theorem.
For any q_0, $q_1 \in M$ there exists an optimal trajectory connecting q_0 to q_1.

Pontryagin Maximum Principle

- A necessay optimality condition — Pontryagin maximum principle (PMP).
- Pontryagin function

$$h_u^\nu = \langle p, u_1 \mathcal{A}_1 + u_2 \mathcal{A}_2 \rangle + \nu \frac{u_1^2 + u_2^2}{2}, \quad p \in T^*M, \ \nu \in \{-1, 0\}.$$

- If $(u(t), q(t))$, $t \in [0, T]$, is an optimal process then

  1. the Hamiltonian system $\dot{p} = -\frac{\partial h_u^\nu}{\partial q}$, $\dot{q} = \frac{\partial h_u^\nu}{\partial p}$;
  2. the maximum condition $h_{u(t)}^\nu(p(t), q(t)) = \max\limits_{u \in \mathbb{R}^2} h_u^\nu(p(t), q(t))$;
  3. the nontriviality condition $(p(t), \nu) \neq (0, 0)$ $\forall t \in [0, T]$.

Abnormal Case of Pontryagin Maximum Principle

- Let $h_i = \langle p, \mathcal{A}_i \rangle$:

$$h_1 = a\cos\theta + b\sin\theta + ck, \quad h_2 = d.$$

- The Pontryagin function $h_u^0 = u_1 h_1 + u_2 h_2$ is unbounded for $h_1^2 + h_2^2 \neq 0$.
- The maximum condition implies $h_1 = h_2 \equiv 0$.
- Let $p = (a, b, c, d)$. The Hamiltonian system implies

$$\dot{a} = 0, \quad \dot{b} = 0, \quad \dot{c} = -u_1(-a\sin\theta + b\cos\theta), \quad \dot{d} = -u_1\, c.$$

- The maximum condition with the Hamiltonian system imply $u_1 \equiv 0$.

Theorem. Abnormal extremal trajectories have the form

$$\gamma(t) = \left(0, 0, 0, \int_0^T u_2(\tau)d\tau\right), \quad u_2(\tau) = \pm 1.$$

Remark. The optimal abnormal trajectories have the form $\gamma(t) = (0, 0, 0, \pm t)$.

Normal Case of Pontryagin Maximum Principle

- Let $h_i = \langle p, \mathcal{A}_i \rangle$: $\qquad h_1 = a\cos\theta + b\sin\theta + ck, \quad h_2 = d.$
- The Pontryagin function $h_u^{-1} = u_1 h_1 + u_2 h_2 - \left(u_1^2 + u_2^2\right)/2.$
- The maximum condition implies $u_1 = h_1$, $u_2 = h_2$.
- Let $p = (a, b, c, d)$. Veritical part of the Hamiltonian system

$$\begin{cases} \dot{a} = 0, \\ \dot{b} = 0, \\ \dot{c} = -(b\cos\theta - a\sin\theta)(kc + a\cos\theta + b\sin\theta), \\ \dot{d} = -c(ck + a\cos\theta + b\sin\theta), \end{cases}$$

- The Hamiltonian system

$$\begin{cases} \dot{x} = h_1\cos\theta, \\ \dot{y} = h_1\sin\theta, \\ \dot{\theta} = h_1 k, \\ \dot{k} = h_2, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = h_1 h_3, \\ \dot{h}_3 = h_1 h_4, \\ \dot{h}_4 = -k h_1(k h_3 + h_1). \end{cases}$$

Normal Case of Pontryagin Maximum Principle

- SR arc-length parameterization $u_1^2 + u_2^2 = 1$.
- The Hamiltonian $H = (h_1^2 + h_2^2)/2 = 1/2$.
- Introduce the polar angle $\alpha \in S^1$: $\quad h_1 = \cos\alpha, \quad h_2 = \sin\alpha$.

Theorem. Naturally parameterized normal extremal trajectories are solutions to the system

$$\begin{cases} \dot{x} = \cos\alpha\cos\theta, \\ \dot{y} = \cos\alpha\sin\theta, \\ \dot{\theta} = k\cos\alpha, \\ \dot{k} = \sin\alpha, \end{cases} \qquad \begin{cases} \dot{\alpha} = h_3, \\ \dot{h}_3 = h_4\cos\alpha, \\ \dot{h}_4 = -k\cos\alpha(kh_3 + \cos\alpha). \end{cases}$$

Normal Case of Pontryagin Maximum Principle

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Can we solve this system analytically?

The question of integrability of the system.

Liouville Integrability of Hamiltonian Systems

Hamiltonian system on a symplectic manifold N, $\dim N = 2d$, is said to be Liouville integrable if it has d functionally independent first integrals in involution, i.e., if there exist functions $f_1 = H, f_2, \dots, f_d \in C^\infty(N)$, constant on the trajectories of the system and such that $\{f_i, f_j\} = 0$, $i, j = 1, \dots, d$, and $f_1, \dots, f_d$ are functionally independent on an open everywhere dense subset of N.

Examples of Nonintegrable SR problems

- Montgomery, R., Shapiro, M., and Stolin, A., A Nonintegrable Sub-Riemannian Geodesic Flow on a Carnot Group, J. Dynam. Control Systems, 1997.
- Bizyaev, I.A., Borisov, A.V., Kilin, A.A. et al. Integrability and nonintegrability of sub-Riemannian geodesic flows on Carnot groups, Regul. Chaot. Dyn., 2016.
- L. V. Lokutsievskiy, Yu. L. Sachkov, Liouville integrability of sub-Riemannian problems on Carnot groups of step 4 or greater, Mat. Sb., 2018.

First Integrals in the 4D Model of V1

The Hamiltonian system

$$\begin{cases} \dot{x} = h_1 \cos\theta, \\ \dot{y} = h_1 \sin\theta, \\ \dot{\theta} = h_1 k, \\ \dot{k} = h_2, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = h_1 h_3, \\ \dot{h}_3 = h_1 h_4, \\ \dot{h}_4 = -k h_1 (k h_3 + h_1). \end{cases}$$

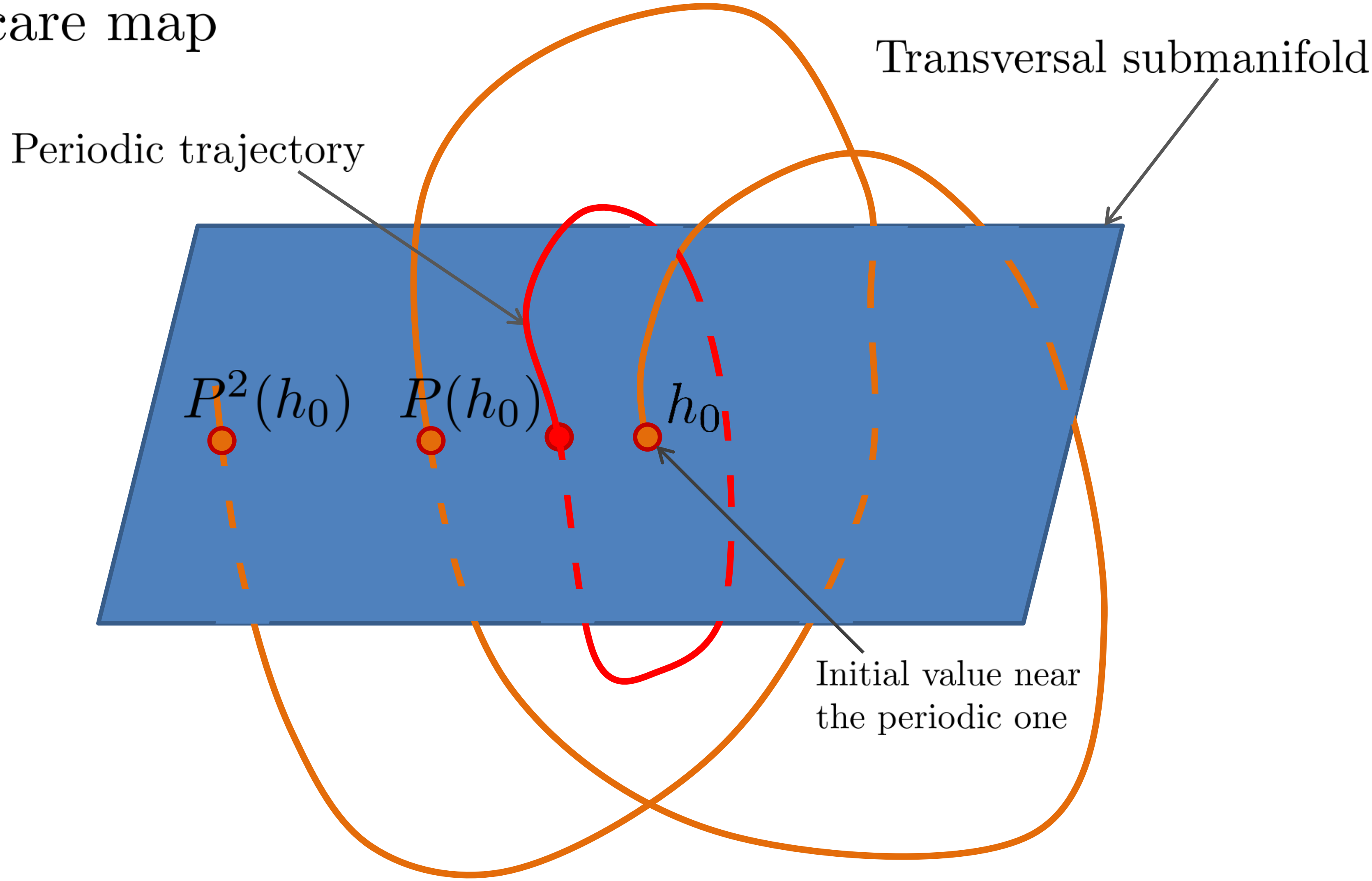
has the following first independent integrals:

1. the Hamiltonian $H = \frac{h_1^2 + h_2^2}{2}$,
2. $a = (h_1 + k h_3)\cos\theta - h_4 \sin\theta$,
3. $b = h_4 \cos\theta + (h_1 + k h_3)\sin\theta$.

To prove Liouville integrability we need to find one more independent first integral.

Numerical Analysis of Liouville Integrability

Poincare map



Numerical Analysis of Liouville Integrability

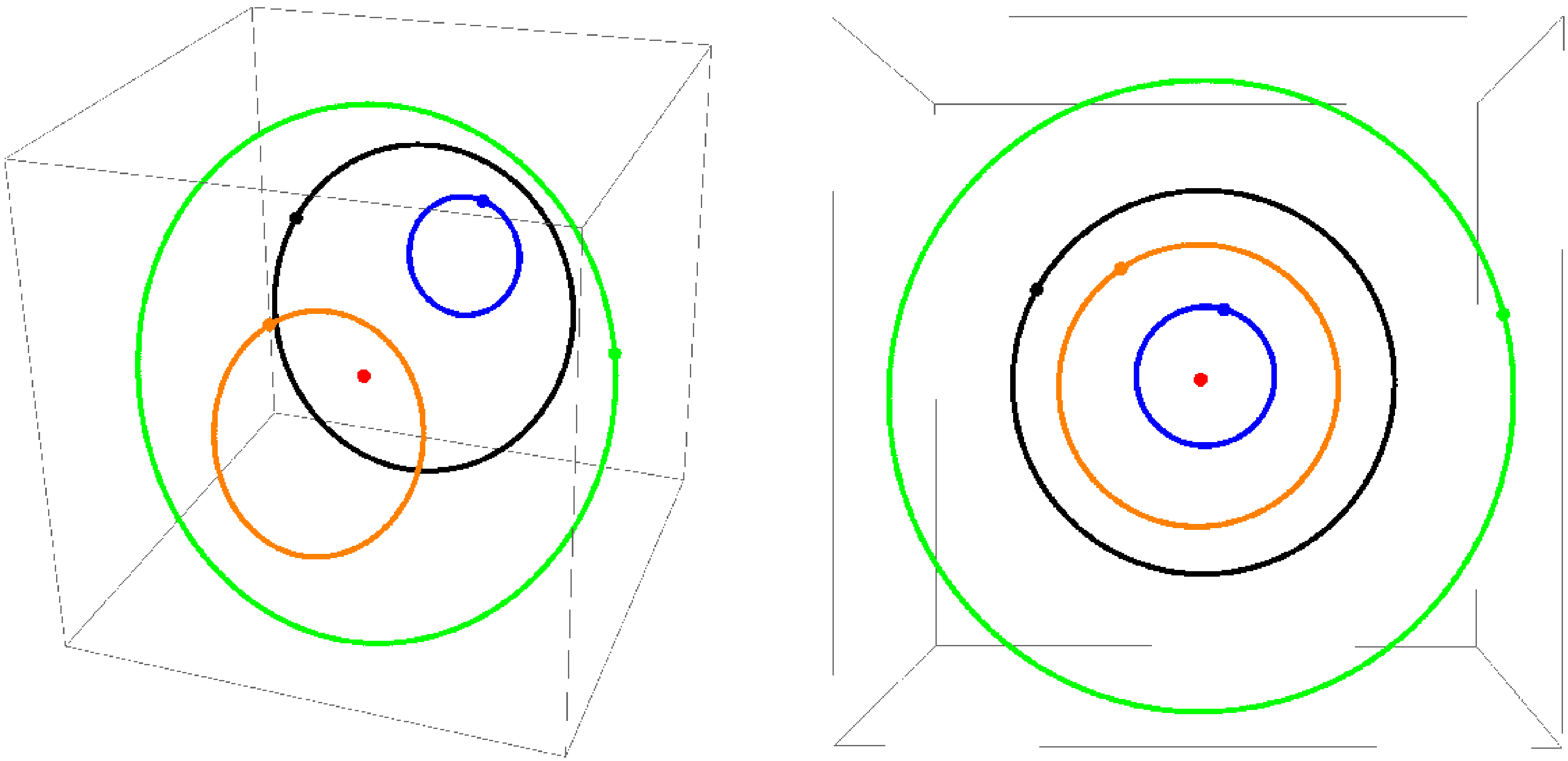
$$\begin{cases} \dot{\alpha} = h_3, \\ \dot{h}_3 = h_4 \cos\alpha, \\ \dot{h}_4 = -k\cos\alpha(kh_3 + \cos\alpha), \\ \dot{k} = \sin\alpha. \end{cases}$$

- One-parametric family of periodic trajectories

$$\alpha(t) = \frac{\pi}{2} + t\,h_3(0),\;\; h_3(t) = h_3(0),\;\; h_4(t) = 0,\;\; k(t) = \frac{\sin(t\,h_3(0))}{h_3(0)}$$

- The period equals $T = |\frac{2\pi}{h_3(0)}|$.
- Transversal submanifold $k = 0$.
- 4 random initial values h_0 from a small neighborhood of the value $\alpha(0) = \frac{\pi}{2}$, $h_3(0) = 1$, $h_4(0) = 0$ have been chosen.
- The Poincare map has been computed for 1000 iterations.

Numerical Analysis of Liouville Integrability



Numerical simulations show Liouville integrability and existance of one more independent first integral.

Conclusion

- The 4-dimensional SR model of V1 is considered
- Proof of controllability and existence of optimal control
- Application of PMP
- Parameterization of abnormal geodesics
- The Hamiltonian system for normal geodesics
- Numerical investigation of Liouville integrability of normal geodesic flow

Thank you for your attention!