Extremal Trajectories of a Spherical Robot on Inhomogeneous Surfaces

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***Abstract*—We consider a kinematic model of a spherical robot on an inhomogeneous surface. We study a problem of the optimal motion of the robot from a given initial configuration to a given final one. The problem is formulated as the problem of optimal rolling of a sphere on a plane with a given external cost. The external cost describes the landscape and encodes the inhomogeneity of the surface. We apply a necessary optimality condition — Pontryagin maximum principle, and characterize the extremals. Finally, we present an example of the rolling along the extremal trajectory, obtained via an interface developed in Wolfram Mathematica.**

***Index Terms*—Plate-ball problem, Inhomogeneous plane, Spherical Robot, Hamiltonian system, Pontryagin maximum principle**

I. Introduction

Spherical mobile robots are an alternative to traditional wheeled robots. They are gaining popularity nowadays due to their manoeuvrability and ability to move in difficult terrain [1]. There are many different designs of spherical mobile robots. They differ in the principle of motion and in the type of actuators used. Some typical designs can be found in [2]–[4].

The dynamics of a mobile robot is determined by a system with nonholonomic constraints that describes the rolling of one body (a sphere) on the surface of another body (rolling surface). The problem of rolling surfaces is of great interest in mechanics, robotics, and control theory, see, e.g., [5]–[9]. A typical dynamic model of the motion of a spherical robot is given by the problem of rolling a sphere on a plane [10].

In this paper, we neglect dynamical effects and consider a rather simple kinematic model. In such a model, the velocity is controlled, in contrast to the acceleration, which is controlled in dynamic models. Nevertheless, such a kinematic model is adequate for motion planning for a spherical robot that is moving slowly. In this case, a trajectory of the kinematic model can be followed by the robot with a necessary precision.

We consider a kinematic model of a spherical mobile robot rolling on a nonhomogeneous surface. The robot is given by a unit sphere that can roll on a surface without slipping and twisting. The inhomogeneous surface is represented by a plane $\mathbb{R}^2$ with a given smooth external cost

$$\mathcal{C} : \mathbb{R}^2 \to [\varepsilon, +\infty) : (x, y) \mapsto \mathcal{C}(x, y), \quad \varepsilon > 0.$$

The external cost encodes the inhomogeneity of the rolling surface. The higher the value of the external cost at the point, the slower the robot moves. Such a model is described by a nonholonomic system in the configuration space

$$G = \mathbb{R}^2 \times \mathrm{SO}_3,$$

where SO_3 is the Lie group of rotations of the space $\mathbb{R}^3$.

A configuration of the robot is given by $g = (x, y, R) \in G$, where $(x, y) \in \mathbb{R}^2$ are Cartesian coordinates of the contact point between the robot and the rolling plane, and $R \in \mathrm{SO}_3$ describes the orientation of the robot in space. By the orientation of the robot we mean the position of orthonormal frame attached to the center of the robot relative to some fixed coordinate system in three-dimensional space. The third coordinate axis is perpendicular to the plane along which the robot is rolling, and the origin lies in this plane, see Fig. 1

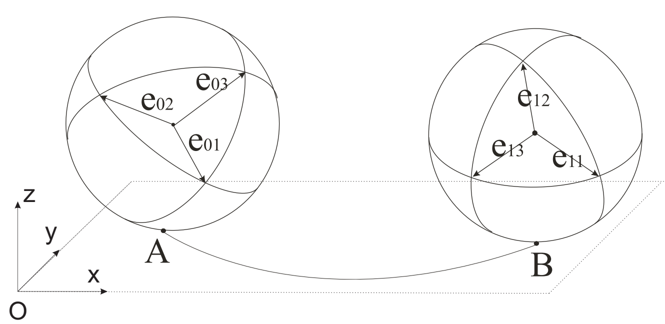

Fig. 1. A model of spherical robot on a plane

The problem with uniform cost $\mathcal{C} = 1$ is the well-known problem of rolling of a sphere on a plane without slipping and twistings. It was stated in [11] by J. Hammersley. Then A. Arthurs and G. Walsh [12] proved integrability of the Hamiltonian system of Pontryagin maximum principle (PMP). V. Jurdjevic in [13], [14] showed that planar projections of extremal trajectories $(x(t), y(t))$ are Euler elasticae (see [15],

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[16]). He gave a description of different qualitative types of extremal trajectories and obtained differential equations for evolution of Euler angles along the extremals. Explicit parametrizations of the extremals by Jacobi elliptic functions were obtained in [17]. Study of optimality of the extremals were iniciated by Yu. Sachkov in [18], where he described continuous and discrete symmetries in the problem and derived equations on Maxwell points, the points reachable by multiple extremals in the same time. It is known that after a Maxwell point an extremal is no longer optimal. The first instance of time, when an extremal reaches a Maxwell point is called the Maxwell time. In such a way, the Maxwell time gives an upper bound on cut time, the instance of time, after which an extremal is no longer optimal. Asymptotic case of rolling of a sphere along the sinusoids of small amplitudes were studied in [19], where, in particular, limiting behaviour of the Maxwell time has been described.

In this paper, we consider the problem with non-uniform external cost. We formulate the problem as an optimal control problem, apply a necessary optimality condition — Pontryagin maximum principle (PMP), and derive the Hamiltonian system of PMP that determines the extremals. We provide a numerical solution to the Hamiltonian system and develop an interface in Wolfram Mathematica that imitates the movement of the spherical robot on a non-homogeneous surface.

II. Statement of the problem

Denote by $(x, y) \in \mathbb{R}^2$ the contact point between the sphere and the plane. The problem of optimal motion of the spherical robot on an inhomogeneous surface is formulated as follows. For any two given configurations g_0, $g_1 \in G$, one aims to find the controls $u_1(t)$, $u_2(t) \in L^\infty([0,T],\mathbb{R})$, such that the corresponding trajectory $\gamma : [0,T] \to G$ transfers the system from g_0 to g_1,

$$\gamma(0) = g_0, \quad \gamma(T) = g_1,$$

and the projection of γ to the plane (x, y) has minimal weighted (by external cost) length

$$\int_0^T \mathcal{C}(x(t), y(t))\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\ \mathrm{d}t \to \min .$$

The velocity of the center of the sphere is controlled. The admissible rolling can be imagined as follows: the sphere is covered from above by a plane parallel to the rolling plane, which moves in directions parallel to the rolling plane. The resulting movement is rolling without twisting or slipping. The dynamics of such a motion is given by

$$\dot{\gamma}(t) = u_1(t)\ \mathcal{A}_1|_{\gamma(t)} + u_2(t)\ \mathcal{A}_2|_{\gamma(t)} ,$$

where $\mathcal{A}_i$ are the basis left–invariant vector fields on the Lie group $G = \mathbb{R}^2 \times \mathrm{SO}_3$, which correspond to infinitesimal rolling along the O_x axis for $i = 1$ and along the O_y axis for $i = 2$. Next we derive the control system explicitly.

Remark 1: The external cost $\mathcal{C}(x, y)$ can be understood as viscosity of the plane. The center of the sphere rolling along a path through a point (x, y) moves with bounded speed $\sqrt{u_1^2(x, y) + u_2^2(x, y)} \leq 1/\mathcal{C}(x, y)$. Restriction of the set of admissible controls by above inequality leads to equivalent formulation of the problem as a time minimization problem. By given boundary configurations, we aim to find a path of the sphere rolling on the inhomogeneously viscous plane from the initial to the final configuration by minimal time.

The Lie group SO_3 is represented by the matrices

$$\mathrm{SO}_3 = \{R : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, RR^T = Id,\, \det R = 1\}.$$

Denote by so_3 the space of skew-symmetric matrices of order 3. It is the tangent space to the group SO_3 at the unit element:

$$\mathrm{so}_3 = \{\rho : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, \rho = -\rho^T\} = T_{Id}\mathrm{SO}_3.$$

The space so_3 is spanned by basis elements ρ_x, ρ_y, ρ_z, where

$$[\rho_x, \rho_y, \rho_z] =$$

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right].$$

Let $R \in \mathrm{SO}_3$ denote the rotation of the space $\mathbb{R}^3$, at which the vectors of the moving frame e_i, specifying the orientation of the sphere, are transformed into unit vectors of the basis coordinate system:

$$R\,e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R\,e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R\,e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Rolling without twisting or slipping is determined by the following control system [13], [14]:

$$\dot{x}(t) = u_1(t),\ \dot{y}(t) = u_2(t),\ \dot{R}(t) = R(t)\left(u_2(t)\rho_x - u_1(t)\rho_y\right).$$

From a geometrical viewpoint, this system is a left-invariant control system on G. Without loss of generality, one can choose the unity $e = (0, 0, Id)$ as the initial condition. Any trajectory with another initial condition is obtained via the left shift on the group. The left-invariant basis vector fields that correspond to infinitesimal rolling along O_x, O_y are given by

$$\mathcal{A}_1 = \frac{\partial}{\partial x} - R\,\rho_y, \qquad \mathcal{A}_2 = \frac{\partial}{\partial y} + R\,\rho_x.$$

To describe the orientation of the sphere, it is convenient, along with the rotation matrix R, to use quaternions [20]. Quaternions are an extension of complex numbers. Just as the rotation of a plane can be described by multiplying by a complex number, quaternions are used to describe the rotations of three-dimensional space. Quaternions form an associative division algebra over real numbers. They are formed by adding to the real numbers three imaginary units i, j, k.

Every quaternion has a form

$$q = q_0 + q_1 i + q_2 j + q_3 k, \quad q_l \in \mathbb{R}.$$

The sum of quaternions is defined as the sum of vectors. The product of a quaternion with a real number is commutative, and in general the product of two quaternions is non-commutative. The rule for multiplying imaginary units is defined as follows: $i^2 = j^2 = k^2 = -1$, $i\ j = -j\ i = k$, $j\ k = -k\ j = i$, $k\ i = -i\ k = j$.

Denote by S^3 the set of quaternions of unit length. This is a three-dimensional sphere in four-dimensional space. A quaternion $q \in S^3$ defines a rotation of the space $I = \mathrm{span}\,(i, j, k)$:

$$q \in S^3 \Rightarrow R_q(a) = qaq^{-1}, \quad a \in I, \;\; R_q \in \mathrm{SO}_3 \cong \mathrm{SO}(I).$$

S^3 is a double cover of SO_3. Since the projection $\pi : S^3 \to \mathrm{SO}_3$ is a local diffeomorphism, any vector field $\mathcal{A}$ on SO_3 has a unique lift to S^3. Therefore, the control system has a unique lift from SO_3 to S^3. Using the expression of rotation matrix via components of quaternions, see [21], we obtain

$$\mathcal{A}_1 = \partial_x + \frac{q_2}{2}\partial_{q_0} + \frac{q_3}{2}\partial_{q_1} - \frac{q_0}{2}\partial_{q_2} - \frac{q_1}{2}\partial_{q_3}, \tag{1}$$

$$\mathcal{A}_2 = \partial_y - \frac{q_1}{2}\partial_{q_0} + \frac{q_0}{2}\partial_{q_1} + \frac{q_3}{2}\partial_{q_2} - \frac{q_2}{2}\partial_{q_3}. \tag{2}$$

Thus, we obtained the following statement of the problem on the Lie group $\bar{G} = \mathbb{R}^2 \times S^3 \ni (x, y, q) = g$. For a given terminal condition g_1 and an external cost $\mathcal{C}(g) := \mathcal{C}(x, y)$, one aims to find a trajectory $\gamma : [0, T] \to \bar{G}$ that satisfies

$$\dot{\gamma}(t) = u_1(t)\ \mathcal{A}_1|_{\gamma(t)} + u_2(t)\ \mathcal{A}_2|_{\gamma(t)}\,, \tag{3}$$

$$\gamma(0) = e,\ \gamma(T) = g_1, \quad (u_1, u_2) \in \mathbb{R}^2 \tag{4}$$

$$\int_0^T \mathcal{C}(\gamma(t))\sqrt{u_1^2(t) + u_2^2(t)}\ \mathrm{d}t \to \min,$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by (1)-(2).

The problem can be simplified. The Cauchy-Schwarz inequality ensures that the original problem is equivalent to the problem of minimizing the action

$$J = \frac{1}{2}\int_0^T \mathcal{C}^2(\gamma(t))\left(u_1^2(t) + u_2^2(t)\right)\ \mathrm{d}t \to \min\,. \tag{5}$$

By virtue of Rashevsky-Chow theorem [9], the system is completely controllable. Filippov theorem guarantees existence of optimal trajectories.

III. Extremal controls and trajectories

In this section, we apply to problem (3)-(5) a necessary optimality condition given by PMP [9], [22].

For $\nu \in \{0, 1\}$ define the Pontryagin function

$$H_u(p, g) = \langle p, u_1\ \mathcal{A}_1|_g + u_2\ \mathcal{A}_2|_g\rangle - \frac{\nu}{2}\mathcal{C}^2(g)\left(u_1^2 + u_2^2\right).$$

The case $\nu = 0$ is called the abnormal case, and the case $\nu = 1$ is called the normal case. Abnormal extremals do not depend on the minimizing functional, they are determined only by the control system. Abnormal extremals in problem (3)-(5) are given by rolling of the sphere along straight lines [18]. Next we consider the normal case of PMP $\nu = 1$.

PMP states that if $(u(t), \gamma(t))$, $t \in [0, T]$ is the optimal control and the corresponding optimal trajectory, then the following conditions hold:

1) the Hamiltonian system $\dot{p} = -\dfrac{\partial H_u}{\partial q}$, $\dot{\gamma} = \dfrac{\partial H_u}{\partial p}$;

2) the maximum condition

$$H_{u(t)}(p(t), \gamma(t)) = \max_{u \in \mathbb{R}^2} H_u(p(t), \gamma(t)) =: H = 1.$$

Natural coordinates in cotangent bundle for left-invariant control problems in Lie groups are given by left invariant Hamiltonians [23]. In our case, we define them as follows:

$$h_i(p, g) = \langle p, \mathcal{A}_i|_g\rangle, \quad g \in \bar{G},\ p \in T_g^*\bar{G}.$$

Application of PMP gives the Pontryagin function

$$H_u(p, g) = u_1 h_1(p, g) + u_2 h_2(p, g) - \frac{1}{2}\mathcal{C}^2(g)\left(u_1^2 + u_2^2\right).$$

The maximum condition gives the (maximized) Hamiltonian

$$H(p, g) = \frac{1}{2\,\mathcal{C}^2(g)}\left(h_1^2(p, g) + h_2^2(p, g)\right) = 1 \tag{6}$$

and expressions for the extremal controls

$$u_1(t) = \frac{h_1(t)}{\mathcal{C}^2(\gamma(t))}, \quad u_2(t) = \frac{h_2(t)}{\mathcal{C}^2(\gamma(t))}. \tag{7}$$

Application of PMP leads to a Hamiltonian systems in cotangent bundle $T^*\bar{G}$ that describes extremals. The subsystem for state variables x, y, q_i is called the *horizontal* part, and the subsystem for conjugate variables $h_1, \ldots, h_5$ is called the *vertical* part of the Hamiltonian system. An extremal control is determined by a solution of the vertical part, while an extremal trajectory is a solution to the horizontal part. We present explicitly the Hamiltonian system of PMP in problem (3)-(5) in the following theorem.

Theorem 1: A normal extremal trajectory $\gamma(t) = (x(t), y(t), q(t))$ in problem (3)-(5) is uniquely determined by the parameters $s_1 = \pm 1$, $h_2^0 \in [-\sqrt{2}\,\mathcal{C}(e), \sqrt{2}\,\mathcal{C}(e)]$, $h_3, \ldots, h_5^0 \in \mathbb{R}$, and given by a solution to the following Hamiltonian system:

$$\begin{cases} \dot{x}(t) = \dfrac{h_1(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{y}(t) = \dfrac{h_2(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{q}_0(t) = \dfrac{-h_2(t)q_1(t) + h_1(t)q_2(t)}{2\mathcal{C}^2(\gamma(t))}, \\ \dot{q}_1(t) = \dfrac{h_2(t)q_0(t) + h_1(t)q_3(t)}{2\mathcal{C}^2(\gamma(t))}, \\ \dot{q}_2(t) = \dfrac{-h_1(t)q_0(t) + h_2(t)q_3(t)}{2\mathcal{C}^2(\gamma(t))}, \\ \dot{q}_3(t) = -\dfrac{h_1(t)q_1(t) + h_2(t)q_2(t)}{2\mathcal{C}^2(\gamma(t))}. \end{cases} \tag{8}$$

$$\begin{cases} \dot{h}_1(t) = \dfrac{1}{\mathcal{C}^3(\gamma(t))}\partial_x\mathcal{C}(\gamma(t)) + \dfrac{h_2(t)h_3(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{h}_2(t) = \dfrac{1}{\mathcal{C}^3(\gamma(t))}\partial_y\mathcal{C}(\gamma(t)) + \dfrac{h_1(t)h_3(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{h}_3(t) = \dfrac{h_1(t)h_4(t) + h_2(t)h_5(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{h}_4(t) = -\dfrac{h_1(t)h_3(t)}{\mathcal{C}^2(\gamma(t))}, \\ \dot{h}_5(t) = -\dfrac{h_2(t)h_3(t)}{\mathcal{C}^2(\gamma(t))}, \end{cases} \tag{9}$$

with the initial condition

$$h_1(0) = s_1\sqrt{2\mathcal{C}^2(e) - (h_2^0)^2},\ h_i(0) = h_i^0,\ i = 2, \ldots, 5.$$

Proof: For a left-invariant control problem with the Hamiltonian H, the vertical part of the Hamiltonian system of PMP has a form, see [9],

$$\dot{h}_i = \{H, h_i\},$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket (see [24]).

The non-zero Lie brackets between $\mathcal{A}_i$ are given by

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, \quad [\mathcal{A}_1, \mathcal{A}_3] = \mathcal{A}_4, \quad [\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_5,$$
$$[\mathcal{A}_1, \mathcal{A}_4] = [\mathcal{A}_2, \mathcal{A}_5] = -\mathcal{A}_3, \quad [\mathcal{A}_4, \mathcal{A}_5] = \mathcal{A}_3,$$
$$[\mathcal{A}_3, \mathcal{A}_4] = \mathcal{A}_5, \quad [\mathcal{A}_3, \mathcal{A}_5] = -\mathcal{A}_4.$$

Using the standard relation between Poisson and Lie brackets $\{h_i, h_j\} = \langle p, [\mathcal{A}_i, \mathcal{A}_j]\rangle$, we obtain vertical part (8).

Substitution of expression (7) to control system (3) gives horizontal part (8).

Restriction to the parameter $h_1(0)$ is due to (6). ■

IV. Computational model of a spherical robot moving along the extremals

Based on the numerical integration of the Hamiltonian system we developed software in Wolfram Mathematica that simulates the rolling of a spherical robot along the extremal trajectories. The input parameters are the external cost $\mathcal{C}(x, y)$ and the initial values of the covector s_1, h_i^0, $i \in \{2, \ldots, 5\}$. The software generates the animation of the resulting motion of the spherical robot. Inhomogenity of the plain is visualized by vertical displacment of a point by the value of the external cost $\mathcal{C}(x, y) - 1$. In such a way, the inhomogeneous plane is represented by the surface $z(x, y) = \mathcal{C}(x, y) - 1$.

See an example of the output in Fig. 2. In this example, the parameters are set as follows. We fix the domain $(x, y) \in [-2, 5] \times [-2, 7]$. We plot a sphere of unit radius in 3D space with verical z-coordinate from the interval $z \in [-3, 4]$. The external cost is chosen $\mathcal{C}(x, y) = 1 + \sin^2 \frac{x}{5} + \cos^2 \frac{y}{5}$. The initial configuration of the sphere is $e = (0, 0, 1)$, the initial value of momentum covector is $s_1 = 1$, $h_2^0 = \frac{1}{\sqrt{2}}$, $h_3(0) = 1$, $h_4(0) = \frac{1}{2}$, $h_5(0) = \frac{1}{2}$. The end time is $T = 60$. In this experiment, the final configuration is $g_1 = (4.68, 3.76, 0.057 + 0.74\ i - 0.49\ j - 0.45\ k)$.

V. Conclusion

We considered the kinematic model of a spherical robot rolling on an inhomogeneous surface. We studied a problem of the optimal motion of the robot from a given initial configuration to a given final one. The problem was formulated as the problem of optimal rolling of a sphere on a plane with a given external cost. The external cost describes the landscape and encodes the inhomogeneity of the surface. We derived optimal control formulation of the problem as a problem on Lie group $\bar{G} = \mathbb{R}^2 \times S^3$. We applied a necessary optimality condition — PMP, and derived the Hamiltonian system of PMP that determines the extremals in Theorem 1. Finally, we developed an interface in Wolfram Mathematica that imitates the movement of the spherical robot on an inhomogeneous surface and presented an example of the resulting rolling along the extremal trajectory in Fig. 2.

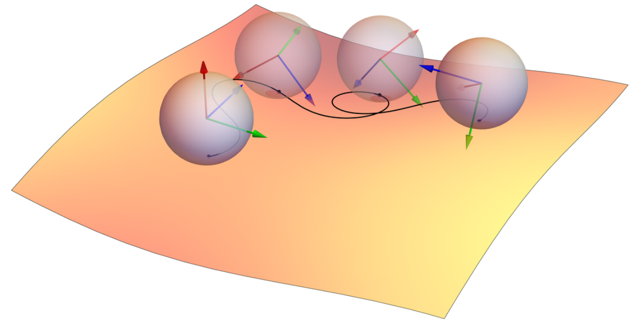

Fig. 2. Rolling of a spherical robot along the extremal trajectory

References

[1] A.V. Borisov, I.S. Mamaev, Y.L. Karavaev, “Mobile robots. Robot Wheel and Robot Ball”, Institute for Computer Studies, 2013, 532 p.

[2] S. Sang, J. Zhao, H. Wu, S. Chen, Q. An, “Modeling and Simulation of a Spherical Mobile Robot”, In: Computer Science and Information Systems, 7(1), (2010), 51–62.

[3] M. Svinin, A. Morinaga and M. Yamamoto, “An analysis of the motion planning problem for a spherical rolling robot driven by internal rotors”, 2012 IEEE/RSJ International Conference on Intelligent Robots and Systems, (2012), 414–419.

[4] Alexander A. Kilin, Elena N. Pivovarova, Tatyana B. Ivanova, “Spherical Robot of Combined Type: Dynamics and Control”, Regul. Chaotic Dyn., 20:6 (2015), 716–728

[5] Z. Li, J. Canny, “Motion of two rigid bodies with rolling constraint”, IEEE Trans. on Robotics and Automation, 6(1), (1990), 62–72.

[6] A. Bicchi, D. Prattichizzo, S. Sastry, “Planning motions of rolling surfaces”, IEEE Conf. on Decision and Control, 1995.

[7] A. Marigo, A. Bicchi, “Rolling bodies with regular surface: the holonomic case”, In: Differential geometry and control: Summer Research Institute on Differential Geometry and Control, June 29–July 19, 1997, Univ. Colorado, Boulder, G. Ferreyra et al., Eds, (1999), 241–256.

[8] A. V. Borisov, I. S. Mamaev, “The rolling motion of a rigid body on a plane and a sphere. Hierarchy of dynamics”, Regul. Chaotic Dyn., 7:2 (2002), 177–200.

[9] A.A. Agrachev, Yu.L. Sachkov, “Control Theory from the Geometric Viewpoint”, Springer–Verlag (2004)

[10] Alexey V. Bolsinov, Alexey V. Borisov, Ivan S. Mamaev, “Rolling of a Ball without Spinning on a Plane: the Absence of an Invariant Measure in a System with a Complete Set of Integrals”, Regul. Chaotic Dyn., 17:6 (2012), 571–579.

[11] J.M. Hammersley, “Oxford commemoration ball”, In: Probability, Statistics and Analysis, pp. 112–142. London Math. Soc. lecture notes, ser. 79, 1983.

[12] A.M. Arthurs, G.R.Walsh, “On Hammersley’s minimum problem for a rolling sphere”, Math. Proc. Cambridge Phil. Soc., 99, pp. 529–534, 1986.

[13] V. Jurdjevic, “The geometry of the plate-ball problem”, Arch. Rat. Mech. Anal., v. 124, 305–328, 1993.

[14] V. Jurdjevic, “Geometric control theory”, Cambridge University Press, 1997.

[15] L. Euler, “Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Sive Solutio Problematis Isoperimitrici Latissimo Sensu Accepti”, Lausanne–Geneva, 1744.

[16] A.E.H. Love, “A Treatise on the Mathematical Theory of Elasticity”, Cambridge Univ. Press, 1927.

[17] A.P. Mashtakov, Yu.L. Sachkov, “Extremal trajectories and the asymptotics of the Maxwell time in the problem of the optimal rolling of a sphere on a plane”, Sb. Math., 202:9, pp. 1347–1371, 2011.

[18] Yu.L. Sachkov, “Maxwell strata and symmetries in the problem of optimal rolling of a sphere over a plane”, Sb. Math., 201:7, 1029–1051, 2010.

[19] A.P. Mashtakov, A.Y. Popov, “Asymptotics of Maxwell Time in the Plate-Ball Problem”, J Math Sci, 195, pp. 336–368, 2013.

[20] V.I. Arnold, “Geometry of complex numbers, quaternions and spins”, MTsNMO, Moscow, 2009.

[21] E.T. Whittaker, G.N. Watson, “A Course of Modern Analysis”, Cambridge University Press, 1996.

[22] M. I. Zelikin, “Optimal control and variational calculus”, Editorial URSS, Moscow, 2004.

[23] A. Agrachev, D. Barilari, U. Boscain, “A Comprehensive Introduction to Sub-Riemannian Geometry”, Cambridge Studies in Advanced Mathematics. — Cambridge: Cambridge University Press, 2019.

[24] C. Laurent–Gengoux, A. Pichereau, P. Vanhaecke, “Poisson structures”, Grundlehren der mathematischen Wissenschaften. — Springer–Verlag, pp. 347, 2013.