Time minimization problem on the group of motions of a plane with admissible control in a half-disk

Alexey Mashtakov

Ailamazyan Program Systems Institute of RAS

Pereslavl-Zalessky, Russia

alexey.mashtakov@gmail.com

Geometric methods in control theory and mathematical physics
(ConfGMDE-2021)

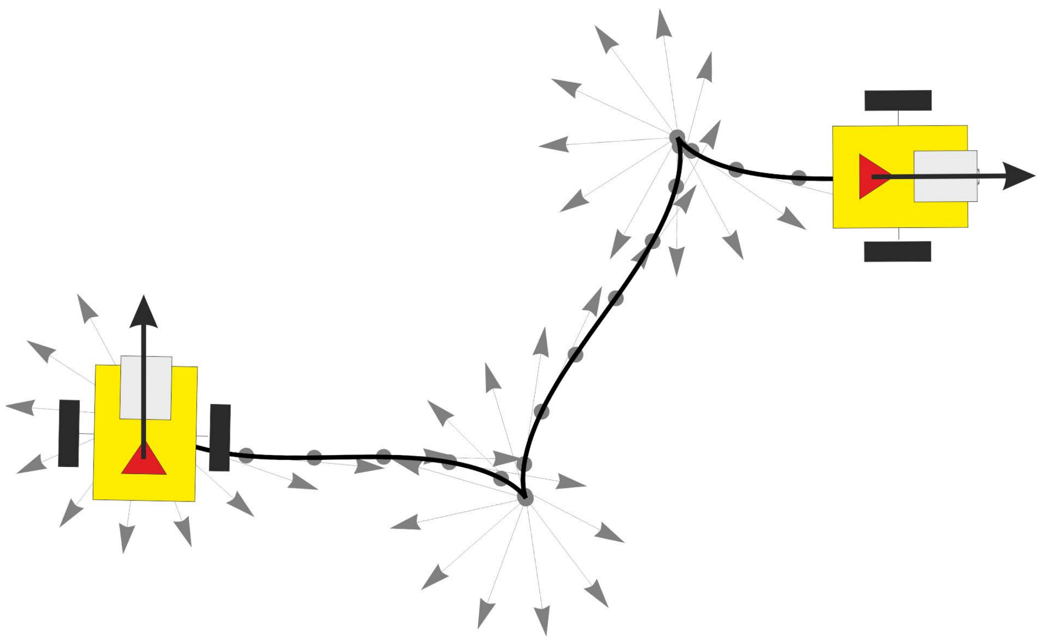
Ryazan, April 28, 2021.

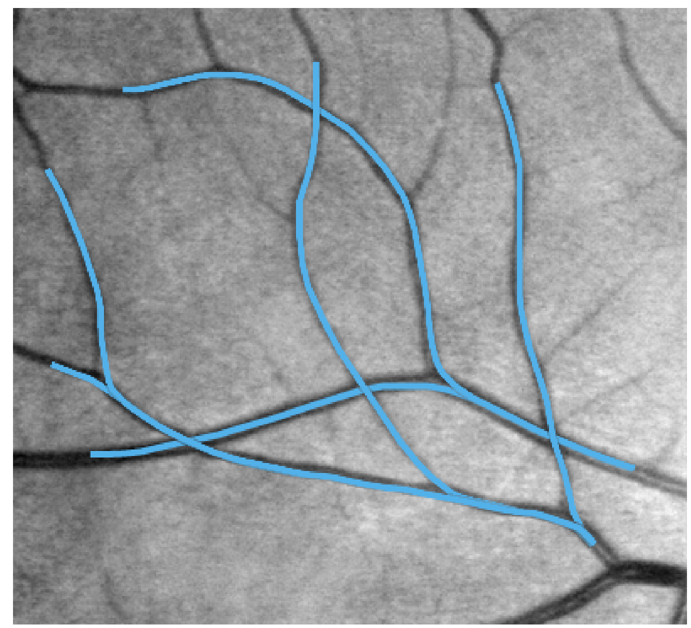
Outline of the Talk

- ◇ Motivation
- ◇ Preliminaries
- ◇ Statement of the problem
- ◇ Controllability
- ◇ Extremal Controls
- ◇ Optimality
- ◇ Conclusion

Motivation: Applications in robotics and image processing

- Motion planning problem for a car-like mobile robot that can move forward and rotate in place
- Extraction of salient curves in images. E.g. vessel tracking on images of human retina.





Preliminaries

- The group of motions of a plane $\mathrm{SE}_2 \equiv M \simeq \mathbb{R}^2_{x,y} \times S^1_\theta \ni q$:

$$qq' = \big((x,y),\theta\big)\big((x',y'),\theta'\big) = \big(R_\theta(x',y') + (x,y), \theta + \theta'\big).$$

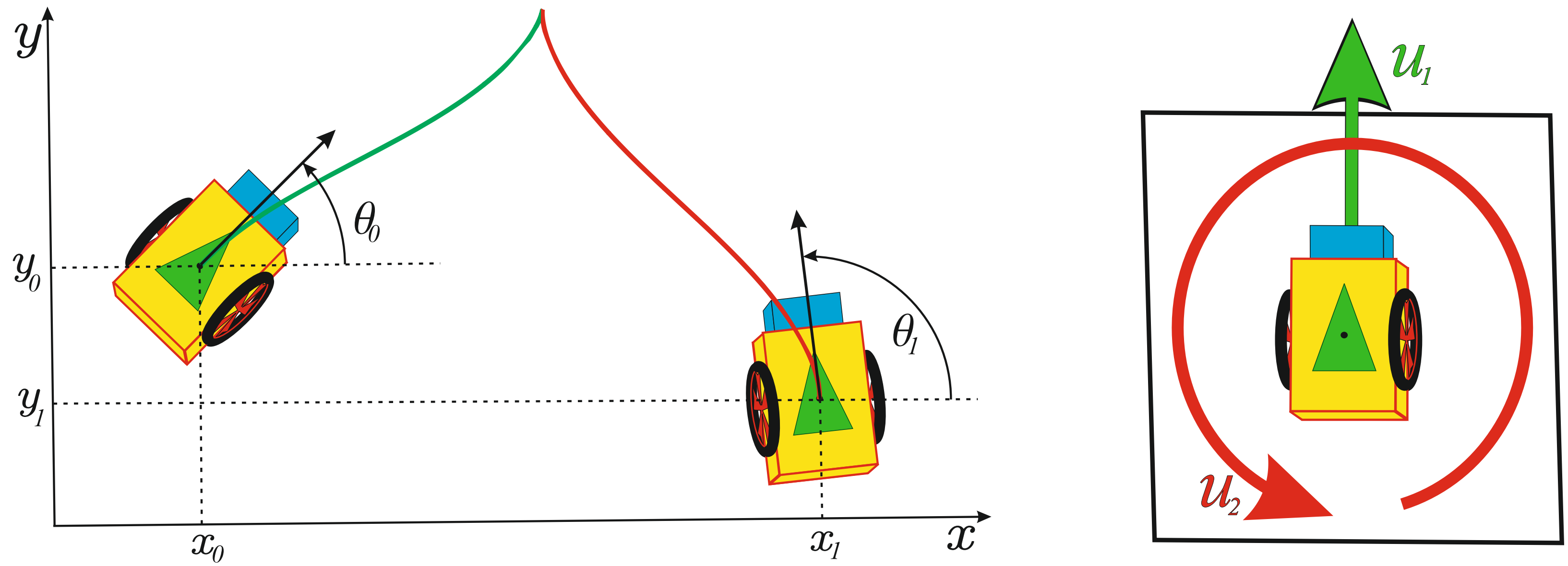
where R_θ is a counter-clockwise planar rotation on angle θ.
The Lie algebra $\mathrm{se}_2 = \mathrm{span}(X_1, X_2, X_3)$, where

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \partial_\theta, \quad X_3 = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}.$$

- By given a dynamics on M, an extremal trajectory is called a trajectory that satisfies the optimality condition — PMP.

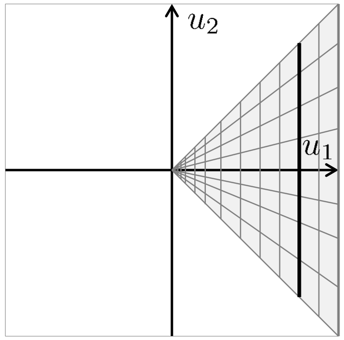
- The wavefront is a set of all points in configuration space M, reachable by all the extremal trajectories in a fixed time T.

Model of a Car on a Plane

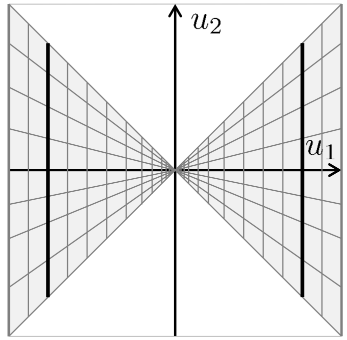


$$\dot{q} = u_1 X_1(q) + u_2 X_2(q),$$

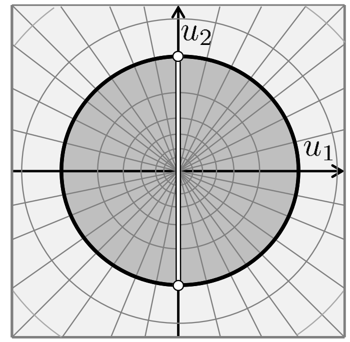
Set of Admissible Controls



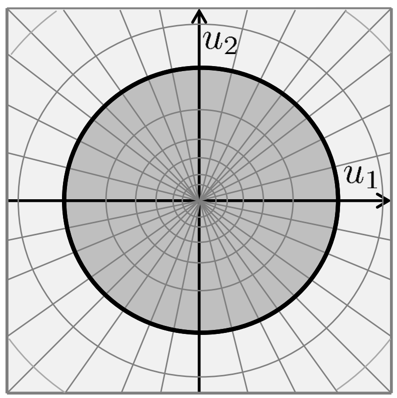
Dubins (1957)



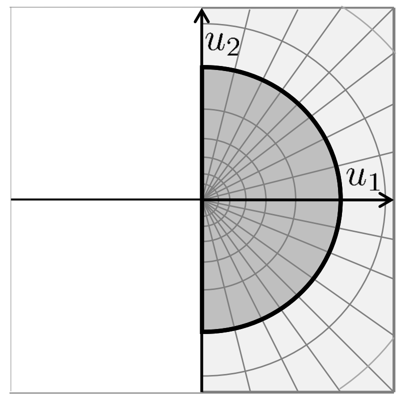
Reeds-Shepp (1990)



Berestovskii (1994)



Sachkov (2010)



Duits (2018)

Statement of the Problem

Consider the following control system (dynamics):

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \begin{array}{l} (x, y, \theta) = q \in \mathrm{SE}_2 = M, \\ u_1^2 + u_2^2 \leq 1,\ u_1 \geq 0. \end{array}$$

By given q_0, $q_1 \in M$ we aim to find the controls $u_1(t)$, $u_2(t)$ such that the corresponding trajectory $\gamma : [0, T] \to M$ transfers the system from q_0 to q_1 by minimal time

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min.$$

Here u_i are $L^\infty([0, T], \mathbb{R})$, and γ is a Lipschitzian curve on M.

Controllability of the System

Theorem. In the time minimization problem for the left-invariant control system on the group of motions of a plane with admissible control in a semicircle, there always exists an optimal trajectory that transfers the system from an arbitrary given initial configuration to an arbitrary given final configuration.

Pontryagin Maximum Principle (PMP)

- A necessary condition of optimality is given by PMP.
- Denote $(p_1, p_2, p_3) \in T^*M$. The Pontryagin function

$$H_u = p_1\sqrt{1-u^2}\cos\theta + p_2\sqrt{1-u^2}\sin\theta + p_3 u.$$

- Let $(u(t), q(t))$, $t \in [0, T]$ be an optimal process. Then

- Hamiltonian system $\dot{p} = -\frac{\partial H_u}{\partial q}$, $\dot{q} = \frac{\partial H_u}{\partial p}$;

- Maximum condition

$$H = \max_{u \in [-1,1]} H_u(p(t), q(t)) \in \{0, 1\}.$$

- Left-invariant Hamiltonians

$$h_1 = p_1\cos\theta + p_2\sin\theta, \quad h_2 = p_3, \quad h_3 = p_1\sin\theta - p_2\cos\theta.$$

Abnormal Extremal Controls and Trajectories

Theorem. Abnormal extremal control exists when $h_1 \leq 0$ and has a form $u_1(t) = 0$, $u_2(t) \in I = [-1, 1]$ — arbitrary $L_\infty([0, T], I)$ function that satisfies the condition

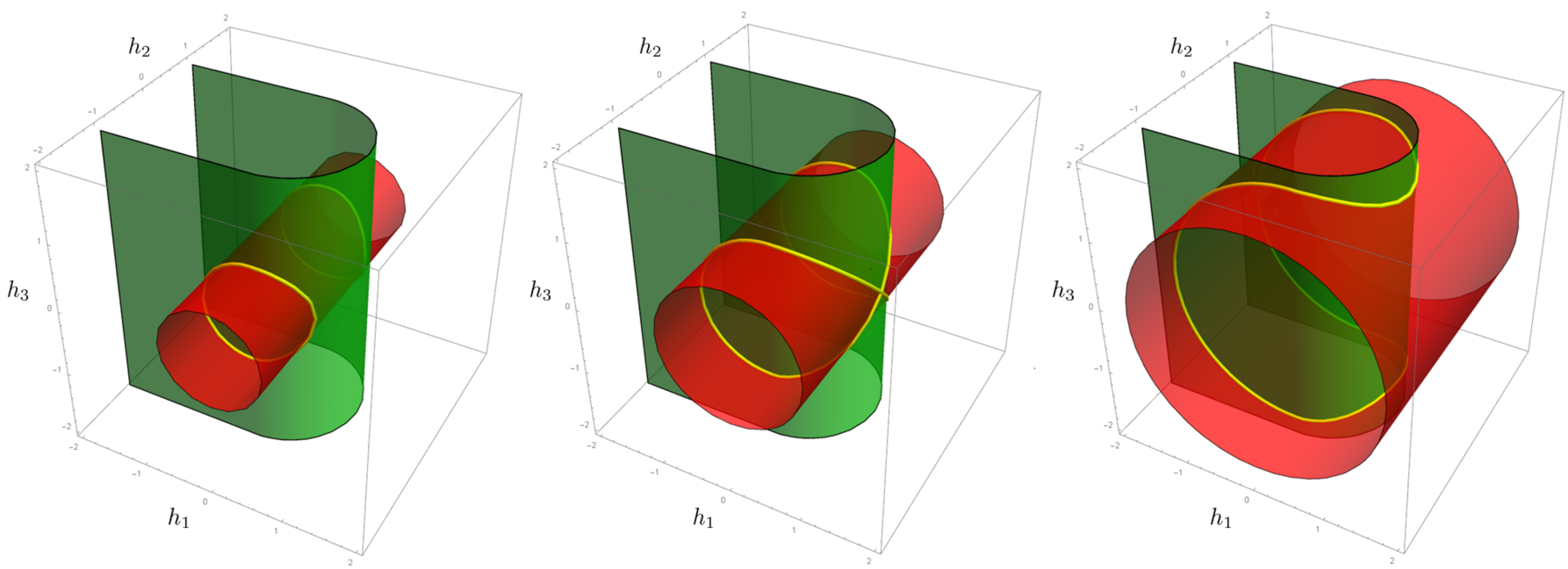
$$h_{10} \cos U_2(t) - h_{30} \sin U_2(t) < 0, \text{ where } U_2(t) = \int_0^t u_2(\tau) d\tau,$$

for all $t \in [0, T]$.

Theorem. Abnormal extremal trajectoriy has a form

$$x(t) = 0, \quad y(t) = 0, \quad \theta(t) = s_2 t.$$

First Integrals of the Hamiltonian System



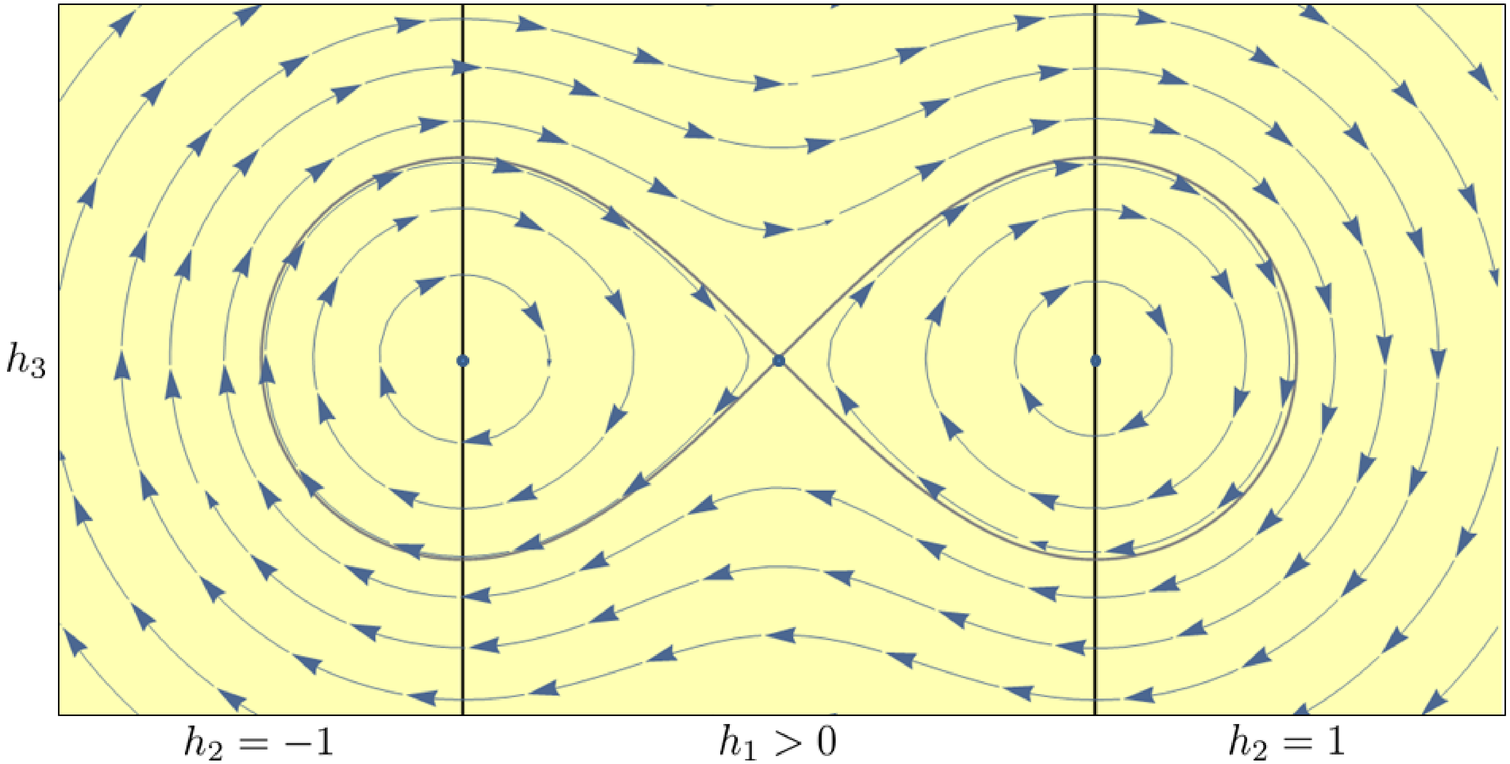
The Hamiltonian

$$H = \begin{cases} |h_2|, & \text{for } h_1 \leq 0, \\ \sqrt{h_1^2 + h_2^2}, & \text{for } h_1 > 0, \end{cases}$$

The Casimir

$$E = h_1^2 + h_3^2.$$

Dynamics of Normal Hamiltonian System



Phase portrait on the level surface $H = 1$ of the Hamiltonian.

Normal Extremal Controls

Theorem. A normal extremal control $(u_1(t), u_2(t))$ is uniquely determined by $h_{10} \in (-\infty, 1]$, $\quad s_2 \in \{-1, 1\}$, $\quad h_{30} \in \mathbb{R}$.

$$\text{Let } E = h_{10}^2 + h_{30}^2, \quad h_{20} = \begin{cases} s_2, & \text{for } h_{10} \leq 0, \\ s_2\sqrt{1 - h_{10}^2}, & \text{for } h_{10} > 0; \end{cases}$$

$$s_1 = \begin{cases} \operatorname{sign} h_{10}, & \text{for } h_{10} \neq 0, \\ -s_2 \operatorname{sign} h_{30}, & \text{for } h_{10} = 0, h_{30} \neq 0; \end{cases}$$

$$s_3^0 = \begin{cases} \operatorname{sign} h_{30}, & \text{for } h_{30} \neq 0, \\ s_2, & \text{for } h_{30} = 0; \end{cases}$$

$$\sigma = (s_1 + 1)/2 \in \{0, 1\}.$$

In the general case $E \notin \{0,1\}$ the control $u_2(t)$ is defined on time intervals formed by splitting the ray $t \geq 0$ by instances $t_0 \in \{0 = t_0^0, t_0^1, t_0^2, \ldots\}$ as

$$u_2(t) = \begin{cases} h_{20} \in [-1,1], & \text{for } t = t_0^0, \\ -s_3^{j-\sigma}\, \mathrm{cn}\left(\xi_{j-\sigma}(t), k\right), & \text{при } t \in (t_0^{j-\sigma}, t_0^{j-\sigma+1}], \\ u_2\left(t_0^{j-\sigma+s_1}\right) \in \{-1,1\}, & \text{при } t \in (t_0^{j-\sigma+s_1}, t_0^{j-\sigma+s_1+1}], \end{cases}$$

$$\text{where } j \in \{2n-1 \mid n \in \mathbb{N}\}, \qquad k = \frac{1}{\sqrt{E}}.$$

The extremal control $u_1(t)$ is given by $u_2(t) = \sqrt{1 - u_1^2(t)}$.

Here $\xi_{j-\sigma}(t) = \frac{1}{k}\left(t - t_0^{j-\sigma}\right) - s_3^{j-\sigma} F(\alpha_{j-\sigma}, k)$,

$$\alpha_{j-\sigma} = \arg\left(-s_3^{j-\sigma}\left(u_2(t_0^{j-\sigma}) + \mathbf{i}\sqrt{1 - u_2^2(t_0^{j-\sigma})}\right)\right) \in (-\pi, \pi],$$

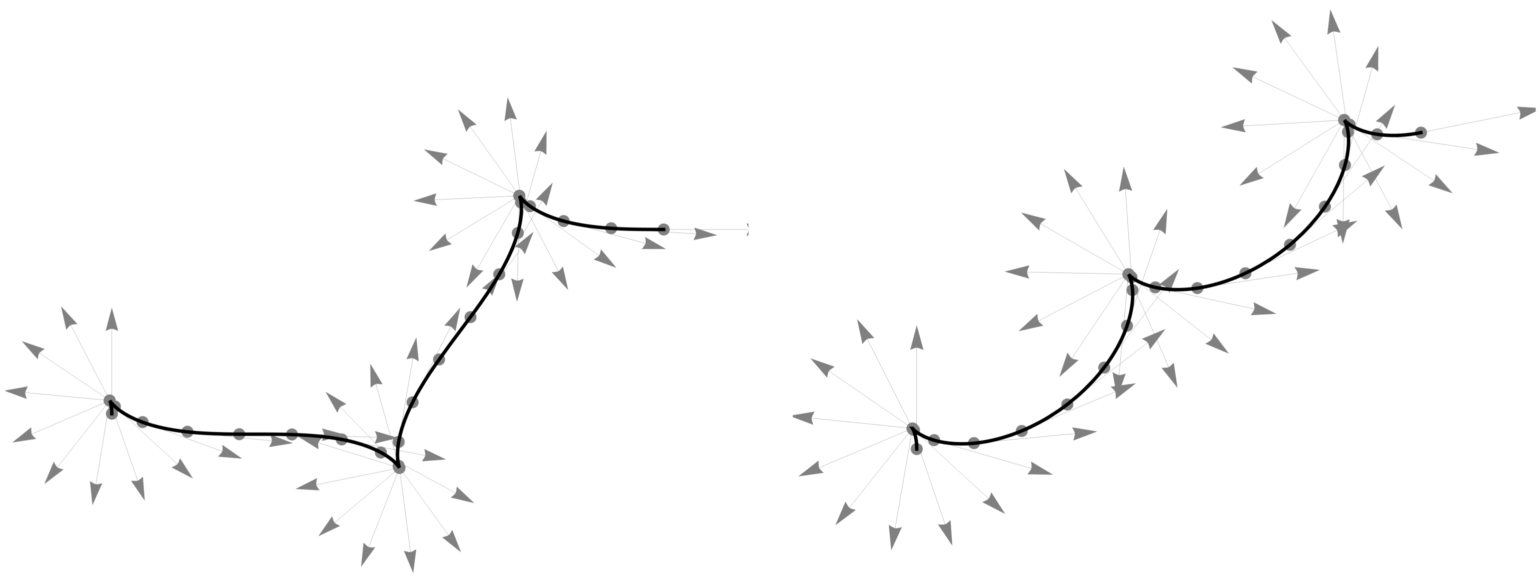
$$s_3^{j-\sigma} = \begin{cases} s_3^0, & \text{for } j-\sigma = 0, \\ s_2 s_3^0, & \text{for } j-\sigma = 1, \\ -s_3^{j-\sigma-2}, & \text{for } E > 1, \\ s_3^{j-\sigma-2}, & \text{for } E < 1. \end{cases}$$

Extremal Trajectories

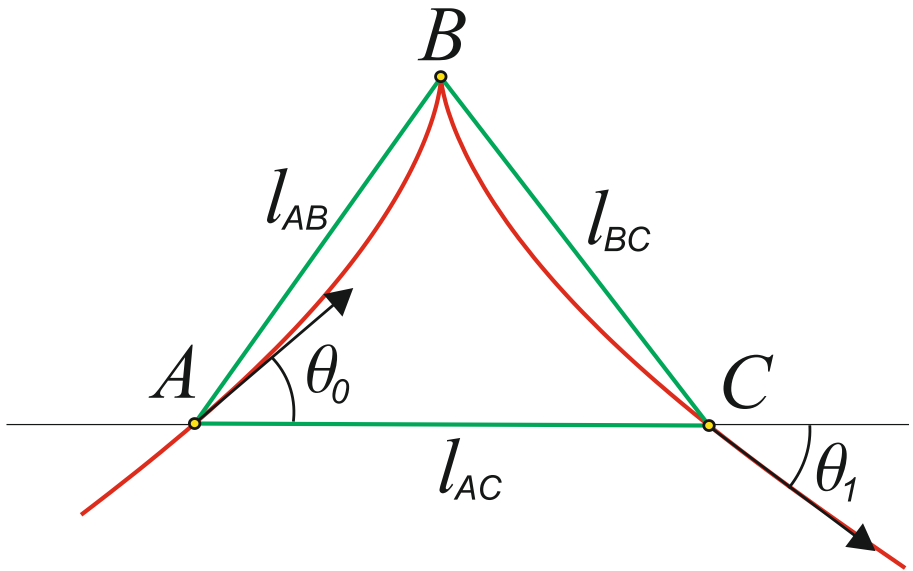
- The extremal trajectories are obtained by integration

$$x(t) = \int_0^t u_1(\tau) \cos\theta(\tau)\,\mathrm{d}\,\tau,\; y(t) = \int_0^t u_1(\tau) \sin\theta(\tau)\,\mathrm{d}\,\tau,\; \theta(t) = \int_0^t u_2(\tau)\,\mathrm{d}\,\tau.$$

- Explicit parametrization by Jacobi elliptic functions



Optimality of Extremal Trajectories



An optimal trajectory does not have internal turn points.

Proof by contradiction. For $\gamma : [0, T] \to \mathrm{SE}_2$ with an internal turn point there exists a shortcut $\gamma_0 : [0, T_0] \to \mathrm{SE}_2$.

$$T_0 = |\theta_0| + l_{AC} + |\theta_1| < T.$$

Structure of Optimal Synthesis

Theorem. Any optimal trajectory has a form

$t \in$	$[0, t_0^1)$	$[t_0^1, t_0^2)$	$[t_0^2, T]$
$x(t)$	0	$x_s(t)$	x_1
$y(t)$	0	$y_s(t)$	y_1
$\theta(t)$	$s_1 t$	$\theta_s(t)$	$\theta_s(t_0^2) + s_2(t - t_0^2)$,

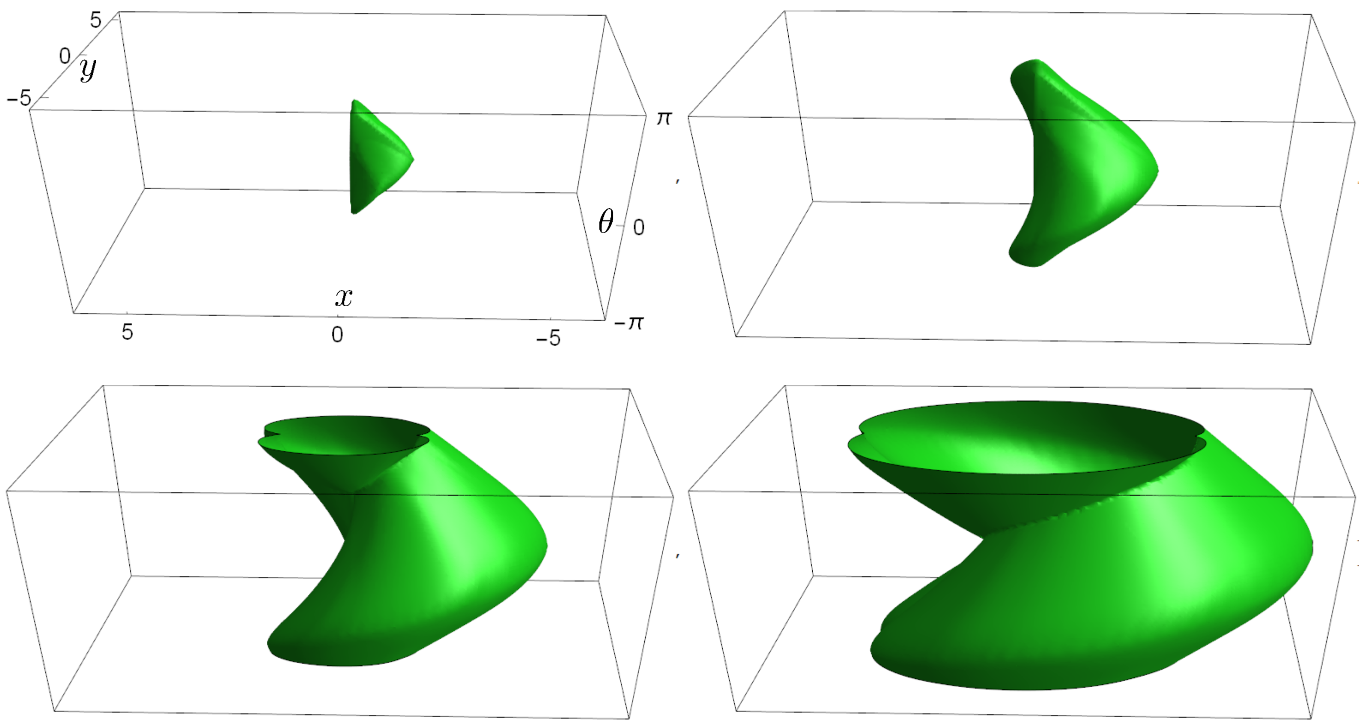
where $0 \leq t_0^1 \leq t_0^2 \leq T$ — control switching points, the signs $s_i = \pm 1$ are determined by initial values, the trajectory

$$(x_s(t), y_s(t), \theta_s(t)) =: q_s(t),$$

$$q_s(t_0^1) = (0, 0, \theta_0^1), \quad q_s(t_0^2) = (x_1, y_1, \theta_0^2)$$

is a sub-Riemannian length minimizer in SE_2 that does not have internal cusps in its planar projection (i.e. for any $t \in (t_0^1, t_0^2)$ the inequality $\dot{x}_s(t)^2 + \dot{y}_s(t)^2 > 0$ holds).

Wavefront along Optimal Trajectories



Wavefronts along optimal trajectories for $T = \frac{\pi}{2},\ \pi,\ \frac{7\pi}{5},\ 2\pi.$

Duits et.al. Optimal Paths for Variants of the 2D and 3D Reeds–Shepp Car with Applications in Image Analysis, JMIV, 2018.

Conclusion

- Solution to the left-invariant control problem, with the set of admissible controls containing zero on the boundary.

- Proof of existence of optimal control

- Explicit formulas for extremal controls and trajectories

- Partial analysis of optimality

- Structure of optimal synthesis

Thank you for your attention!