

1 Conjugate time in the sub-Riemannian problem  
2 on the Cartan group<sup>\*†</sup>

3 Yu. L. Sachkov

Sobolev Institute of Mathematics  
Academic Koptug avenue 4, 630090  
Novosibirsk, Russia

and

Ailamazyan Program Systems Institute of RAS  
Pereslavl-Zalessky, Russia  
e-mail: yusachkov@gmail.com

4 March 18, 2021

5 **Abstract**

6 The Cartan group is the free nilpotent Lie group of rank 2 and step 3.  
7 We consider the left-invariant sub-Riemannian problem on the Cartan  
8 group defined by an inner product in the first layer of its Lie algebra. This  
9 problem gives a nilpotent approximation of an arbitrary sub-Riemannian  
10 problem with the growth vector  $(2, 3, 5)$ .

11 In previous works we described a group of symmetries of the sub-  
12 Riemannian problem on the Cartan group, and the corresponding Maxwell  
13 time — the first time when symmetric geodesics intersect one another. It  
14 is known that geodesics are not globally optimal after the Maxwell time.

15 In this work we study local optimality of geodesics on the Cartan  
16 group. We prove that the first conjugate time along a geodesic is not less  
17 than the Maxwell time corresponding to the group of symmetries. We  
18 characterize geodesics for which the first conjugate time is equal to the  
19 first Maxwell time. Moreover, we describe continuity of the first conjugate  
20 time near infinite values.

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<sup>\*</sup>The work is supported by the Mathematical Center in Akademgorodok under Agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation

<sup>†</sup>MSC2010: 49K15, 53C17, 93C15

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# 1 Introduction

## 1.1 Problem statement

This work deals with the nilpotent sub-Riemannian problem with the growth vector  $(2, 3, 5)$ . This problem evolves on the *Cartan group*, which is the connected simply connected free nilpotent Lie group of rank 2 and step 3.

The Lie algebra of the Cartan group is the 5-dimensional nilpotent Lie algebra  $\mathfrak{g} = \text{span}(X_1, X_2, X_3, X_4, X_5)$  with the multiplication table

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad \text{ad } X_4 = \text{ad } X_5 = 0,$$

see Fig. 1.

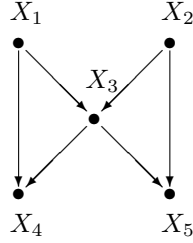


Figure 1: The Cartan Lie algebra  $\mathfrak{g}$

We consider the sub-Riemannian problem on the Cartan group  $G$  for the left-invariant sub-Riemannian structure generated by the orthonormal frame  $X_1, X_2$ :

$$\dot{g} = u_1 X_1(g) + u_2 X_2(g), \quad g \in G, \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad (1.1)$$

$$g(0) = g_0, \quad g(t_1) = g_1, \quad (1.2)$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min, \quad (1.3)$$

see books [1, 28] as a reference on sub-Riemannian geometry.

In appropriate coordinates  $g = (x, y, z, v, w)$  on the Cartan group  $G \cong \mathbb{R}^5$ , the problem is stated as follows:

$$\dot{g} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v} \\ \dot{w} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \\ -\frac{x^2+y^2}{2} \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \\ 0 \end{pmatrix}, \quad g \in \mathbb{R}^5, \quad u \in \mathbb{R}^2, \quad (1.4)$$

$$g(0) = g_0, \quad g(t_1) = g_1, \quad (1.5)$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min. \quad (1.6)$$

Admissible trajectories  $g(t)$  are Lipschitzian, and admissible controls  $u(t)$  are measurable and locally bounded.

1 Since the problem is invariant under left shifts on the Cartan group, we can  
2 assume that the initial point is identity of the group:  $g_0 = \text{Id} = (0, 0, 0, 0, 0)$ .

3 Problem (1.4)–(1.6) has the following geometric model called a generalized  
4 Dido's problem. Take two points  $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$  connected by a curve  
5  $\gamma_0 \subset \mathbb{R}^2$ , a number  $S \in \mathbb{R}$ , and a point  $c \in \mathbb{R}^2$ . One has to find the shortest  
6 curve  $\gamma \subset \mathbb{R}^2$  that connects the points  $(x_0, y_0)$  and  $(x_1, y_1)$ , such that the domain  
7 bounded by  $\gamma_0$  and  $\gamma$  has algebraic area  $S$  and center of mass  $c$ .

Optimal control problem (1.4)–(1.6) is the sub-Riemannian (SR) length minimization problem for the distribution

$$\Delta_g = \text{span}(X_1(g), X_2(g)), \quad g \in G,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{x^2 + y^2}{2} \frac{\partial}{\partial w}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2 + y^2}{2} \frac{\partial}{\partial v},$$

endowed with the inner product  $\langle \cdot, \cdot \rangle$  in which  $X_1, X_2$  are orthonormal:

$$\langle X_i, X_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$

The distribution  $\Delta$  has the flag

$$\Delta \subset \Delta^2 = \Delta + [\Delta, \Delta] \subset \Delta^3 = \Delta^2 + [\Delta, \Delta^2] = TG$$

and the growth vector  $(2, 3, 5) = (\dim \Delta_g, \dim \Delta_g^2, \dim \Delta_g^3)$ , where

$$\Delta_g^2 = \text{span}(X_1(g), X_2(g), X_3(g)), \quad \Delta_g^3 = \text{span}(X_1(g), \dots, X_5(g)),$$

$$X_3 = [X_1, X_2] = \frac{\partial}{\partial z} + x \frac{\partial}{\partial v} + y \frac{\partial}{\partial w},$$

$$X_4 = [X_1, X_3] = \frac{\partial}{\partial v}, \quad X_5 = [X_2, X_3] = \frac{\partial}{\partial w}.$$

8 Thus  $(\Delta, \langle \cdot, \cdot \rangle)$  is a nilpotent SR structure with the growth vector  $(2, 3, 5)$ . It  
9 is a local quasihomogeneous nilpotent approximation [1, 5, 12] to an arbitrary SR  
10 structure on a 5-dimensional manifold with the growth vector  $(2, 3, 5)$ . Examples  
11 of such structures include models for:

- 12 • a pair of bodies rolling one on another without slipping or twisting [3, 27],  
13 in particular, the sphere rolling on a plane [22],
- 14 • a car with 2 off-hooked trailers [24, 40],
- 15 • electric charge moving in a magnetic field [6].

16 Such a nilpotent SR structure is unique, up to homomorphism of the Lie group  $G$ .  
17 Generalized Dido's problem (1.4)–(1.6) is a model of the nilpotent SR problem  
18 with the growth vector  $(2, 3, 5)$ , see other models in [6, 33].

19 The paper continues the study of this problem started in works [32–36]. The  
20 main result of these works is an upper bound of the cut time (i.e., the time of  
21 loss of *global* optimality) along extremal trajectories of the problem. The aim  
22 of this paper is to investigate the first conjugate time (i.e., the time of loss of  
23 *local* optimality) along the trajectories. We show that the function that gives  
24 the upper bound of the cut time provides the lower bound of the first conjugate  
25 time. In order to state this main result exactly, we recall necessary facts from  
26 the previous works [32–36].

## 1.2 Previously obtained results

Problem (1.4)–(1.6) was considered first by R. Brockett and L. Dai [18]: they proved integrability of geodesics in this problem in Jacobi’s functions.

The following results were obtained in [32–36], if not stated otherwise.

Existence of optimal solutions of problem (1.4)–(1.6) is implied by the Rashevsky-Chow and Filippov theorems [4].

### 1.2.1 Pontryagin maximum principle

By the Cauchy-Schwartz inequality, the sub-Riemannian length minimization problem (1.6) is equivalent to the energy minimization problem:

$$\int_0^{t_1} \frac{u_1^2 + u_2^2}{2} dt \rightarrow \min. \quad (1.7)$$

The Pontryagin maximum principle [4, 30] was applied to the resulting optimal control problem (1.4), (1.5), (1.7).

Abnormal extremal trajectories are one-parameter subgroups  $g_t$  tangent to the distribution  $\Delta$ :

$$\dot{g} = u_1 X_1 + u_2 X_2, \quad u_1, u_2 \equiv \text{const}. \quad (1.8)$$

They project to straight lines in the plane  $(x, y)$ , thus are optimal.

Normal extremals satisfy the Hamiltonian system

$$\dot{\lambda} = \vec{H}(\lambda), \quad \lambda \in T^*G, \quad (1.9)$$

where  $H = \frac{1}{2}(h_1^2 + h_2^2)$ ,  $h_i(\lambda) = \langle \lambda, X_i \rangle$ . In coordinates  $(h_1, \dots, h_5; g)$  on  $T^*G$ , this system reads as

$$\dot{h}_1 = -h_2 h_3, \quad (1.10)$$

$$\dot{h}_2 = h_1 h_3, \quad (1.11)$$

$$\dot{h}_3 = h_1 h_4 + h_2 h_5, \quad (1.12)$$

$$\dot{h}_4 = 0, \quad (1.13)$$

$$\dot{h}_5 = 0, \quad (1.14)$$

$$\dot{g} = h_1 X_1 + h_2 X_2. \quad (1.15)$$

Normal extremal controls are  $u_1 = h_1$ ,  $u_2 = h_2$ .

Since the vertical part (1.10)–(1.14) of this system is homogeneous, we can restrict to the level surface  $\{H = 1/2\}$ , which corresponds to length parameterization of extremal trajectories.

Abnormal extremal trajectories (1.8) are simultaneously normal.

### 1.2.2 Integrability

There are 3 independent Casimir functions [1] on the Lie coalgebra  $\mathfrak{g}^*$ : the Hamiltonians  $h_4$ ,  $h_5$ , and

$$E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4.$$

1 Symplectic leaves of maximal dimension are two-dimensional:

$$\text{connected components of } \{p \in \mathfrak{g}^* \mid h_4, h_5, E \equiv \text{const}\}, \quad (1.16)$$

2 thus the vertical subsystem (1.10)–(1.14) is Liouville integrable. The symplectic  
3 foliation on  $\mathfrak{g}^*$  consists of:

- 4 • two-dimensional parabolic cylinders (1.16) in the domain  $\{h_4^2 + h_5^2 \neq 0\}$ ,
- 5 • two-dimensional affine planes  $\{h_4 = h_5 = 0, h_3 = \text{const} \neq 0\}$ , and
- 6 • points  $(h_1, h_2) = \text{const}$  in the plane  $\{h_3^2 + h_4^2 + h_5^2 = 0\}$ .

7 **Remark 1.** For free nilpotent Lie coalgebras of rank  $\geq 3$  and step  $\geq 3$ , and  
8 of rank  $\geq 2$  and step  $\geq 4$ , typical symplectic leaves have dimension at least 4,  
9 and the corresponding Hamiltonian systems are not Liouville integrable [15, 25].

10 Introduce coordinates  $(\theta, c, \alpha, \beta)$  on the level surface  $\{\lambda \in T^*M \mid H = \frac{1}{2}\}$  by  
11 the following formulas:

$$h_1 = \cos \theta, \quad h_2 = \sin \theta, \quad h_3 = c, \quad h_4 = \alpha \sin \beta, \quad h_5 = -\alpha \cos \beta, \quad (1.17)$$

12 then  $E = \frac{c^2}{2} - \alpha \cos(\theta - \beta)$ .

The cylinder

$$C = \{\lambda \in \mathfrak{g}^* \mid H(\lambda) = 1/2\}$$

has the following stratification depending on its intersection with the symplectic leaves:

$$\begin{aligned} C &= \bigsqcup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j, \\ C_1 &= \{\lambda \in C \mid \alpha > 0, E \in (-\alpha, \alpha)\}, \\ C_2 &= \{\lambda \in C \mid \alpha > 0, E \in (\alpha, +\infty)\}, \\ C_3 &= \{\lambda \in C \mid \alpha > 0, E = \alpha, \theta - \beta \neq \pi\}, \\ C_4 &= \{\lambda \in C \mid \alpha > 0, E = -\alpha\}, \\ C_5 &= \{\lambda \in C \mid \alpha > 0, E = \alpha, \theta - \beta = \pi\}, \\ C_6 &= \{\lambda \in C \mid \alpha = 0, c \neq 0\}, \\ C_7 &= \{\lambda \in C \mid \alpha = c = 0\}. \end{aligned}$$

13 Intersections of symplectic leaves with the level surface of the Hamiltonian  
14  $\{H = 1/2\}$  are shown in Figs. 2–7.

15 It is obvious from Figs. 2–7 that extremal control  $u(t) = (u_1(t), u_2(t)) =$   
16  $(h_1(t), h_2(t))$  has the following nature:

- 17 •  $u(t) \equiv \text{const}$  in the cases  $\lambda \in C_4 \cup C_5 \cup C_7$ ,
- 18 •  $u(t)$  is periodic in the cases  $\lambda \in C_1 \cup C_2 \cup C_6$ ,
- 19 •  $u(t)$  is nonperiodic but asymptotically constant (has finite limits as  $t \rightarrow$   
20  $\pm\infty$ ) in the case  $\lambda \in C_3$ .

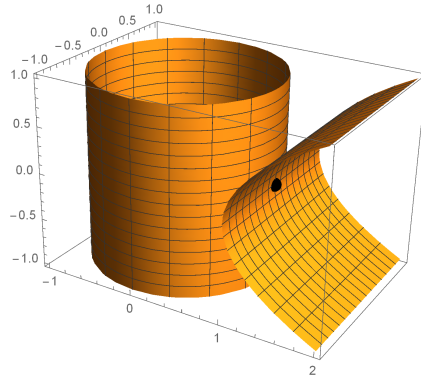


Figure 2:  $E = -\alpha < 0$ :  $\lambda \in C_4$

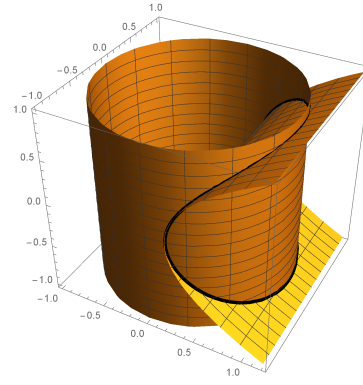


Figure 3:  $E \in (-\alpha, \alpha)$ ,  $\alpha > 0$ :  $\lambda \in C_1$

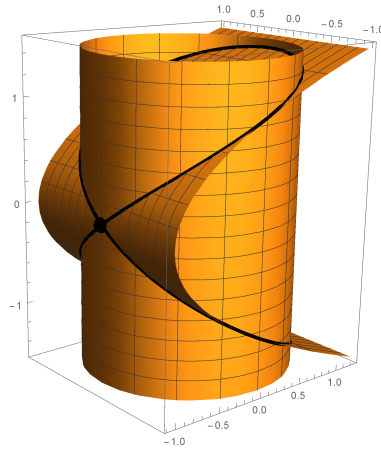


Figure 4:  $E = \alpha > 0$ :  $\lambda \in C_3 \cup C_5$

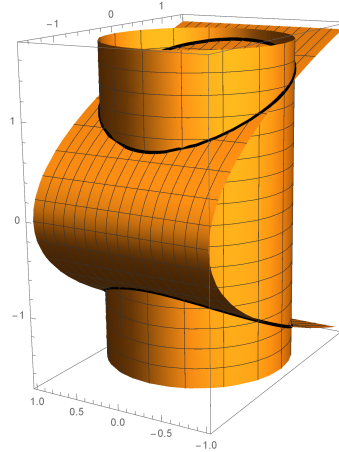


Figure 5:  $E > \alpha > 0$ :  $\lambda \in C_2$

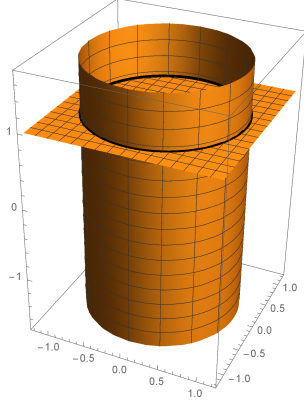


Figure 6:  $\alpha = 0, c \neq 0$ :  $\lambda \in C_6$

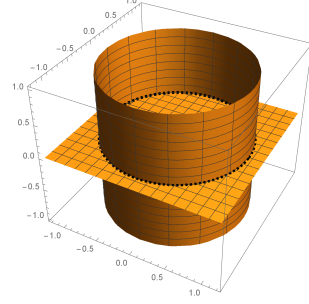


Figure 7:  $\alpha = c = 0$ :  $\lambda \in C_7$

### 1.2.3 Continuous symmetries

The Lie algebra of infinitesimal symmetries of the distribution  $\Delta$  was described by E. Cartan [20]: it is the 14-dimensional Lie algebra  $\mathfrak{g}_2$  — the unique non-compact real form of the complex exceptional simple Lie algebra  $\mathfrak{g}_2^{\mathbb{C}}$  [31]. See a modern exposition and explicit construction of these symmetries in [33].

The Lie algebra of infinitesimal symmetries of the SR structure  $(\Delta, \langle \cdot, \cdot \rangle)$  is 6-dimensional: it contains 5 basis right-invariant vector fields on  $G$  plus a vector field

$$X_0 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - w \frac{\partial}{\partial v} + v \frac{\partial}{\partial w},$$

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1,$$

its flow is a simultaneous rotation in the planes  $(x, y)$  and  $(v, w)$ :

$$e^{sX_0}(x, y, z, v, w) = (x \cos s - y \sin s, x \sin s + y \cos s, z, v \cos s - w \sin s, v \sin s + w \cos s).$$

There exists also a vector field  $Y \in \text{Vec } G$ , an infinitesimal symmetry of  $\Delta$ , such that

$$[Y, X_1] = -X_1, \quad [Y, X_2] = -X_2, \quad [Y, X_0] = 0,$$

this is the vector field

$$Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z} + 3v \frac{\partial}{\partial v} + 3w \frac{\partial}{\partial w},$$

its flow is given by dilations:

$$e^{rY}(x, y, z, v, w) = (e^r x, e^r y, e^{2r} z, e^{3r} v, e^{3r} w).$$

Introduce the Hamiltonians  $h_0(\lambda) = \langle \lambda, X_0(g) \rangle$ ,  $h_Y(\lambda) = \langle \lambda, Y(g) \rangle$ ,  $\lambda \in T^*G$ , the corresponding Hamiltonian vector fields  $\vec{h}_0, \vec{h}_Y \in \text{Vec}(T^*G)$ , and the



vector field  $Z = \vec{h}_Y + \sum_{i=1}^5 h_i \frac{\partial}{\partial h_i} \in \text{Vec}(T^*G)$ . Then the rotation  $\vec{h}_0$  is a symmetry of the normal Hamiltonian vector field:

$$\begin{aligned} [\vec{h}_0, \vec{H}] &= 0, & \vec{h}_0 H &= 0, \\ e^{s\vec{h}_0} \circ e^{t\vec{H}}(\lambda) &= e^{t\vec{H}} \circ e^{s\vec{h}_0}(\lambda), & \lambda \in T^*G, \quad s, t \in \mathbb{R}, \end{aligned}$$

and the dilation  $Z$  is a generalized symmetry of  $\vec{H}$ :

$$\begin{aligned} [Z, \vec{H}] &= -\vec{H}, & ZH &= 0, \\ e^{rZ} \circ e^{t\vec{H}}(\lambda) &= e^{t'\vec{H}} \circ e^{rZ}(\lambda), & t' &= te^r, \quad \lambda \in T^*G, \quad r, t \in \mathbb{R}. \end{aligned}$$

Thus

$$e^{sX_0} \circ \text{Exp}(\lambda, t) = \text{Exp}(e^{s\vec{h}_0}(\lambda), t), \quad (1.18)$$

$$e^{rY} \circ \text{Exp}(\lambda, t) = \text{Exp}(e^{rZ}(\lambda), t'), \quad t' = te^r. \quad (1.19)$$

#### 1.2.4 Pendulum and elasticæ

On the level surface  $\{\lambda \in T^*G \mid H(\lambda) = 1/2\}$  the vertical subsystem (1.10)–(1.14) of the normal Hamiltonian system takes in coordinates (1.17) the form of a *generalized pendulum*:

$$\ddot{\theta} = -\alpha \sin(\theta - \beta), \quad \alpha, \beta = \text{const}. \quad (1.20)$$

See the phase portrait of pendulum (1.20) at Fig. 8 and Fig. 9 for the cases  $\alpha > 0$  and  $\alpha = 0$  respectively.

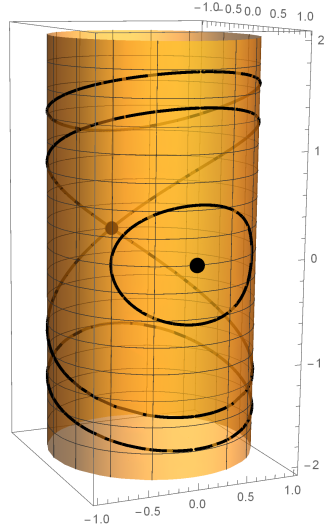


Figure 8: Phase portrait of pendulum (1.20),  $\alpha > 0$

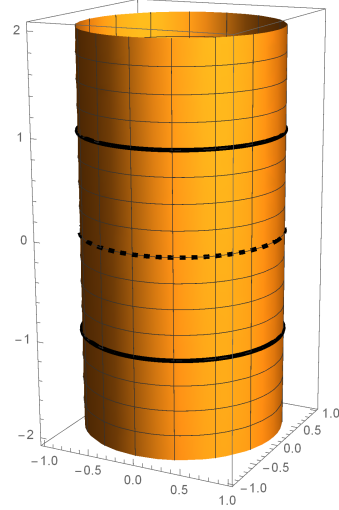


Figure 9: Phase portrait of pendulum (1.20),  $\alpha = 0$

Projections of extremal trajectories to the plane  $(x, y)$  satisfy the ODEs

$$\dot{x} = \cos \theta, \quad \dot{y} = \sin \theta.$$

1 Thus they are Euler elasticae [21,26] — stationary configurations of elastic rod  
 2 in the plane. Elasticae have different shapes depending on different types of  
 3 motion of pendulum (1.20):

- 4 • for pendulum at the stable equilibrium with the minimal energy  $E =$   
 5  $-\alpha < 0$  ( $\lambda \in C_4$ ), elasticae are straight lines,
- 6 • for oscillating pendulum with low energy  $E \in (-\alpha, \alpha)$ ,  $\alpha > 0$  ( $\lambda \in C_1$ ),  
 7 elasticae are periodic curves with inflection points,
- 8 • for pendulum with critical energy  $E = \alpha > 0$  ( $\lambda \in C_3 \cup C_5$ ), elasticae are  
 9 straight lines or critical non-periodic curves,
- 10 • for rotating pendulum with high energy  $E > \alpha > 0$  ( $\lambda \in C_2$ ), elasticae are  
 11 periodic curves without inflection points,
- 12 • for pendulum uniformly rotating without gravity ( $\alpha = 0$ ,  $c \neq 0$ ,  $\lambda \in C_6$ ),  
 13 elasticae are circles,
- 14 • and for stationary pendulum without gravity ( $\alpha = c = 0$ ,  $\lambda \in C_7$ ) elasticae  
 15 are straight lines.

16 See the plots of Euler elasticae in [21, 23, 32]. The pendulum is Kirchhoff's  
 17 kinetic analog of elasticae [23].

### 18 1.2.5 Exponential mapping

19 The family of all normal extremals is parametrized by points of the phase cylin-  
 20 der of the pendulum

$$C = \left\{ \lambda \in \mathfrak{g}^* \mid H(\lambda) = \frac{1}{2} \right\} = \{ (\theta, c, \alpha, \beta) \mid \theta \in S^1, c \in \mathbb{R}, \alpha \geq 0, \beta \in S^1 \},$$

21 and is given by the *exponential mapping*

$$\begin{aligned} \text{Exp} : C \times \mathbb{R}_+ &\rightarrow G, \\ \text{Exp}(\lambda, t) &= g_t = (x_t, y_t, z_t, v_t, w_t). \end{aligned}$$

### 22 1.2.6 Discrete symmetries of the exponential mapping

23 The quotient of the generalized pendulum (1.20) modulo rotation  $X_0$  and dila-  
 24 tion  $Y$  is the standard pendulum

$$\dot{\theta} = c, \quad \dot{c} = -\sin \theta, \quad (\theta, c) \in S^1 \times \mathbb{R}. \quad (1.21)$$

The field of directions of this equation has obvious discrete symmetries — re-  
 flections in the coordinate axes and in the origin

$$\begin{aligned} \varepsilon^1 : (\theta, c) &\mapsto (\theta, -c), \\ \varepsilon^2 : (\theta, c) &\mapsto (-\theta, c), \\ \varepsilon^3 : (\theta, c) &\mapsto (-\theta, -c). \end{aligned}$$

These reflections generate a dihedral group

$$D_2 = \{\text{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\} = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

1 The action of reflections extends naturally to Euler elasticae  $(x_t, y_t)$ , so that  
 2 modulo rotations in the plane  $(x, y)$ :

- 3 •  $\varepsilon^1$  is the reflection of an elastica in the center of its chord,
- 4 •  $\varepsilon^2$  is the reflection of an elastica in the middle perpendicular to its chord,
- 5 •  $\varepsilon^3$  is the reflection of an elastica in its chord.

Further, the action of reflections naturally extends to the preimage of the exponential mapping:

$$\varepsilon^i : C \times \mathbb{R}_+ \rightarrow C \times \mathbb{R}_+, \quad i = 1, 2, 3,$$

and to its image:

$$\delta^i : G \rightarrow G, \quad i = 1, 2, 3,$$

6 so that

$$\delta^i \circ \text{Exp}(\lambda, t) = \text{Exp} \circ \varepsilon^i(\lambda, t), \quad (\lambda, t) \in C \times \mathbb{R}_+, \quad i = 1, 2, 3, \quad (1.22)$$

7 and  $\varepsilon^i$  preserves time:  $\varepsilon^i(\lambda, t) = (*, t)$ . In such a case we say that the pair of  
 8 mappings  $(\varepsilon^i, \delta^i)$  is a *symmetry of the exponential mapping*.

Combining reflections and rotations, we obtain a group  $\text{Sym}$  of symmetries of the exponential mapping:

$$e^{s\vec{h}_0}, \quad e^{s\vec{h}_0} \circ \varepsilon^i : C \times \mathbb{R}_+ \rightarrow C \times \mathbb{R}_+, \quad (1.23)$$

$$e^{sX_0}, \quad e^{sX_0} \circ \delta^i : G \rightarrow G. \quad (1.24)$$

9 Notice that

$$\varepsilon^3(\theta, c, \alpha, \beta, t) = (-\theta, -c, \alpha, -\beta, t), \quad \lambda = (\theta, c, \alpha, \beta) \in C, \quad t \in \mathbb{R}_+, \quad (1.25)$$

10 i.e., the action of  $\varepsilon^3$  does not depend on  $t$ .

### 11 1.2.7 Integration of the normal Hamiltonian system

12 The equation of pendulum (1.20) is integrable in Jacobi's functions, thus the  
 13 normal Hamiltonian system (1.9) is integrable in Jacobi's functions as well.

14 In order to parametrize extremal trajectories explicitly, in work [32] were  
 15 introduced elliptic coordinates  $(\varphi, k, \alpha, \beta)$  on the sets  $C_1, C_2, C_3$  in the following  
 16 way:

- 17 •  $\varphi \in \mathbb{R}$  is the time of motion of pendulum (1.20) from a chosen initial curve  
 18 to a current point, and

- $k$  is a reparametrized energy  $E$ :

$$\lambda \in C_1 \quad \Rightarrow \quad k = \sqrt{\frac{E + \alpha}{2\alpha}} \in (0, 1),$$

$$\lambda \in C_2 \quad \Rightarrow \quad k = \sqrt{\frac{2\alpha}{E + \alpha}} \in (0, 1),$$

$$\lambda \in C_3 \quad \Rightarrow \quad k = 1.$$

In the elliptic coordinates  $(\varphi, k, \alpha, \beta)$  on  $\cup_{i=1}^3 C_i$  the vertical part of the Hamiltonian system (1.20) rectifies:

$$\dot{\varphi} = 1, \quad \dot{k} = \dot{\alpha} = \dot{\beta} = 0.$$

1 In [32] these coordinates were used for explicit parametrization of extremal  
2 trajectories in terms of Jacobi functions  $\text{sn}(u, k)$ ,  $\text{cn}(u, k)$ ,  $\text{dn}(u, k)$ ,  $E(u, k) =$   
3  $\int_0^u \text{dn}^2(t, k) dt$  [42].

#### 4 1.2.8 Optimality of normal extremal trajectories

Short arcs of normal extremal trajectories are (globally) optimal, thus they are SR geodesics. But long arcs of geodesics are, in general, not optimal. The instant at which a geodesic loses its optimality is called the *cut time*:

$$t_{\text{cut}}(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is optimal for } s \in [0, t]\}.$$

A geodesic is called *locally optimal* if it is optimal w.r.t. all trajectories with the same endpoints, in some neighborhood in the topology of  $C([0, t_1], G)$  (or, which is equivalent, in the topology of  $G$ ). The instant when a geodesic loses its local optimality is called the *first conjugate time*:

$$t_{\text{conj}}^1(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is locally optimal for } s \in [0, t]\}.$$

5 There are 3 reasons for the loss of global optimality for SR geodesics [1]:

- 6 1. intersection points of different geodesics of equal length (such points are  
7 called *Maxwell points*),
- 8 2. conjugate points,
- 9 3. abnormal trajectories.

As we mentioned above, all abnormal trajectories in the SR problem on the Cartan group are optimal. Maxwell points in this problem were studied in detail in [34–36]. A typical reason for Maxwell points is a symmetry of the exponential mapping. Along each geodesic  $g_t = \text{Exp}(\lambda, t)$ ,  $\lambda \in C$ , was found the first Maxwell time corresponding to the group of symmetries  $\text{Sym}$  (1.23), (1.24). The *first Maxwell time*

$$t_{\text{MAX}}^1 : C \rightarrow (0, +\infty]$$

10 corresponding to the group  $\text{Sym}$  was described. Thus the following upper bound  
11 of the cut time was proved:

**Theorem 1** ([36], Theorem 6.1). *For any  $\lambda \in C$*

$$t_{\text{cut}}(\lambda) \leq t_{\text{MAX}}^1(\lambda). \quad (1.26)$$

The first Maxwell time corresponding to the group of symmetries  $\text{Sym}$  is explicitly defined as follows:

$$\begin{aligned}\lambda \in C_1 &\Rightarrow t_{\text{MAX}}^1(\lambda) = \min \left( \frac{2}{\sqrt{\alpha}} p_1^z(k), \frac{2}{\sqrt{\alpha}} p_1^V(k) \right), \\ \lambda \in C_2 &\Rightarrow t_{\text{MAX}}^1(\lambda) = \frac{2k}{\sqrt{\alpha}} p_1^V(k), \\ \lambda \in C_6 &\Rightarrow t_{\text{MAX}}^1(\lambda) = \frac{4}{|c|} p_1^V(0), \\ \lambda \in C_i, i = 3, 4, 5, 7 &\Rightarrow t_{\text{MAX}}^1(\lambda) = +\infty.\end{aligned}\quad (1.27)$$

- <sup>1</sup> Here  $p_1^z = p_1^z(k) \in (K, 3K)$  is the first positive root of the equation  $f_z(p, k) = 0$ ,  
<sup>2</sup> where

$$f_z(p, k) = \text{sn } p \text{ dn } p - (2 \text{E}(p) - p) \text{cn } p, \quad (1.28)$$

$p_1^V = p_1^V(k)$  is the first positive root of the equation  $f_V(p, k) = 0$ , where

$$\begin{aligned}f_V(p) = \frac{4}{3} \text{sn } p \text{ dn } p (-p - 2(1 - 2k^2 + 6k^2 \text{cn } p^2)(2 \text{E}(p) - p) + (2 \text{E}(p) - p)^3 \\ + 8k^2 \text{cn } p \text{ sn } p \text{ dn } p) + 4 \text{cn } p (1 - 2k^2 \text{sn } p^2)(2 \text{E}(p) - p)^2, \\ p_1^V(k) \in [2K, 4K) \quad \text{for } \lambda \in C_1,\end{aligned}\quad (1.29)$$

and

$$\begin{aligned}f_V(p) = \frac{4}{3} \{ 3 \text{dn } p (2 \text{E}(p) - (2 - k^2)p)^2 + \text{cn } p [8 \text{E}^3(p) - 4 \text{E}(p)(4 + k^2) \\ - 12 \text{E}^2(p)(2 - k^2)p + 6 \text{E}(p)(2 - k^2)^2 p^2 \\ + p(16 - 4k^2 - 3k^4 - (2 - k^2)^3 p^2)] \text{sn } p - \\ - 2 \text{dn } p (-4k^2 + 3(2 \text{E}(p) - (2 - k^2)p)^2) \text{sn } p^2 + \\ + 12k^2 \text{cn } p (2 \text{E}(p) - (2 - k^2)p) \text{sn } p^3 - 8k^2 \text{sn } p^4 \text{dn } p \}, \\ p_1^V(k) \in (K, 2K) \quad \text{for } \lambda \in C_2,\end{aligned}\quad (1.30)$$

and  $p_1^V(0) \in (\pi/2, \pi)$  is the first positive root of the function

$$f_V^0(p) = [(32p^2 - 1) \cos 2p - 8p \sin 2p + \cos 6p]/512.$$

- <sup>3</sup> Here  $K = K(k)$  is the complete elliptic integral of the first kind.

### <sup>4</sup> 1.3 Result of this paper

- <sup>5</sup> In this article we study local optimality of SR geodesics on the Cartan group  
<sup>6</sup> and estimate the first conjugate time. The main result is the following bound.

- <sup>7</sup> **Theorem 2.** *For any  $\lambda \in C$*

$$t_{\text{conj}}^1(\lambda) \geq t_{\text{MAX}}^1(\lambda). \quad (1.31)$$

- <sup>8</sup> In Sections 4–6 we prove inequality (1.31),  $\lambda \in C_i$  for all  $i = 1, \dots, 7$ .  
<sup>9</sup> Theorem 2 follows immediately from Theorems 4, 5, 7, 8, 9.

1 Theorem 2 is interesting from two points of view. First, it describes locally  
 2 optimal geodesics in a sharp way; we will see in Sec. 7 that for certain geodesics  
 3 inequality (1.31) turns into equality. Second, Th. 2 is an important step in the  
 4 study of global optimality in the SR problem on the Cartan group. Namely, we  
 5 conjectured in [36] that inequality (1.26) is in fact equality:

**Conjecture 1.** *For any  $\lambda \in C$*

$$t_{\text{cut}}(\lambda) = t_{\text{MAX}}^1(\lambda).$$

6 In a forthcoming paper [7], this conjecture is proved with the use of Th. 2  
 7 via a symmetry method [1, 10, 19, 38].

## 8 1.4 Methods of this paper

Consider the Jacobian of the exponential map

$$J_0 = \frac{\partial(x, y, z, v, w)}{\partial(\theta, c, \alpha, \beta, t)} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \cdots & \frac{\partial x}{\partial t} \\ \vdots & \ddots & \vdots \\ \frac{\partial w}{\partial \theta} & \cdots & \frac{\partial w}{\partial t} \end{vmatrix}.$$

9 A point  $g_t = \text{Exp}(\lambda, t)$  on a strictly normal geodesic is *conjugate* if and only  
 10 if  $(\lambda, t)$  is a critical point of the exponential mapping, that is why  $g_t$  is the  
 11 corresponding critical value:  $J_0(\lambda, t) = 0$ . So inequality (1.31) is equivalent to  
 12 the following one:

$$J_0(\lambda, t) \neq 0 \text{ for } t \in (0, t_{\text{MAX}}^1(\lambda)). \quad (1.32)$$

13 This inequality is proved in Secs. 4–6 for  $\lambda \in C = \cup_{i=1}^7 C_i$  as follows:

- 14 1. First generic cases  $\lambda \in C_1$  and  $\lambda \in C_2$  are considered directly and inde-  
 15 pendently.
- 16 2. The Jacobian of the exponential mapping is computed w.r.t. the coordi-  
 17 nates  $(\varphi, k, \alpha, \beta)$ .
- 18 3. On the basis of continuous symmetries (rotations and dilations), the  $5 \times 5$   
 19 Jacobian  $\frac{\partial(x, y, z, v, w)}{\partial(\varphi, k, \alpha, \beta, t)}$  is reduced to a  $3 \times 3$  Jacobian  $\frac{\partial(P, Q, R)}{\partial(\varphi, k, t)}$ , where  
 20  $P, Q, R$  are invariants of the continuous symmetries.
- 21 4. The Jacobian  $\frac{\partial(P, Q, R)}{\partial(\varphi, k, t)}$  is computed explicitly via parametrization of  
 22 the exponential mapping, and is simplified to a function  $J_1$ .
- 23 5. We prove the inequality

$$J_1 \neq 0 \text{ for } t \in (0, t_{\text{MAX}}^1(\lambda))$$

basically by three methods which we explain below:

- 24 (a) homotopy invariance of the index of the second variation,
- 25 (b) method of comparison function,
- 26 (c) “*divide et impera*” method.

- 1      6. In such a way, inequality (1.32) is proved for the generic cases  $\lambda \in C_1$  and  
2       $\lambda \in C_2$  (Secs. 4 and 5 respectively).
- 3      7. For  $\lambda \in C_4 \cup C_5 \cup C_7$ , the geodesics are globally optimal since they project  
4      to straight lines in the plane of the distribution, and the required bound  
5      follows trivially.
- 6      8. For  $\lambda \in C_3$ , the geodesics are globally optimal since they project to length  
7      minimizers on the Engel group [10], and the required bound follows.
- 8      9. Bound (1.32) is proved for the special case  $\lambda \in C_6$  by a limit passage from  
9      the generic case  $\lambda \in C_2$  via the homotopy invariance of the index of the  
10     second variation.
- 11    10. We conclude that bound (1.32) is proved for all  $\lambda \in C$ .

12    **Homotopy invariance of index of second variation.** Under certain non-  
13    degeneracy conditions (see Sec. 3), for each strictly normal geodesic one can  
14    define the index of the second variation equal to the number of conjugate points  
15    with account of their multiplicity. A geodesic does not contain conjugate points  
16    if and only if its index vanishes. The index of the second variation is preserved  
17    under homotopies of geodesics such that their endpoints are not conjugate. So  
18    we can prove absence of conjugate points on a geodesic if we construct its homo-  
19    topy to a “simple” geodesic without conjugate points, in such a way that end-  
20    points of all geodesics in the continuous family are not conjugate. In Sec. 5 we  
21    construct such a homotopy from a geodesic  $g_t = \text{Exp}(\lambda, t)$ ,  $\lambda = (\varphi, k, \alpha, \beta) \in C_2$ ,  
22    to a geodesic  $\bar{g}_t = \text{Exp}(\bar{\lambda}, t)$ ,  $\bar{\lambda} = (\bar{\varphi}, \bar{k}, \alpha, \beta) \in C_2$ , where  $\bar{k}$  is close to 0. The  
23    geodesic  $\bar{g}_t$  is “simple” because for  $\bar{k} \rightarrow 0$  the exponential mapping is asymp-  
24    totically expressed by trigonometric functions, not Jacobi ones as for general  
25     $k \in (0, 1)$ . See details on the homotopy invariance of the index of the second  
26    variation in Sec 3.

**Comparison functions.** Let  $f_0(t)$ ,  $f_1(t)$  be real-analytic functions on an in-  
terval  $t \in (a, b)$ . The function  $f_1(t)$  is called a *comparison function* for  $f_0(t)$   
if

$$\begin{aligned} & \left( \frac{f_0(t)}{f_1(t)} \right)' \cdot f_1^2(t) \geq 0 \quad (\text{or } \leq 0), \\ & f_1(t) \neq 0, \quad t \in (a, b), \end{aligned} \tag{1.33}$$

27    and equality in (1.33) is possible only for isolated values of  $t$ . Then  $\frac{f_0(t)}{f_1(t)}$   
28    increases (or decreases) for  $t \in (a, b)$ .

29    In such a way one can often bound  $\frac{f_0(t)}{f_1(t)}$ , and after that bound  $f_0(t)$  in the  
30    case when it is not monotone.

31    **“Divide et impera”.** For a given function  $f_0(t)$ , we find a sequence of func-  
32    tions  $f_1(t)$ ,  $f_2(t)$ ,  $\dots$ ,  $f_N(t)$  such that  $f_i$  is more simple than  $f_{i-1}$ ; in order  
33    to construct  $f_i$ , we find a comparison function  $\tilde{f}_{i-1}$  for  $f_{i-1}$ . In such a way  
34    we divide the analytical complexity of the function  $f_0(t)$  into several parts and

bound the obtained parts step by step. Moreover, at each step we divide the function under study by an appropriate divisor.

The method works perfectly for trigonometric quasipolynomials as follows. Consider a trigonometric quasipolynomial

$$f_0(t) = t^N \tilde{f}_0(t) + o(t^N), \quad t \rightarrow \infty,$$

where  $\tilde{f}_0(t)$  is a trigonometric function. Compute successfully:

$$\begin{aligned} f_1(t) &= \left( \frac{f_0}{\tilde{f}_0} \right)' \cdot \tilde{f}_0^2 = t^{N-1} \tilde{f}_1(t) + o(t^{N-1}), \quad t \rightarrow \infty, \\ &\dots, \\ f_{N-1}(t) &= \left( \frac{f_{N-2}}{\tilde{f}_{N-2}} \right)' \cdot \tilde{f}_{N-2}^2 = t \tilde{f}_{N-1}(t) + o(t), \quad t \rightarrow \infty, \\ f_N(t) &= \left( \frac{f_{N-1}}{\tilde{f}_{N-1}} \right)' \cdot \tilde{f}_{N-1}^2, \end{aligned}$$

and  $f_N(t)$  is a trigonometric function.

We determine the sign of  $f_N(t)$  on an interval  $t \in (a, b)$  under investigation, and on this basis determine monotonicity of  $\frac{f_{N-1}}{\tilde{f}_{N-1}}$ . Then we determine the signs of  $\tilde{f}_{N-1}$  and  $f_{N-1}$  and so on. In such a way we obtain successfully bounds for  $f_{N-1}, f_{N-2}, \dots, f_0$ .

**Projection to lower-dimensional SR minimizers.** There is a general construction of projecting left-invariant SR structures to quotient groups such that optimality of projected geodesics in the quotient group implies optimality of the initial geodesic.

Let  $(\Delta, \langle \cdot, \cdot \rangle)$  be a left-invariant SR structure on a Lie group  $G$ . Let  $G_0 \subset G$  be a closed normal subgroup whose Lie algebra intersects trivially with  $\Delta$ . Consider the quotient  $\pi : G \rightarrow H = G/G_0$ . Then  $\pi_*\Delta \subset TH$  is a left-invariant distribution on  $H$ , and  $\pi_{*g} : \Delta_g \rightarrow (\pi_*\Delta)_{\pi(g)}$  is an isomorphism. Denote by  $\pi_*\langle \cdot, \cdot \rangle$  the inner product in  $\pi_*\Delta$  induced by  $\pi_*$ . Then  $(\pi_*\Delta, \pi_*\langle \cdot, \cdot \rangle)$  is a left-invariant SR structure on  $H$ , of the same rank as  $(\Delta, \langle \cdot, \cdot \rangle)$ . If  $g_t \in G$  is a horizontal curve for  $\Delta$ , then  $\pi(g_t)$  is a horizontal curve for  $\pi_*\Delta$ . Moreover, if  $\pi(g_t)$  is optimal for  $(\pi_*\Delta, \pi_*\langle \cdot, \cdot \rangle)$ , then  $g_t$  is optimal for  $(\Delta, \langle \cdot, \cdot \rangle)$ .

## 1.5 Structure of this paper

In Sec. 2 we establish invariance of the cut time and the first conjugate time w.r.t. symmetries of the problem (rotation  $X_0$ , dilation  $Y$ , and reflection  $\varepsilon^3$ ).

In Sec. 3 we recall the results on homotopy invariance of the index of the second variation in the form we require in Secs. 5, 6.

In Secs. 4 and 5 we prove Th. 2 for the generic cases  $\lambda \in C_1$  and  $\lambda \in C_2$  respectively. And in Sec. 6 we prove Th. 2 for the special cases  $\lambda \in \cup_{i=3}^7 C_i$ .

In Sec. 7 we present cases when the first conjugate time coincides with the first Maxwell time. In Sec. 8 we show that the first conjugate time is continuous near infinite values, out of certain abnormal geodesics.



1 Finally, in Sec. 9 we present a numerical evidence of two-sided bounds of  
 2 the first conjugate time.

### 3 1.6 Related works

4 As we already mentioned, this work is an essential step towards construction  
 5 of optimal synthesis and description of the cut time and cut locus for the left-  
 6 invariant SR structure on the Cartan group. So far this goal was achieved just  
 7 for few left-invariant SR structures on Lie groups.

8 In the case  $\dim G = 3$ , growth vector  $(2, 3)$ , the following left-invariant SR  
 9 structures were completely studied:

- 10 • Heisenberg group: A.M. Vershik, V.Y. Gershkovich [41], R. Brockett [17],
- 11 • axisymmetric metrics on  $SO(3)$ ,  $SL(2)$ : U. Boscain and F. Rossi [16],
- 12 V.N. Berestovskii and I.A. Zubareva [13, 14],
- 13 •  $SE(2)$ ,  $SH(2)$ : Ya. Butt, A. Bhatti and the author [19, 38].

14 The free nilpotent SR structure with the growth vector  $(3, 6)$  was studied by  
 15 O. Myasnychenko [29].

16 The SR problem on the Engel group (growth vector  $(2, 3, 4)$ ) was studied by  
 17 A.A. Ardentov and the author [11].

## 18 2 Symmetries of cut time and conjugate time

19 The exponential mapping is preserved by 3 symmetries: rotations  $e^{s\vec{h}_0}$ , dila-  
 20 tions  $e^{sZ}$ , reflection  $\varepsilon^3$ . Thus the cut time and the conjugate time are also  
 21 preserved by these symmetries, see Cor. 1 below. This is proved via the follow-  
 22 ing statement.

**Lemma 1.** *Let there exist homeomorphisms  $F : C \rightarrow C$ ,  $f : G \rightarrow G$  and a  
 number  $a > 0$  such that*

$$f \circ \text{Exp}(\lambda, t) = \text{Exp}(F(\lambda), at), \quad (\lambda, t) \in C \times \mathbb{R}_+.$$

23 *Then  $t_{\text{cut}}(F(\lambda)) = a t_{\text{cut}}(\lambda)$  and  $t_{\text{conj}}^1(F(\lambda)) = a t_{\text{conj}}^1(\lambda)$  for all  $\lambda \in C$ .*

24 *Proof.* Let a geodesic  $g_t = \text{Exp}(\lambda, t)$ ,  $t \in [0, t_1]$ , be optimal in a neighborhood  
 25  $O \subset G$ . We prove that the geodesic  $\tilde{g}_t = \text{Exp}(F(\lambda), at) = f(g_t)$ ,  $t \in [0, t_1]$ , is  
 26 optimal in the neighborhood  $f(O)$ .

By contradiction, let there exist a geodesic better than  $\tilde{g}_t$ ,  $t \in [0, t_1]$ :

$$\begin{aligned} \{\hat{g}_t = \text{Exp}(\hat{\lambda}, t) \mid t \in [0, \hat{t}]\} &\subset f(O), \\ \hat{g}_{\hat{t}} &= \tilde{g}_{t_1}, \quad \hat{t} < t_1. \end{aligned}$$

Consider then the geodesic

$$\{\tilde{g}_t = \text{Exp}(F^{-1}(\hat{\lambda}), t/a) = f^{-1}(\hat{g}_t) \mid t \in [0, \hat{t}]\} \subset O.$$

We have

$$\tilde{g}_{\hat{t}} = f^{-1}(\hat{g}_{\hat{t}}) = f^{-1}(\tilde{g}_{t_1}) = f^{-1} \circ f(g_{t_1}) = g_{t_1},$$

27 thus  $\tilde{g}_t$ ,  $t \in [0, \hat{t}]$ , is better than  $g_t$ ,  $t \in [0, t_1]$ , a contradiction.  $\square$

**Corollary 1.** *For any  $s \in \mathbb{R}$  and any  $\lambda \in C$  there hold the equalities*

$$\begin{aligned} t_{\text{cut}} \circ e^{s\vec{h}_0}(\lambda) &= e^{-s} t_{\text{cut}} \circ e^{sZ}(\lambda) = t_{\text{cut}} \circ \varepsilon^3(\lambda) = t_{\text{cut}}(\lambda), \\ t_{\text{conj}}^1 \circ e^{s\vec{h}_0}(\lambda) &= e^{-s} t_{\text{conj}}^1 \circ e^{sZ}(\lambda) = t_{\text{conj}}^1 \circ \varepsilon^3(\lambda) = t_{\text{conj}}^1(\lambda). \end{aligned}$$

*Proof.* Apply Lemma 1 in the cases:

$$\begin{aligned} F &= e^{s\vec{h}_0}, \quad f = e^{sX_0}, \quad a = 1, & \text{see (1.18),} \\ F &= e^{sZ}, \quad f = e^{sY}, \quad a = e^s, & \text{see (1.19),} \\ F &= \varepsilon^3, \quad f = \delta^3, \quad a = 1, & \text{see (1.22), (1.25).} \end{aligned}$$

1

□

## 2 3 Conjugate points and homotopy

3 In this section we recall some necessary facts from the theory of conjugate points  
4 in optimal control problems. We will need these facts in Secs. 5, 6. For details  
5 see, e.g., [2, 4, 39].

Consider an optimal control problem of the form

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \quad (3.1)$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ fixed}, \quad (3.2)$$

$$J = \int_0^{t_1} \varphi(q(t), u(t)) dt \rightarrow \min, \quad (3.3)$$

where  $M$  is a finite-dimensional analytic manifold,  $f(q, u)$  and  $\varphi(q, u)$  are respectively analytic in  $(q, u)$  families of vector fields and functions on  $M$  depending on the control parameter  $u \in U$ , and  $U$  an open subset of  $\mathbb{R}^m$ . Admissible controls are  $u(\cdot) \in L_\infty([0, t_1], U)$ , and admissible trajectories  $q(\cdot)$  are Lipschitzian. Let

$$h_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u), \quad \lambda \in T^*M, \quad q = \pi(\lambda) \in M, \quad u \in U,$$

6 be the normal Hamiltonian of PMP for problem (3.1)–(3.3).

7 Fix a triple  $(\tilde{u}(t), \lambda_t, q(t))$  consisting of a normal extremal control  $\tilde{u}(t)$ , the  
8 corresponding extremal  $\lambda_t$ , and a strictly normal extremal trajectory  $q(t)$  for  
9 problem (3.1)–(3.3).

10 Let the following hypotheses hold:

11 **(H1)** For all  $\lambda \in T^*M$  and  $u \in U$ , the quadratic form  $\frac{\partial^2 h_u}{\partial u^2}(\lambda)$  is negative  
12 definite.

**(H2)** For any  $\lambda \in T^*M$ , the function  $u \mapsto h_u(\lambda)$ ,  $u \in U$ , has a maximum point  $\bar{u}(\lambda) \in U$ :

$$h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \quad \lambda \in T^*M.$$

13

14 **(H3)** The extremal control  $\tilde{u}(\cdot)$  is a corank one critical point of the endpoint  
15 mapping.

<sup>1</sup> **(H4)** The Hamiltonian vector field  $\vec{H}(\lambda)$ ,  $\lambda \in T^*M$ , is forward complete, i.e.,  
<sup>2</sup> all its trajectories are defined for  $t \in [0, +\infty)$ .

<sup>3</sup> An instant  $t_* > 0$  is called a conjugate time (for the initial instant  $t = 0$ )  
<sup>4</sup> along the extremal  $\lambda_t$  if the restriction of the second variation of the endpoint  
<sup>5</sup> mapping to the kernel of its first variation is degenerate, see [4] for details. In  
<sup>6</sup> this case the point  $q(t_*) = \pi(\lambda_{t_*})$  is called conjugate for the initial point  $q_0$   
<sup>7</sup> along the extremal trajectory  $q(\cdot)$ .

<sup>8</sup> Under hypotheses **(H1)**–**(H4)**, we have the following:

- <sup>9</sup> 1. Normal extremal trajectories lose their local optimality (both strong and  
<sup>10</sup> weak) at the first conjugate point, see [4].
- <sup>11</sup> 2. An instant  $t > 0$  is a conjugate time if and only if the exponential mapping  
<sup>12</sup>  $\text{Exp}_t = \pi \circ e^{t\vec{H}}$  is degenerate, see [2].
- <sup>13</sup> 3. Along each normal extremal trajectory, conjugate times are isolated one  
<sup>14</sup> from another, see [39].

<sup>15</sup> We will apply the following statement for the proof of absence of conjugate  
<sup>16</sup> points via homotopy.

<sup>17</sup> **Theorem 3** (Corollary 2.2 [37]). *Let  $(u^s(t), \lambda_t^s)$ ,  $t \in [0, +\infty)$ ,  $s \in [0, 1]$ , be a*  
<sup>18</sup> *continuous in parameter  $s$  family of normal extremal pairs in optimal control*  
<sup>19</sup> *problem (3.1)–(3.3) satisfying hypotheses **(H1)**–**(H4)**. Let the corresponding*  
<sup>20</sup> *extremal trajectories  $q^s(t)$  be strictly normal.*

<sup>21</sup> *Let  $s \mapsto t_1^s$  be a continuous function,  $s \in [0, 1]$ ,  $t_1^s \in (0, +\infty)$ . Assume that*  
<sup>22</sup> *for any  $s \in [0, 1]$  the instant  $t = t_1^s$  is not a conjugate time along the extremal  $\lambda_t^s$ .*

<sup>23</sup> *If the extremal trajectory  $q^0(t) = \pi(\lambda_t^0)$ ,  $t \in (0, t_1^0]$ , does not contain conju-*  
<sup>24</sup> *gate points, then the extremal trajectory  $q^1(t) = \pi(\lambda_t^1)$ ,  $t \in (0, t_1^1]$ , also does not*  
<sup>25</sup> *contain conjugate points.*

<sup>26</sup> One easily checks that the sub-Riemannian problem (1.4), (1.5), (1.7) satis-  
<sup>27</sup> fies all hypotheses **(H1)**–**(H4)**, so the results cited in this section are applicable  
<sup>28</sup> to this problem.

## <sup>29</sup> 4 Conjugate time for $\lambda \in C_1$

<sup>30</sup> In this section we prove Th. 2 in the case  $\lambda \in C_1$ :

<sup>31</sup> **Theorem 4.** *If  $\lambda \in C_1$ , then  $t_{\text{conj}}^1(\lambda) \geq t_{\text{MAX}}^1(\lambda)$ .*

Consider the Jacobian of the exponential mapping

$$J = \frac{\partial(x, y, z, v, w)}{\partial(t, \varphi, k, \alpha, \beta)}.$$

<sup>32</sup> We show that  $J > 0$  for  $t \in (0, t_{\text{MAX}}^1(\lambda))$ ,  $\lambda \in C_1$ .

## 4.1 Transformation of Jacobian $J$

As was shown in [32], the exponential mapping in problem (1.1)–(1.3) has a two-parameter group of symmetries

$$S_{X_0, Y} = \{e^{sX_0} \circ e^{rY} \mid s \in S^1, r \in \mathbb{R}\} \subset \text{Diff}(G).$$

Consider in the domain

$$G_1 = \{g \in G \mid r > 0\}, \quad r = \sqrt{x^2 + y^2},$$

coordinates corresponding to the action the group  $S_{X_0, Y}$ :

$$P = \frac{z}{2r^2}, \quad Q = \frac{xv + yw}{r^4}, \quad R = \frac{-yv + xw}{r^4}, \quad r, \quad \chi = \arctan \frac{y}{x}. \quad (4.1)$$

We have

$$X_0 P = X_0 Q = X_0 R = X_0 r = 0, \quad X_0 \chi = 1, \quad (4.2)$$

$$Y P = Y Q = Y R = Y \chi = 0, \quad Y r = 1. \quad (4.3)$$

Differentiating formulas of coordinates (4.1) with account of (4.2), (4.3), we compute the Jacobian of transformation to these coordinates:

$$\frac{\partial(P, Q, R, r, \chi)}{\partial(x, y, z, v, w)} = \frac{1}{2r^9}.$$

Thus if  $g_t = \text{Exp}(\lambda, t) \in G_1$ ,  $(\bar{P}, \bar{Q}, \bar{R}, \bar{r}, \bar{\chi})(\lambda, t) = (P, Q, R, r, \chi)(g_t)$ , then

$$J = \frac{\partial(x, y, z, v, w)}{\partial(P, Q, R, r, \chi)} \cdot \frac{\partial(\bar{P}, \bar{Q}, \bar{R}, \bar{r}, \bar{\chi})}{\partial(t, \varphi, k, \alpha, \beta)} = 2r^9 \cdot \frac{\partial(\bar{P}, \bar{Q}, \bar{R}, \bar{r}, \bar{\chi})}{\partial(t, \varphi, k, \alpha, \beta)}. \quad (4.4)$$

In order to compute derivatives in (4.4) w.r.t.  $\alpha$  and  $\beta$ , notice that in the coordinates  $(t, \varphi, k, \alpha, \beta)$  on  $N = C \times \mathbb{R}_+$  we have

$$\vec{h}_0 = \frac{\partial}{\partial \beta}, \quad Z = \frac{\partial}{\partial t} - 2\alpha \frac{\partial}{\partial \alpha}, \quad (4.5)$$

see the remark at the end of Sec. 4.2 [34]. Further, for any function  $f : N \rightarrow \mathbb{R}$  of the form  $f = l \circ \text{Exp}$  we have

$$\begin{aligned} \vec{h}_0 f|_\nu &= \frac{d}{ds} \Big|_{s=0} f \circ e^{s\vec{h}_0}(\nu) = \frac{d}{ds} \Big|_{s=0} l \circ \text{Exp} \circ e^{s\vec{h}_0}(\nu) \\ &= \frac{d}{ds} \Big|_{s=0} l \circ e^{sX_0} \circ \text{Exp}(\nu) = X_0 l|_g, \quad g = \text{Exp}(\nu), \quad \nu \in N. \end{aligned}$$

We used here Eq. (1.18).

We get similarly a differentiation rule

$$Z f|_\nu = Y l|_g, \quad g = \text{Exp}(\nu), \quad \nu \in N.$$

In view of (4.2), (4.3), the functions  $\bar{P}$ ,  $\bar{Q}$ ,  $\bar{R}$  are invariants of the group of symmetries  $S_{X_0, Y}$ , whence

$$\vec{h}_0 \bar{P} = \vec{h}_0 \bar{Q} = \vec{h}_0 \bar{R} = Z \bar{P} = Z \bar{Q} = Z \bar{R} = 0.$$

Similarly, in view of the equalities  $X_0 r = 0$ ,  $X_0 \chi = 1$ ,  $Y r = 1$ ,  $Y \chi = 0$ , we get

$$\vec{h}_0 \bar{r} = 0, \quad \vec{h}_0 \bar{\chi} = 1, \quad Z \bar{r} = \bar{r}, \quad Z \bar{\chi} = 0.$$

We obtain from representations (4.5) that

$$\frac{\partial f}{\partial \alpha} = f_\alpha = -\frac{1}{2\alpha}(Zf - tf_t), \quad \frac{\partial f}{\partial \beta} = f_\beta = \vec{h}_0 f, \quad f \in C^\infty(N).$$

Now we are able to transform the Jacobian:

$$\begin{aligned} \frac{\partial(\bar{P}, \bar{Q}, \bar{R}, \bar{r}, \bar{\chi})}{\partial(t, \varphi, k, \alpha, \beta)} &= \begin{vmatrix} \bar{P}_t & \bar{P}_\varphi & \bar{P}_k & \bar{P}_\alpha & \bar{P}_\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\chi}_t & \bar{\chi}_\varphi & \bar{\chi}_k & \bar{\chi}_\alpha & \bar{\chi}_\beta \end{vmatrix} = -\frac{1}{2\alpha} \begin{vmatrix} \bar{P}_t & \bar{P}_\varphi & \bar{P}_k & Z\bar{P} & \vec{h}_0 \bar{P} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\chi}_t & \bar{\chi}_\varphi & \bar{\chi}_k & Z\bar{\chi} & \vec{h}_0 \bar{\chi} \end{vmatrix} \\ &= -\frac{1}{2\alpha} \begin{vmatrix} \bar{P}_t & \bar{P}_\varphi & \bar{P}_k & 0 & 0 \\ \bar{Q}_t & \bar{Q}_\varphi & \bar{Q}_k & 0 & 0 \\ \bar{R}_t & \bar{R}_\varphi & \bar{R}_k & 0 & 0 \\ * & * & * & \bar{r} & 0 \\ * & * & * & 0 & 1 \end{vmatrix} = -\frac{r}{2\alpha} \frac{\partial(\bar{P}, \bar{Q}, \bar{R})}{\partial(t, \varphi, k)}. \end{aligned}$$

1 In view of equality (4.4), we obtain

$$J = -\frac{r^{10}}{\alpha} \cdot \frac{\partial(\bar{P}, \bar{Q}, \bar{R})}{\partial(t, \varphi, k)} \quad (4.6)$$

2 in the case  $r > 0$ .

### 3 4.2 Computation of Jacobian $J$

4 Immediate computation on the basis of explicit parametrization of the expo-  
5 nential mapping (Subsec. 5.3 [32]) gives the following result:

$$\frac{\partial(\bar{P}, \bar{Q}, \bar{R})}{\partial(t, \varphi, k)} = \frac{16k}{3(1-k^2)} \cdot \frac{1}{r^{10} \Delta^2} \cdot J_1, \quad (4.7)$$

where

$$\begin{aligned} \Delta &= 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau, \\ p &= \sqrt{\alpha} \frac{t}{2}, \quad \tau = \sqrt{\alpha} \left( \varphi + \frac{t}{2} \right), \\ J_1 &= a_0 + \xi a_1 + \xi^2 a_2, \quad \xi = \operatorname{sn}^2 \tau, \end{aligned} \quad (4.8)$$

$$a_0 = f_V \cdot a_{01}, \quad a_2 = f_z \cdot a_{21}, \quad (4.9)$$

$$a_1 = -a_0 - a_2/k^2, \quad (4.10)$$

the functions  $f_z$  and  $f_V$  are defined in (1.28) and (1.29),

$$\begin{aligned} a_{01} = & (3 \operatorname{cn} p (-3E_2^2(p) + 4E_2(p)p - 8E_2(p)k^2p - p^2 + 2k^2(-4 + 3E_2^2(p) \\ & + 4E_2(p)(-1 + 2k^2)p + p^2) \operatorname{sn}^2(p) + 8k^4 \operatorname{sn}^4(p)) \\ & + 3 \operatorname{dn} p \operatorname{sn} p (-E_2^3(p) - 2p + E_2^2(p)(2 - 4k^2)p + 8k^2p(1 + (1 - 2k^2) \operatorname{sn}^2(p)) \\ & - E_2(p)(-2 + p^2 + 4k^2(-2 + 3 \operatorname{sn}^2 p))) / 4, \end{aligned} \quad (4.11)$$

$$\begin{aligned} a_{21} = & -(k^2(\operatorname{cn} p (-E_2^5(p) + E_2^4(p)(2 - 4k^2)p - E_2^2(p)(9 - 64k^2 + 64k^4)p \\ & + p^3 - E_2^3(p)(-8 + 16k^2 + p^2)) + \operatorname{dn} p (E_2^2(p)(-4 + 5E_2^2(p) + 48k^2) \\ & + 2E_2(p)(1 - 4E_2^2(p) + 8(-6 + E_2^2(p))k^2 + 64k^4)p \\ & + (2 + 3E_2^2(p))p^2) \operatorname{sn} p - 16k^2 \operatorname{cn} p (5E_2(p) - 2p + 4k^2p) \operatorname{sn}^2 p \\ & - 4k^2 \operatorname{dn} p (-12 + 10E_2^2(p) + 8E_2(p)(-1 + 2k^2)p + p^2) \operatorname{sn}^3 p \\ & + 16k^4 \operatorname{cn} p (5E_2(p) - 2p + 4k^2p) \operatorname{sn}^4 p - 48k^4 \operatorname{dn} p \operatorname{sn}^5 p), \end{aligned} \quad (4.12)$$

$$E_2(p, k) = 2E(p, k) - p.$$

1 Now from equalities (4.6), (4.7) we get a factorization

$$J = -\frac{16k}{3(1 - k^2)\alpha\Delta^2} \cdot J_1 \quad (4.13)$$

2 under the assumption  $r > 0$ .

3 **Remark 2.** The function  $t \mapsto r^2 = x^2 + y^2$  is real analytic and is not identically  
4 zero, thus its roots are isolated. The both sides of equality (4.13) are continuous  
5 in  $t$ , so this equality holds for all  $(\lambda, t) \in C_1 \times \mathbb{R}_+$ .

6 Summing up, we have the following statement.

7 **Lemma 2.** For all  $(\lambda, t) \in C_1 \times \mathbb{R}_+$  we have  $\operatorname{sgn} J = -\operatorname{sgn} J_1$ .

### 8 4.3 Estimate of Jacobian $J_1$

**Lemma 3.** Consider the function (4.8)

$$J_1 = a_0 + \xi \cdot a_1 + \xi^2 \cdot a_2, \quad \xi \in [0, 1],$$

9 where the coefficients  $a_0, a_1, a_2$  are given by equalities (4.9), (4.11), (4.12),  
10 (4.10), and  $k \in (0, 1)$ .

11 Then  $J_1 < 0$  for all  $p \in (0, p_1(k))$ , where the function  $p_1(k)$  is given by  
12 Eq. (1.27).

13 *Proof.* 1) Let us show that  $a_0(p) < 0$  for  $p \in (0, p_1(k))$ .

We have from (4.9) that  $a_0 = f_V \cdot a_{01}$ . Immediate computation shows that

$$\frac{\partial}{\partial p} \left( \frac{a_{01}}{f_z} \right) f_z^2 = \frac{3}{4} x_2, \quad (4.14)$$

$$x_2 = k^2(\operatorname{cn} p E_4(p)p - \alpha_0)^2 + (1 - k^2)(E_2(p)p + \beta_0)^2, \quad (4.15)$$

$$\begin{aligned} \alpha_0 = & (1 + \operatorname{sn}^2(p) - 2k^2 \operatorname{sn}^2 p) E_2^2(p) - 4(2k^2 - 1) \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p E_2(p) \\ & + 4(2k^2 - 1) \operatorname{sn}^2 p \operatorname{dn}^2 p, \end{aligned}$$

$$\beta_0 = (2k^2 \operatorname{sn}^2 p - 1) E_2^2(p) + 8k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p E_2(p) - 8k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p,$$

$$E_4(p) = \operatorname{cn} p E_2(p) - 2 \operatorname{sn} p \operatorname{dn} p, \quad E_2(p) = 2E(p) - p.$$

Equality (4.15) means that  $x_2 \geq 0$ . Moreover, the identity  $x_2 \equiv_p 0$  is possible only when  $\text{cn } pE_4(p)p - \alpha_0 \equiv_p 0$ ,  $pE_2(p) + \beta_0 \equiv_p 0$ , which is impossible in view of the expansions

$$\begin{aligned} \text{cn } pE_4(p)p - \alpha_0 &= -2p^2 + o(p^2), \\ pE_2(p) + \beta_0 &= -4/45k^2p^6 + o(p^6), \quad p \rightarrow 0. \end{aligned}$$

1 Thus  $x_2 \geq 0$ , moreover,  $x_2(p) = 0$  only at isolated points  $p$ .

Then equality (4.14) means that the function  $p \mapsto \frac{a_{01}(p)}{f_z(p)}$  strictly increases at each interval where  $f_z(p) \neq 0$ . Determine the sign of  $f_z(p)$  at the first such interval  $p \in (0, p_z^1)$ :

$$f_z(p) = \frac{p^3}{3} + o(p^3), \quad p \rightarrow 0 \quad \Rightarrow \quad f_z(p) > 0 \text{ for } p \in (0, p_z^1),$$

2 see the plots of  $f_z(p)$  at Figs. 3–4 [36].

Determine similarly the sign of  $a_{01}(p)$  for small  $p > 0$ :

$$a_{01}(p) = \frac{4}{1575}k^2(1 - k^2)p^{10} = o(p^{10}), \quad p \rightarrow 0,$$

3 thus  $a_{01}(p) > 0$  for  $p \rightarrow +0$ . Consequently,  $\frac{a_{01}}{f_z} > 0$  for  $p \rightarrow +0$ . But  $\frac{a_{01}}{f_z}$

4 strictly increases at the interval  $(0, p_z^1)$ , thus  $\frac{a_{01}}{f_z} > 0$  when  $p \in (0, p_z^1)$ . Since  
5  $f_z(p) > 0$ , then

$$a_{01}(p) > 0 \text{ for } p \in (0, p_z^1). \quad (4.16)$$

6 Finally, let us determine the sign of  $f_V(p)$  at the first interval of sign-  
7 definiteness  $p \in (0, p_1^V)$ :

$$f_V(p) = -\frac{4}{45}p^6 + o(p^6) < 0 \text{ as } p \rightarrow +0, \quad (4.17)$$

8 see the plots of  $f_V(p)$  at Figs. 8, 9 [36].

9 Now conditions (4.9), (4.16), (4.17) imply the required inequality:

$$a_0(p) < 0 \text{ for } p \in (0, p_1(k)). \quad (4.18)$$

2) Let us show that  $a_2(p) > 0$  for  $p \in (0, p_1(k))$ . Condition (4.9) gives

$a_2 = f_z \cdot a_{21}$ . Immediate differentiation yields the equalities

$$\begin{aligned}
\frac{\partial}{\partial p} \left( \frac{a_{21}(p)}{f_V(p)} \right) f_V^2(p) &= -\frac{4}{3} k^2 x_1, \\
x_1 &= k^2 (\text{cn}^2 p p^2 + \beta_1 p + \gamma_1)^2 + (1 - k^2)(p^2 + \delta_1 p + \varepsilon_1)^2, \\
\beta_1 &= -\text{cn}^2 p E_2^3(p) + 6 \text{cn} p \text{sn} p \text{dn} p E_2^2(p) - (8 - 10 \text{cn}^2(p) \\
&\quad - 4k^2 \text{sn}^2(p)(2 - 3 \text{cn}^2(p))) E_2(p) + 4 \text{cn} p \text{sn} p \text{dn} p (2k^2 \text{sn}^2 p - 1), \\
\gamma_1 &= 8 \text{cn} p \text{dn} p E_2^3(p)(-1 + 2k^2) \text{sn} p + E_2^4(p)(3 - \text{sn}^2 p + 2k^2(-2 + \text{sn}^2 p)) \\
&\quad - 4 \text{dn}^2 p \text{sn}^2 p (3 + 8k^4(2 + \text{sn}^2 p) - 4k^2(5 + \text{sn}^2 p)) \\
&\quad - 4 \text{cn} p \text{dn} p E_2(p) \text{sn} p (-7 + 8k^2(5 + \text{sn}^2 p - 2k^2(2 + \text{sn}^2 p))) \\
&\quad + E_2^2(p)(-15 + 23 \text{sn}^2 p + 8k^2(10 - 10 \text{sn}^2 p - 3 \text{sn}^4 p \\
&\quad + k^2(-8 + 4 \text{sn}^2 p + 6 \text{sn}^4 p))), \\
\delta_1 &= -E_2^3(p) - (2 - 4k^2 \text{sn}^2 p) E_2(p), \\
\varepsilon_1 &= 16 \text{cn} p \text{dn} p E_2^3(p) k^2 \text{sn} p + E_2^4(p)(1 + 2k^2(-2 + \text{sn}^2 p)) \\
&\quad + 32 \text{cn} p \text{dn} p E_2(p) k^2 \text{sn} p (-3 + 2k^2(2 + \text{sn}^2 p)) \\
&\quad - 16 \text{dn}^2 p k^2 \text{sn}^2 p (-3 + 2k^2(2 + \text{sn}^2 p)) \\
&\quad + E_2^2(p)(1 + 16k^2(3 - 4 \text{sn}^2 p + k^2(-4 + 2 \text{sn}^2 p + 3 \text{sn}^4 p))).
\end{aligned} \tag{4.19}$$

Thus  $x_1 \geq 0$ , moreover,  $x_1$  vanishes at isolated points in view of the following expansions as  $p \rightarrow 0$ :

$$\begin{aligned}
p^2 \text{cn}^2 p + \beta_1 p + \gamma_1 &= \frac{4}{1575} (1 - k^2) p^{10} + o(p^{10}) \neq 0, \\
p^2 + \delta_1 p + \varepsilon_1 &= \frac{4}{1575} k^2 p^{10} + o(p^{10}) \neq 0.
\end{aligned}$$

Consequently, the function  $\frac{a_{21}(p)}{f_V(p)}$  strictly decreases at each interval where  $f_V(p) \neq 0$ . Taking into account the signs:

$$\begin{aligned}
f_V(p) &< 0 \text{ for } p \in (0, p_1^V), \\
a_{21}(p) &= \frac{16}{1488375} k^4 (1 - k^2) p^{15} + o(p^{15}) > 0 \text{ as } p \rightarrow +0,
\end{aligned}$$

- 1 we conclude that  $\frac{a_{21}(p)}{f_V(p)} < 0$  as  $p \rightarrow +0$ , thus  $\frac{a_{21}(p)}{f_V(p)} < 0$  for  $p \in (0, p_1^V)$ .
- 2 Whence  $a_{21}(p) > 0$  for  $p \in (0, p_1^V)$ , so

$$a_2(p) > 0 \text{ for } p \in (0, p_1(k)). \tag{4.20}$$

3) Let us prove that  $a_0 + a_1 + a_2 < 0$  for  $p \in (0, p_1(k))$ . We obtain from (4.10) the equalities

$$\begin{aligned}
a_0 + a_1 + a_2 &= (1 - k^2) f_z \cdot a_{021}, \\
a_{021} &= -a_{21}/k^2.
\end{aligned}$$



Immediate differentiation gives

$$\frac{\partial}{\partial p} \left( \frac{a_{021}}{f_V} \right) f_V^2 = \frac{4}{3} x_1,$$

where the function  $x_1$  is defined by Eq. (4.19). It was shown in item 2) of this proof that  $x_1(p)$  is nonnegative and vanishes at isolated points. Thus the function  $\frac{a_{021}(p)}{f_V(p)}$  strictly increases at each interval where  $f_V(p) \neq 0$ . Taking into account the sign

$$a_{021} = -\frac{16}{1488375} k^2 (1 - k^2) p^{15} + o(p^{15}) < 0 \text{ as } p \rightarrow 0,$$

we conclude that  $\frac{a_{021}(p)}{f_V(p)} > 0$  for  $p \in (0, p_1(k))$ , thus  $a_{021}(p) < 0$  for  $p \in (0, p_1(k))$  and

$$a_0 + a_1 + a_2 < 0 \text{ for } p \in (0, p_1(k)). \quad (4.21)$$

4) Let us make use of the bounds on the coefficients  $a_0, a_1, a_2$  proved in items 1)–3) and show that the quadratic trinomial  $J_1(\xi) = a_0 + a_1\xi + a_2\xi^2$  is negative at the segment  $\xi \in [0, 1]$ .

Choose any  $k \in (0, 1)$  and  $p \in (0, p_1(k))$ . At the endpoints of the segment  $\xi \in [0, 1]$  we have:

$$J_1(0) = a_0 < 0 \text{ by (4.18),}$$

$$J_1(1) = a_0 + a_1 + a_2 < 0 \text{ by (4.21).}$$

But the equality  $a_2 > 0$  (see (4.20)) means that the function  $\xi \mapsto J_1(\xi)$  is convex. Thus

$$J_1(\xi) \leq \xi a_0 + (1 - \xi)(a_0 + a_1 + a_2) < 0 \text{ for } \xi \in [0, 1].$$

Lemma 3 is proved. □

#### 4.4 Proof of the bound of conjugate time for $\lambda \in C_1$

Theorem 4 follows immediately from Lemmas 2 and 3.

### 5 Conjugate time for $\lambda \in C_2$

In this section we prove Theorem 2 for  $\lambda \in C_2$ :

**Theorem 5.** *If  $\lambda \in C_2$ , then  $t_{\text{conj}}^1(\lambda) \geq t_{\text{MAX}}^1(\lambda)$ .*

#### 5.1 Computation of Jacobian $J$

We use the elliptic coordinates  $(\psi, k, \alpha, \beta)$  in the domain  $C_2$ , see Subsec. 1.2. For a fixed  $\lambda = (\psi, k, \alpha, \beta) \in C_2$ , conjugate times are roots  $t > 0$  of the Jacobian

$$J = \frac{\partial(x, y, z, v, w)}{\partial(t, \psi, k, \alpha, \beta)}.$$

We compute this Jacobian in the domain  $G_1 = \{g \in G \mid r^2 = x^2 + y^2 > 0\}$  in the same way as in Subsec 4.2:

$$J = -\frac{r^{10}}{\alpha} \frac{\partial(\bar{P}, \bar{Q}, \bar{R})}{\partial(t, \psi, k)} = -\frac{64}{3k^4(1-k^2)\Delta^2\alpha} J_1, \quad (5.1)$$

$$J_1 = a_0 + a_1\xi + a_2\xi^2, \quad \xi = \sin^2 u_2, \quad (5.2)$$

$$p = \frac{\sqrt{\alpha}t}{2k}, \quad \tau = \sqrt{\alpha} \left( \psi + \frac{t}{2k} \right),$$

$$u_1 = \text{am}(p, k), \quad u_2 = \text{am}(\tau, k),$$

$$a_0 = \frac{1}{16} f_V \cdot a_{01}, \quad (5.3)$$

$$a_0 + a_1 + a_2 = (1 - k^2)a_0 = \frac{1 - k^2}{16} f_V \cdot a_{01}, \quad (5.4)$$

$$a_2 = f_z \cdot a_{21}, \quad (5.5)$$

$$f_z = \frac{2}{k} \left( \sqrt{1 - k^2 \sin^2 u_1} ((2 - k^2)F(u_1) - 2E(u_1)) + k^2 \cos u_1 \sin u_1 \right),$$

$$f_V = \frac{4}{3} \left\{ 3\sqrt{1 - k^2 \sin^2 u_1} (2E(u_1) - (2 - k^2)F(u_1))^2 \right.$$

$$+ \cos u_1 [8E^3(u_1) - 4E(u_1)(4 + k^2) - 12E^2(u_1)(2 - k^2)F(u_1)$$

$$+ 6E(u_1)(2 - k^2)^2 F^2(u_1) + F(u_1)(16 - 4k^2 - 3k^4$$

$$- (2 - k^2)^3 F^2(u_1))] \sin u_1 - 2\sqrt{1 - k^2 \sin^2 u_1} (-4k^2 + 3(2E(u_1)$$

$$- (2 - k^2)F(u_1))^2 \sin^2 u_1 + 12k^2 \cos u_1 (2E(u_1)$$

$$\left. - (2 - k^2)F(u_1)) \sin^3 u_1 - 8k^2 \sin^4 u_1 \sqrt{1 - k^2 \sin^2 u_1} \right\}, \quad (5.6)$$

$$k^4 a_0 + k^2 a_1 + a_2 = (1 - k^2)a_2 = (1 - k^2)f_z \cdot a_{21},$$

$$a_1 = -k^2 a_0 - a_2, \quad (5.7)$$

- 1 where the coefficients  $a_{01}$  and  $a_{21}$  are defined explicitly in (A.1) and (A.2).
- 2 **Remark 3.** Equality (5.1) is valid for all  $(\lambda, t) \in C_2 \times \mathbb{R}_+$  by the same argument
- 3 as in remark at the end of Subsec. 4.2.

## 4 5.2 Conjugate time as $k \rightarrow 0$

Compute the asymptotics of the Jacobian  $J_1$  (5.2) as  $k \rightarrow 0$ :

$$f_z = f_z^0 k^3 + o(k^3),$$

$$f_z^0 = \frac{1}{16} (4p - \sin 4p),$$

$$f_V = \frac{k^8}{512} f_V^0 + o(k^8),$$

$$f_V^0 = (32u_1^2 - 1) \cos 2u_1 + \cos 6u_1 - 8u_1 \sin 2u_1,$$

$$\begin{aligned}
a_{01} &= \frac{3}{2048} k^8 a_{01}^0 + o(k^8), \\
a_{01}^0 &= 64u_1^3 \sin 2u_1 + 48u_1^2 \cos 2u_1 - 44u_1 \sin 2u_1 - 4u_1 \cos u_1 \sin 2u_1 \\
&\quad + 3 \cos 2u_1 - 3 \cos 6u_1, \\
a_{21} &= \frac{1}{4194304} k^{17} a_{21}^0 + o(k^{17}), \\
a_{21}^0 &= 45u_1 + 608u_1^3 - 512u_1^5 + 16u_1(28u_1^2 - 3) \cos 4u_1 + 3u_1 \cos 8u_1 \\
&\quad + 12 \sin 4u_1 - 432u_1^2 \sin 4u_1 + 256u_1^4 \sin 4u_1 - 6 \sin 8u_1,
\end{aligned}$$

thus

$$\begin{aligned}
a_0 &= k^{16} a_0^0 + o(k^{16}), \\
a_0^0 &= \frac{3}{2^{24}} f_V^0 \cdot a_{01}^0,
\end{aligned} \tag{5.8}$$

$$a_2 = k^{20} a_2^0 + o(k^{20}), \tag{5.9}$$

$$a_2^0 = f_z^0 \cdot a_{21}^0, \tag{5.10}$$

$$a_1 = -k^2 a_0 - a_2 = -k^{18} a_0^0 + o(k^{18}).$$

Then

$$\begin{aligned}
J_1 &= k^{16} a_0^0 + o(k^{16}) + \xi(-k^{18} a_0^0 + o(k^{18})) + \xi^2(k^{20} a_2^0 + o(k^{20})) \\
&= k^{16} a_0^0 + o(k^{16}) = \frac{3}{2^{24}} k^{16} f_V^0 \cdot a_{01}^0 + o(k^{16}).
\end{aligned} \tag{5.11}$$

- 1 Thus, in order to bound the first positive root of  $J_1$  as  $k \rightarrow 0$ , we have to bound  
2 the first positive roots of  $f_V^0$  and  $a_{01}^0$ .

Let us denote by  $F[a]$  the first positive root of a function  $a(t)$ :

$$F[a] := \inf\{t > 0 \mid a(t) = 0\}.$$

- 3 **Lemma 4.** *We have  $F[f_V^0] \in (\frac{5}{8}\pi, \frac{3}{4}\pi)$ , moreover, the function  $f_V^0$  has a simple*  
4 *root at the point  $F[f_V^0]$ . If  $u_1 \in (0, F[f_V^0])$ , then  $f_V^0(u_1) < 0$ .*

*Proof.* Introduce the function

$$g(u_1) = \frac{2f_V^0(u_1)}{\sin 2u_1}, \quad u_1 \neq \frac{\pi n}{2}.$$

We have

$$g'(u_1) = -\frac{8(\sin 4u_1 - 4u_1)^2}{\sin^2 2u_1}, \quad u_1 \neq \frac{\pi n}{2},$$

thus  $g(u_1)$  decreases at intervals  $u_1 \in \left(\frac{\pi n}{2}, \frac{\pi(n+1)}{2}\right)$ . Since  $\lim_{u_1 \rightarrow 0} g(u_1) = 0$ , then  $g(u_1) < 0$  and  $f_V^0(u_1) < 0$  when  $u_1 \in (0, \frac{\pi}{2})$ . Evaluate  $f_V^0$  at several points:

$$\begin{aligned}
f_V^0\left(\frac{\pi}{2}\right) &= -8\pi^2 < 0, \\
f_V^0\left(\frac{3\pi}{4}\right) &= 6\pi > 0, \\
f_V^0\left(\frac{5\pi}{8}\right) &= \frac{4 + 10\pi - 25\pi^2}{2\sqrt{2}} < 0.
\end{aligned}$$

- 1 By monotonicity of  $g$ , it follows that  $F[f_V^0] \in (\frac{5}{8}\pi, \frac{3}{4}\pi)$ . It is obvious from the  
2 above that  $f_V^0(u_1) < 0$  for  $u_1 \in (0, F[f_V^0])$ .

Let us prove that the first root is simple:

$$(f_V^0(u_1))' = (1/2 \sin 2u_1 g(u_1))' = 2 \cot 2u_1 f_V^0(u_1) + 1/2 \sin 2u_1 g'(u_1).$$

- 3 If  $u_1 = F[f_V^0]$ , then  $f_V^0(u_1) = 0$ ,  $g'(u_1) < 0$ ,  $\sin 2u_1 < 0$ , thus  $(f_V^0)'(u_1) > 0$ .  $\square$

- 4 **Lemma 5.** We have  $F[a_{01}^0] \in (\frac{3}{4}\pi, \pi)$ . If  $u_1 \in (0, F[a_{01}^0])$ , then  $a_{01}^0(u_1) < 0$ .

- 5 *Proof.* We have  $a_{01}^0(\pi) = 48\pi^2 > 0$ , thus it remains to prove that  $a_{01}^0(u_1) < 0$   
6 for  $u_1 \in (0, \frac{3}{4}\pi]$ .

We prove this bound by the method “*divide et impera*”. Let us apply this method to the function

$$x_0(u_1) = a_{01}^0(u_1).$$

We get:

$$x_0 = 64u_1^3 \tilde{x}_0 + o(u_1^3), \quad u_1 \rightarrow \infty,$$

$$\tilde{x}_0 = \sin 2u_1,$$

$$x_1 = \left( \frac{x_0}{\tilde{x}_0} \right)' \tilde{x}_0^2 = -96u_1^2 \tilde{x}_1 + o(u_1^2), \quad u_1 \rightarrow \infty,$$

$$\tilde{x}_1 = \cos 4u_1,$$

$$x_2 = \left( \frac{x_1}{\tilde{x}_1} \right)' \tilde{x}_1^2 = \frac{1}{2}u_1 \tilde{x}_2 + o(u_1), \quad u_1 \rightarrow \infty,$$

$$\tilde{x}_2 = 192 - 2 \cos u_1 + 10 \cos 3u_1 - 6 \cos 5u_1 - \cos 7u_1 - 192 \cos 8u_1 - \cos 9u_1,$$

$$x_3 = \left( \frac{x_2}{\tilde{x}_2} \right)' \tilde{x}_2^2 = -32 \cos 4u_1 \left( \cos \frac{u_1}{2} + \cos \frac{3u_1}{2} \right)^2 \sin^4 \left( \frac{u_1}{2} \right) x_4,$$

$$x_4 = 69678 + 127442 \cos u_1 + 130632 \cos 2u_1 + 133678 \cos 3u_1 + 145244 \cos 4u_1 \\ + 126302 \cos 5u_1 + 110040 \cos 6u_1 + 93731 \cos 7u_1 + 74782 \cos 8u_1 \\ + 55823 \cos 9u_1 + 37200 \cos 10u_1 + 18576 \cos 11u_1 + 72 \cos 12u_1.$$

Let  $u_1 \in (0, \pi/16)$ . Then it is easy to see that

$$69678 + 55823 \cos 9u_1 + 72 \cos 12u_1 > 0,$$

$$\cos u_1 + \cos 10u_1 > 0,$$

$$\cos u_1 + \cos 11u_1 > 0,$$

$$\cos iu_1 > 0, \quad i \in \{1, 2, \dots, 8\},$$

thus  $x_4 > 0$ , so  $x_3 < 0$ . Further, it is easy to show that

$$192 - 192 \cos 8u_1 > 0,$$

$$-2 \cos u_1 + 10 \cos 3u_1 - 6 \cos 5u_1 - \cos 7u_1 - \cos 9u_1 > 0,$$

thus  $\tilde{x}_2 > 0$ . The function  $\frac{x_2}{\tilde{x}_2}$  decreases, and by virtue of the asymptotics

$$\frac{x_2}{\tilde{x}_2} = -\frac{3}{13}u_1 + o(u_1), \quad u_1 \rightarrow 0,$$

we get  $\frac{x_2}{\tilde{x}_2} < 0$  and  $x_2 < 0$ . Similarly, the function  $\frac{x_1}{\tilde{x}_1}$  decreases, and by virtue of the asymptotics

$$\frac{x_1}{\tilde{x}_1} = -360u_1^4 + o(u_1^4), \quad u_1 \rightarrow 0,$$

we get  $\frac{x_1}{\tilde{x}_1} < 0$  and  $x_1 < 0$ . Finally, the function  $\frac{x_0}{\tilde{x}_0}$  decreases, and by virtue of the asymptotics

$$\frac{x_0}{\tilde{x}_0} = -30u_1^3 + o(u_1^3), \quad u_1 \rightarrow 0,$$

1 we get  $\frac{x_0}{\tilde{x}_0} < 0$  and  $x_0 < 0$ .

2 We prove similarly, going by steps of length  $\pi/16$ , that  $x_0 = a_{01}^0 < 0$  for  
3  $u_1 \in [\frac{\pi}{16}, \frac{3\pi}{4}]$ .

4 Thus  $F[a_{01}^0] \in (\frac{3\pi}{4}, \pi)$ , and  $a_{01}^0 < 0$  for  $u_1 \in (0, F[a_{01}^0])$ .  $\square$

5 The following lemma will be used in the proof of Lemma 8.

6 **Lemma 6.** *If  $u_1 > 0$ , then  $a_{21}^0(u_1) < 0$ .*

*Proof.* We apply the method “*divide et impera*”. We have:

$$\begin{aligned} x_0 &= a_{21}^0 = -512u_1^5 + o(u_1^5), & u_1 \rightarrow \infty, \\ x_1 &= x_0' = 512u_1^4\tilde{x}_1 + o(u_1^4), & u_1 \rightarrow \infty, \\ \tilde{x}_1 &= 2\cos 4u_1 - 5 < 0, \\ x_2 &= \left(\frac{x_1}{\tilde{x}_1}\right)' \tilde{x}_1^2 = 1024u_1^3\tilde{x}_2 + o(u_1^3), & u_1 \rightarrow \infty, \\ \tilde{x}_2 &= 48 - 25\cos 4u_1 + 4\cos 8u_1 > 0, \\ x_3 &= \left(\frac{x_2}{\tilde{x}_2}\right)' \tilde{x}_2^2 = 1536u_1^2\tilde{x}_3 + o(u_1^2), & u_1 \rightarrow \infty, \\ \tilde{x}_3 &= (2\cos 4u_1 - 5)(-781 + 96\cos 4u_1 - 141\cos 8u_1 + 16\cos 12u_1) > 0, \\ x_4 &= \left(\frac{x_3}{\tilde{x}_3}\right)' \tilde{x}_3^2 = 6144u_1\tilde{x}_4 + o(u_1), & u_1 \rightarrow \infty, \\ \tilde{x}_4 &= -(2\cos 4u_1 - 5)^2(48 - 25\cos 4u_1 + 4\cos 8u_1) \\ &\quad (-25136 - 27745\cos 4u_1 - 3880\cos 8u_1 + 53\cos 12u_1 + 8\cos 16u_1) \\ &\quad \sin^2 2u_1 > 0, & u_1 \neq \frac{\pi n}{2}, \\ x_5 &= \left(\frac{x_4}{\tilde{x}_4}\right)' \tilde{x}_4^2 = 786432(5 - 2\cos 4u_1)^4(48 - 25\cos 4u_1 + 4\cos 8u_1)^2 \\ &\quad (98009 - 18812\cos 4u_1 + 18073\cos 8u_1 - 2564\cos 12u_1 + 64\cos 16u_1) \\ &\quad \sin^{12} 2u_1 > 0, & u_1 \neq \frac{\pi n}{2}. \end{aligned}$$

7 The function  $\frac{x_4}{\tilde{x}_4}$  increases on  $\mathbb{R}$ , and since  $\lim_{u_1 \rightarrow 0} \frac{x_4}{\tilde{x}_4} = 0$ , then  $\frac{x_4}{\tilde{x}_4} > 0$  for

8  $u_1 > 0$ . Let  $u_1 > 0$ , then  $x_4 > 0$ , thus  $\frac{x_3}{\tilde{x}_3}$  increases. Since  $\lim_{u_1 \rightarrow 0} \frac{x_3}{\tilde{x}_3} = 0$ , then

9  $\frac{x_3}{\tilde{x}_3} > 0$ ,  $x_3 > 0$ , and  $\frac{x_2}{\tilde{x}_2}$  increases for  $u_1 > 0$ . By virtue of  $\lim_{u_1 \rightarrow 0} \frac{x_2}{\tilde{x}_2} = 0$ , we

10 have  $\frac{x_2}{\tilde{x}_2} > 0$ ,  $x_2 > 0$ , and  $\frac{x_1}{\tilde{x}_1}$  increases as  $u_1 > 0$ . Since  $\lim_{u_1 \rightarrow 0} \frac{x_1}{\tilde{x}_1} = 0$ , then

1  $\frac{x_1}{\bar{x}_1} > 0$ ,  $x_1 < 0$ , and  $x_0$  decreases as  $u_1 > 0$ . Finally, we have  $x_0(0) = 0$ , thus  
2  $x_0(u_1) = a_{21}^0(u_1) < 0$  for  $u_1 > 0$ .  $\square$

3 Now we are able to bound the Jacobian  $J_1$  for  $k \rightarrow 0$  as follows. Denote  
4 by  $u_1 = u_V^1(k)$ ,  $k \in (0, 1)$ , the first positive root of the function  $f_V(u_1, k)$ , see  
5 (5.6).

6 **Lemma 7.** *For any  $\varepsilon > 0$  there exists  $\bar{k} = \bar{k}(\varepsilon) \in (0, 1)$  such that for any  
7  $k \in (0, \bar{k}]$ , any  $u_1 \in (0, u_V^1(k) - \varepsilon]$  and any  $\xi \in [0, 1]$  we have  $J_1(u_1, \xi, k) > 0$ .*

*Proof.* By virtue of (5.11),

$$J_1 = k^{16}a_0^0 + o(k^{16}), \quad k \rightarrow 0,$$

$$a_0^0 = \frac{3}{2^{24}}f_V^0 \cdot a_{01}^0,$$

8 moreover,

$$|J_1(u_1, \xi, k) - k^{16}a_0^0(u_1)| \leq f(k) = o(k^{16}), \quad k \rightarrow 0, \quad (5.12)$$

9 where the function  $f(k)$  does not depend on  $\xi$  since  $\xi \in [0, 1]$ .

10 By Lemmas 4, 5, if  $u \in (0, u_V^1(0))$ , then  $a_0^0(u_1) > 0$ .

Define a function:

$$\tilde{J}(u_1, \xi, k) = \begin{cases} J_1(u_1, \xi, k)/k^{16}, & k > 0, \\ a_0^0(u_1), & k = 0. \end{cases}$$

We have  $\lim_{k \rightarrow 0} J_1/k^{16} = a_0^0$ , thus the function  $\tilde{J}$  is continuous on the set  $\mathbb{R}_{u_1} \times [0, 1]_\xi \times [0, 1)_k$ . Then there exists an open set  $O_1 \subset \mathbb{R}_{u_1} \times [0, 1]_\xi \times [0, 1)_k$  such that:

$$O_1 \supset I := \{(u_1, \xi, k) \in \mathbb{R} \times [0, 1] \times [0, 1) \mid u_1 \in (0, u_V^1(0)), k = 0, \xi \in [0, 1]\},$$

$$\tilde{J}|_{O_1} > 0.$$

11 By virtue of inequality (5.12), the neighborhood  $O_1$  does not depend on  $\xi$  and  
12 projects to the axis  $\xi$  onto the segment  $[0, 1]$ . Then the set  $O_2 = O_1 \setminus I \subset$   
13  $\mathbb{R}_{u_1} \times [0, 1]_\xi \times (0, 1)_k$  is open and  $J_1|_{O_2} > 0$ .

Now we study the sign of  $J_1$  in the neighborhood of  $(u_1, k) = (0, 0)$ . To this end, compute the asymptotics as  $u_1^2 + k^2 \rightarrow 0$ :

$$a_0 = \frac{4}{70875}k^{16}u_1^{16} + o(u_1^2 + k^2)^{16},$$

$$a_2 = -\frac{4}{4465125}k^{20}u_1^{20} + o(u_1^2 + k^2)^{20},$$

$$a_1 = -k^2a_0 - a_2 = -\frac{4}{70875}k^{18}u_1^{16} + o(u_1^2 + k^2)^{17},$$

whence

$$J_1 = \frac{4}{70875}k^{16}u_1^{16} + o(u_1^2 + k^2)^{16} > 0 \text{ as } (u_1, k) \rightarrow (0, 0).$$

Thus there exists an open set  $O_3 \subset (0, +\infty)_{u_1} \times [0, 1]_\xi \times (0, 1)_k$  such that the closure of  $O_3$  contains a neighborhood of the set  $\{(u_1, \xi, k) \mid u_1 = k = 0, \xi \in [0, 1]\}$  in  $[0, +\infty)_{u_1} \times [0, 1]_\xi \times [0, 1)_k$  and  $J_1|_{O_3} > 0$ . The set  $O_3$  does not depend on  $\xi$  and projects to the axis  $\xi$  onto the segment  $[0, 1]$  by the same argument as  $O_1, O_2$ .

Take any  $\varepsilon > 0$ . The set  $\text{cl}(O_2) \cup \text{cl}(O_3)$  contains a neighborhood of the set  $[0, u_V^1(0) - \varepsilon]_{u_1} \times [0, 1]_\xi \times \{k = 0\}$ . Thus there exists  $\bar{k} = \bar{k}(\varepsilon) \in (0, 1)$  such that the set  $\{(u_1, \xi, k) \in \mathbb{R} \times [0, 1] \times (0, 1) \mid k \in (0, \bar{k}], u_1 \in (0, u_V^1(k) - \varepsilon], \xi \in [0, 1]\}$  is contained in  $O_2 \cup O_3$ , thus the Jacobian  $J_1$  is positive on it:

$$\forall k \in (0, \bar{k}] \quad \forall u_1 \in (0, u_V^1(k) - \varepsilon] \quad \forall \xi \in [0, 1] \quad J_1(u_1, \xi, k) > 0.$$

□

The bound of the Jacobian  $J_1$  of Lemma 7 can be reformulated in terms of conjugate points along the corresponding geodesics.

**Corollary 2.** *Take any  $\varepsilon > 0$  and choose  $\bar{k} = \bar{k}(\varepsilon) \in (0, 1)$  according to Lemma 7. Take any  $\lambda = (\psi, k, \alpha, \beta) \in C_2$  with  $k \in (0, \bar{k}]$ . Define functions:*

$$\begin{aligned} u_1(k, \varepsilon) &= u_V^1(k) - \varepsilon, \\ p(k, \varepsilon) &= F(u_1(k, \varepsilon), k), \\ t(k, \varepsilon) &= 2kp(k, \varepsilon). \end{aligned}$$

Then the geodesic

$$\gamma_\lambda^\varepsilon = \{\text{Exp}(\lambda, s) \mid s \in (0, t(k, \varepsilon))\}$$

does not contain conjugate points.

### 5.3 Conjugate time for arbitrary $k \in (0, 1)$

Now we can prove a bound for the Jacobian  $J_1$  via homotopy from an arbitrary  $k \in (0, 1)$  to a small  $k$  close to 0.

**Lemma 8.** *For any  $k \in (0, 1)$ , any  $u_1 \in (0, u_V^1(k))$  and any  $\xi \in (0, 1)$  we have  $J_1(u_1, \xi, k) > 0$ .*

*Proof.* Take any  $\lambda_1 = (\psi_1, k_1, \alpha_1, \beta_1) \in C_2$  such that  $\xi_1 \in (0, 1)$ , where

$$\begin{aligned} \xi_1 &= \text{sn}^2(\tau_1, k_1), \\ \tau_1 &= \sqrt{\alpha_1} \left( \psi_1 + \frac{t_1}{2k_1} \right), \\ t_1 &= \frac{2k_1}{\sqrt{\alpha_1}} F(u_V^1(k_1), k_1). \end{aligned}$$

Take any  $k \in (0, 1)$ , and let  $u_1 = u_V^1(k)$ ,  $\xi = \xi_1$ . Since  $f_V(u_1, k) = 0$ , thus  $a_0 = a_0 + a_1 + a_2 = 0$  by virtue of (5.3), (5.4), so we have

$$J_1 = a_1\xi + a_2\xi^2 = -a_2\xi(1 - \xi). \quad (5.13)$$

By virtue of (5.5), we have

$$a_2(u_1, k) < 0, \quad (5.14)$$

thus  $J_1(u_V^1(k), \xi, k) > 0$ .

Recall equalities (5.9), (5.10) and define a function

$$\tilde{a}_2(u_1, k) = \begin{cases} \frac{a_2(u_1, k)}{k^{20}} = f_z^0(u_1) a_{21}^0(u_1) + o(1), & k \in (0, 1), \\ f_z^0(u_1) a_{21}^0(u_1), & k = 0. \end{cases}$$

This function is continuous on the set  $\mathbb{R}_{u_1} \times [0, 1)_k$ . Moreover,

$$\tilde{a}_2(u_V^1(k), k) < 0 \text{ for all } k \in [0, 1)$$

by virtue of (5.14) and Lemma 6. Thus there exists  $\varepsilon > 0$  (fix it) such that for all  $k \in (0, 1)$  and all  $u_1 \in [u_V^1(k) - \varepsilon, u_V^1(k)]$  we have  $a_2(u_1, k) < 0$ , whence  $J_1(u_1, \xi_1, k) > 0$  in view of (5.13).

For the fixed  $\varepsilon > 0$ , choose  $\bar{k} = \bar{k}(\varepsilon) \in (0, 1)$  by Lemma 7. Now we construct a homotopy of geodesics from  $k_1$  to  $\bar{k}$ .

A parameter of the homotopy is  $\mu \in [\bar{k}, k_1]$ , we assume that  $\bar{k} < k_1$ , otherwise there is nothing to prove. Define a continuous family of initial points of extremals:

$$\begin{aligned} \lambda_\mu &= (\psi_\mu, k = \mu, \alpha_1, \beta_1) \in C_2, \quad \mu \in [\bar{k}, k_1], \\ t_\mu &= \frac{2\mu}{\sqrt{\alpha_1}} F(u_V^1(\mu), \mu), \\ \psi_\mu &= \frac{1}{\sqrt{\alpha_1}} F(\text{am}(\tau_1, k_1), \mu) - \frac{t_\mu}{2\mu}, \end{aligned}$$

so that

$$\xi_1 = \text{sn}^2 \left( \sqrt{\alpha_1} \left( \psi_\mu + \frac{t_\mu}{2\mu} \right), \mu \right).$$

Further, consider a continuous one-parameter family of geodesics:

$$\begin{aligned} \gamma_\mu &= \{\text{Exp}(\lambda_\mu, s) \mid s \in (0, t(\mu, \varepsilon)]\}, \\ t(\mu, \varepsilon) &= \frac{2\mu}{\sqrt{\alpha_1}} F(u_V^1(\mu) - \varepsilon, \mu). \end{aligned}$$

By Cor. 2, the geodesic  $\gamma_{\bar{k}}$  does not contain conjugate points.

Further, for any  $\mu \in [\bar{k}, k_1]$  and  $u_1 = u_V^1(\mu) - \varepsilon$  we have  $J_1(u_1, \xi_1, \mu) > 0$  as was proved above. That is, the terminal time  $s = t(\mu, \varepsilon)$  is not conjugate for the geodesic  $\gamma_\mu$ .

By Th. 3, the curve  $\gamma_{k_1}$  is free of conjugate points.

Since  $J_1(u_1, \xi_1, k) > 0$  for  $u_1 \in (0, u_V^1(k) - \varepsilon]$  and small  $k$ , then  $J_1(u_1, \xi_1, k_1) > 0$  for  $u_1 \in (0, u_V^1(k_1))$ .  $\square$

In the following two lemmas we clarify the sign of the Jacobian  $J_1$  for the remaining special values of the parameters  $\xi, u_1$ .

**Lemma 9.** *Let  $\xi \in \{0, 1\}$ ,  $k \in (0, 1)$ , and  $u_1 \in (0, u_V^1(k))$ . Then  $J_1(u_1, \xi, k) > 0$ .*



*Proof.* Take any  $\lambda_1 = (\psi_1, k_1, \alpha_1, \beta_1) \in C_2$  such that

$$\begin{aligned}\xi_1 &= \text{sn}^2 \left( \sqrt{\alpha_1} \left( \psi_1 + \frac{t_1}{2k_1} \right), k_1 \right) = 0, \\ t_1 &= \frac{2k_1}{\sqrt{\alpha_1}} F(u_V^1(k_1), k_1),\end{aligned}$$

the case  $\xi_1 = 1$  is considered similarly. Take any sufficiently small  $\varepsilon > 0$ . Then  $\bar{t} = t_1 - \varepsilon$  is close to  $t_1 > \bar{t}$ . Thus  $\bar{\xi} = \text{sn}^2(\sqrt{\alpha_1}(\psi_1 + \frac{\bar{t}}{2k_1}), k_1) \neq \xi_1 = 0$ ,  $\bar{\xi}$  is close to  $\xi_1$ , thus  $\bar{\xi} \in (0, 1)$ .

By Lemma 8, the geodesic  $\{\text{Exp}(\lambda_1, s) \mid s \in (0, \bar{t}]\}$  is free of conjugate points. Since  $\bar{t} = t_1 - \varepsilon$  is arbitrarily close to  $t_1$ , the geodesic  $\{\text{Exp}(\lambda_1, s) \mid s \in (0, t_1)\}$  is also free of conjugate points.  $\square$

**Lemma 10.** Let  $k \in (0, 1)$ ,  $u_1 = u_V^1(k)$ ,  $\xi \in \{0, 1\}$ . Then  $J_1(u_1, \xi, k) = 0$ .

*Proof.* By virtue of (5.7),

$$J_1 = a_0(1 - k^2\xi) - a_2\xi(1 - \xi),$$

which vanishes for  $u_1 = u_V^1(k)$  and  $\xi \in \{0, 1\}$  in view of (5.3).  $\square$

## 5.4 Final bound for the conjugate time in $C_2$

Now we summarize our study of the first conjugate time for  $\lambda \in C_2$ .

**Theorem 6.** Let  $\lambda = (\psi, k, \alpha, \beta) \in C_2$  and

$$\begin{aligned}t_1 &= \frac{2k}{\sqrt{\alpha}} F(u_V^1(k), k) = t_{\text{MAX}}^1(\lambda), \\ \xi &= \text{sn}^2 \left( \sqrt{\alpha} \left( \psi + \frac{t_1}{2k} \right), k \right).\end{aligned}$$

If  $\xi \in \{0, 1\}$ , then  $t_{\text{conj}}^1(\lambda) = t_1$ .

If  $\xi \in (0, 1)$ , then  $t_{\text{conj}}^1(\lambda) \geq t_1$ .

*Proof.* Apply Lemmas 8–10.  $\square$

Now Th. 5 follows immediately from Th. 6.

## 6 Conjugate time for $\lambda \in \cup_{i=3}^7 C_i$

**Theorem 7.** If  $\lambda \in C_4 \cup C_5 \cup C_7$ , then  $t_{\text{cut}}(\lambda) = t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) = +\infty$ .

*Proof.* If  $\lambda \in C_4 \cup C_5 \cup C_7$ , then  $(x_t, y_t)$  is a straight line, thus  $\text{Exp}(\lambda, t)$  is optimal for  $t \in [0, +\infty)$ .  $\square$

**Theorem 8.** If  $\lambda \in C_3$ , then  $t_{\text{cut}}(\lambda) = t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) = +\infty$ .

1 *Proof.* We project the SR problem on the Cartan group to the SR problem on  
2 the Engel group studied in [8–11]. For  $\lambda \in C_3$ , the projection to the Engel  
3 group is optimal, thus the geodesic in the Cartan group is optimal as well. Let  
4 us prove this in detail.

The quotient  $H = G/e^{\mathbb{R}X_5}$  is the four-dimensional simply connected nilpotent Lie group called the Engel group [28]. The quotient mapping in coordinates is

$$\pi : (x, y, z, v, w) \mapsto (x, y, z, v), \quad G \rightarrow H.$$

The Engel algebra is

$$\begin{aligned} \mathfrak{h} = \pi_* \mathfrak{g} &= \text{span}(Y_1, Y_2, Y_3, Y_4), & Y_i &= \pi_* X_i, \quad i = 1, \dots, 4, \\ [Y_1, Y_2] &= Y_3, & [Y_1, Y_3] &= Y_4, & \text{ad } Y_4 &= 0. \end{aligned}$$

The left-invariant SR problem on the Engel group

$$\dot{h} = u_1 Y_1 + u_2 Y_2, \quad h \in H, \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad (6.1)$$

$$h(0) = \text{Id}, \quad h(t_1) = h_1, \quad (6.2)$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min \quad (6.3)$$

5 was studied in detail in [8–11]. In particular, it was shown that if an extremal  
6 control  $u(t)$  in problem (6.1)–(6.3) is non-periodic (i.e., if  $\lambda \in C_3$ ), then it is  
7 optimal for  $t \in [0, +\infty)$ . The corresponding geodesic  $h_t \in H$  is asymptotic to  
8 the abnormal trajectory  $e^{tY_2}$ .

9 Further, if a projection  $h_t = \pi(g_t) \in H$  is optimal for problem (6.1)–(6.3) on  
10 the Engel group  $H$ , then the trajectory  $g_t \in G$  is optimal for problem (1.4)–(1.6)  
11 on the Cartan group. The lift  $g_t \in G$  is asymptotic to the abnormal trajectory  
12  $e^{tX_2}$ . But an arbitrary geodesic in  $G$  corresponding to a non-periodic control  
13 (i.e., with  $\lambda \in C_3$ ) can be obtained by a rotation  $e^{sX_0}$  of  $g_t$ , which preserves  
14 optimality (see Subsubsec. 1.2.3).

15 Thus geodesics  $\text{Exp}(\lambda, t)$ ,  $\lambda \in C_3$ ,  $t \in [0, +\infty)$ , are optimal for problem (1.4)–  
16 (1.6). So we have  $t_{\text{cut}}(\lambda) = t_{\text{conj}}^1(\lambda) = +\infty = t_{\text{MAX}}^1(\lambda)$  for  $\lambda \in C_3$ .  $\square$

17 **Theorem 9.** *If  $\lambda \in C_6$ , then  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$ .*

*Proof.* Let  $\bar{\lambda} = (\bar{\theta}, \bar{h}_3, \bar{h}_4, \bar{h}_5) \in C_6$ , where  $\bar{h}_3 \neq 0$ ,  $\bar{h}_4 = \bar{h}_5 = 0$ . Take any  $\hat{\mu} > 0$  and consider a continuous curve

$$\begin{aligned} \lambda_\mu &= (\bar{\theta}, \bar{h}_3, h_4, h_5), & h_4 &= \mu, & h_5 &= 0, & \mu &\in [0, \hat{\mu}], \\ \lambda_0 &= \bar{\lambda}, & \lambda_\mu &\in C_2, & \mu &\in (0, \hat{\mu}]. \end{aligned}$$

Notice that the function  $\mu \mapsto t_{\text{MAX}}^1(\lambda_\mu)$ ,  $\mu \in [0, \hat{\mu}]$ , is continuous since

$$\lim_{\mu \rightarrow +0} t_{\text{MAX}}^1(\lambda_\mu) = \lim_{\mu \rightarrow +0} \frac{2k}{\sqrt{\alpha}} p_1^V(k) = \frac{4}{|\bar{h}_3|} p_1^V(0) = t_{\text{MAX}}^1(\lambda_0).$$

Take any sufficiently small  $\varepsilon > 0$  and define a continuous family of geodesics:

$$\begin{aligned} g_{t,\mu} &= \text{Exp}(\lambda_\mu, t), & t &\in (0, t_{1,\mu}], \\ t_{1,\mu} &= t_{\text{MAX}}^1(\lambda_\mu) - \varepsilon. \end{aligned}$$

1 If  $\mu \in (0, \widehat{\mu}]$ , then  $\lambda_\mu \in C_2$ , and the geodesic  $g_{t,\mu}$  does not contain conjugate  
2 points by Th. 5.

3 Conjugate instants along  $g_{t,0}$  are isolated one from another (see Sec. 3), thus  
4 we can choose an arbitrarily small  $\varepsilon > 0$  such that the instant  $t_{1,0} = t_{\text{MAX}}^1(\bar{\lambda}) - \varepsilon$   
5 is not conjugate along  $g_{t,0}$ . Then, by Th. 3, the geodesic  $g_{t,0}$ ,  $t \in (0, t_{1,0}]$ , also  
6 does not contain conjugate points.

7 Since  $\varepsilon > 0$  can be chosen arbitrarily small, then the geodesic  $g_{t,0}$ ,  $t \in$   
8  $(0, t_{\text{MAX}}^1(\bar{\lambda}))$ , does not contain conjugate points.

9 It was proved in [36] that the instant  $t_{\text{MAX}}^1(\bar{\lambda})$  is conjugate for  $g_{t,0}$ , thus  
10  $t_{\text{conj}}^1(\bar{\lambda}) = t_{\text{MAX}}^1(\bar{\lambda})$ .  $\square$

## 11 7 Cases when $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$

12 In this section we present cases when the lower bound of the first conjugate time  
13 given by Th. 2 turns into equality.

### 14 7.1 Cases when $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) = +\infty$

15 **Proposition 1.** *If  $\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7$ , then  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) = t_{\text{cut}}(\lambda) =$*   
16  *$+\infty$ .*

17 *Proof.* Apply Theorems 7, 8.  $\square$

### 18 7.2 Cases when $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) < +\infty$

19 **Proposition 2.** *Let  $\lambda = (\varphi, k, \alpha, \beta) \in C_1$ . The equality  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) <$*   
20  *$+\infty$  holds in the following cases:*

- 21 (1)  $k \in \{k_0, k_1\}$ ,
- 22 (2)  $k \in (0, k_1) \cup (k_0, 1)$ ,  $\text{cn } \tau = 0$ ,  $\tau = \sqrt{\alpha}(\varphi + t_{\text{MAX}}^1(\lambda)/2)$ ,
- 23 (3)  $k \in (k_1, k_0)$ ,  $\text{sn } \tau = 0$ ,  $\tau = \sqrt{\alpha}(\varphi + t_{\text{MAX}}^1(\lambda)/2)$ .

24 Recall that  $k_1 \approx 0.8$  and  $k_0 \approx 0.9$  are the only roots of the equation  $p_z^1(k) =$   
25  $p_1^V(k)$ ,  $k \in (0, 1)$ , see [36].

26 *Proof.* We prove that the instant  $t_1 = t_{\text{MAX}}^1(\lambda)$  is conjugate in cases (1)–(3).

By virtue of equalities (4.8)–(4.10), we have

$$J = (1 - \xi)f_V a_{01} - \frac{\xi(1 - k^2\xi)}{k^2}f_z a_{21}, \quad p = \sqrt{\alpha}t_1/2.$$

27 If  $k \in \{k_0, k_1\}$ , then  $f_V(p, k) = f_z(p, k) = 0$ , thus  $J = 0$ . So the instant  $t_1$  is  
28 conjugate.

If  $\text{cn } \tau = 0$ , then  $\xi = 1$ , thus

$$J = -\frac{1 - k^2}{k^2}f_z a_{21}.$$

For  $k \in (0, k_1) \cup (k_0, 1)$  we have from [36]

$$t_{\text{MAX}}^1(\lambda) = \frac{2}{\sqrt{\alpha}} p_1^z(k),$$

1 thus  $f_z(p) = f_z(p_1^z(k)) = 0$ . So  $J = 0$ .

If  $\text{sn } \tau = 0$ , then  $\xi = 0$ , thus

$$J = f_V a_{01}.$$

For  $k \in (k_1, k_0)$  we have from [36]

$$t_{\text{MAX}}^1(\lambda) = \frac{2}{\sqrt{\alpha}} p_1^V(k),$$

2 thus  $f_V(p) = f_z(p_1^V(k)) = 0$ . So  $J = 0$ .

3 We proved that in all cases (1)–(3) the instant  $t_1 = t_{\text{MAX}}^1(\lambda)$  is conjugate. By  
4 Th. 2, all instants  $t \in (0, t_{\text{MAX}}^1(\lambda))$  are not conjugate, thus  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$ .  
5  $\square$

6 **Remark 4.** The condition  $\text{cn } \tau = 0$  (resp.  $\text{sn } \tau = 0$ ) in Propos. 2 means that  
7 the corresponding inflectional elastic arc  $(x_t, y_t)$  is centered at inflection point  
8 (resp. at vertex), see [35].

**Proposition 3.** Let  $\lambda = (\psi, k, \alpha, \beta) \in C_2$ . The equality  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) < +\infty$  holds if

$$\text{sn}^2 \tau \in \{0, 1\}, \quad \tau = \sqrt{\alpha}(\psi + t_{\text{MAX}}^1(\lambda)/(2k)).$$

9 *Proof.* See Th. 6.  $\square$

10 **Remark 5.** The condition  $\text{sn}^2 \tau \in \{0, 1\}$  in Propos. 3 means that the corre-  
11 sponding non-inflectional elastic arc  $(x_t, y_t)$  is centered at vertex, see [35].

12 **Proposition 4.** If  $\lambda \in C_6$ , then  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda) < +\infty$ .

13 *Proof.* See Th. 9.  $\square$

14 **Remark 6.** There is a numerical evidence that the equality  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$   
15 holds only in the cases listed in Propositions 1–4.

16 We plot elasticae terminating at the instant  $t_1 = t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$  in the  
17 following cases:

- 18 •  $\lambda \in C_1$ ,  $k = k_0$  (Fig. 10), we draw the initial elastica plus elasticae ob-  
19 tained from it by the reflections  $\varepsilon^1, \varepsilon^2, \varepsilon^3$ ,
- 20 •  $\lambda \in C_1$ ,  $k = k_1$  (Fig. 11),
- 21 •  $\lambda \in C_1$ ,  $\text{cn } \tau = 0$ ,  $k \in (0, k_1)$  and  $k \in (k_0, 1)$  (Figs. 12 and 13),
- 22 •  $\lambda \in C_1$ ,  $\text{sn } \tau = 0$ ,  $k \in (k_1, k_0)$  (Fig. 14),
- 23 •  $\lambda \in C_2$ ,  $\text{sn } \tau = 0$  (Fig. 16),

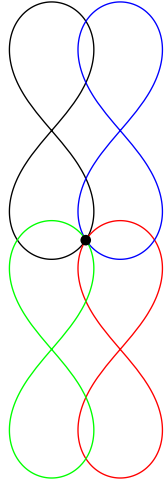


Figure 10:  $\lambda \in C_1$ ,  $k = k_0$

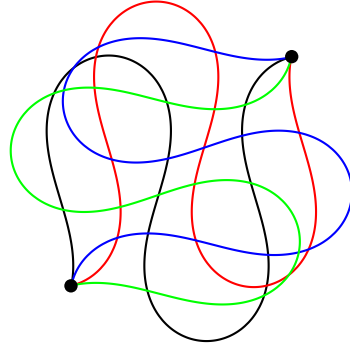


Figure 11:  $\lambda \in C_1$ ,  $k = k_1$

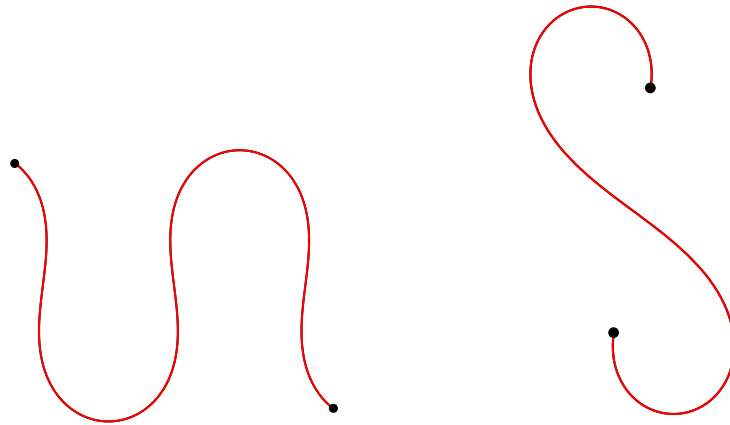


Figure 12:  $\lambda \in C_1$ ,  $\text{cn } \tau = 0$ ,  $k \in (0, k_1)$

Figure 13:  $\lambda \in C_1$ ,  $\text{cn } \tau = 0$ ,  $k \in (k_0, 1)$

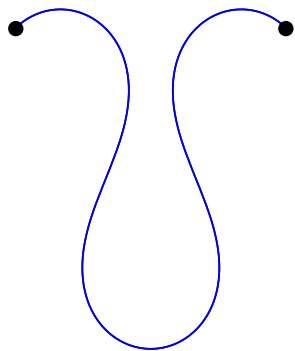


Figure 14:  $\lambda \in C_1, \operatorname{sn} \tau = 0$

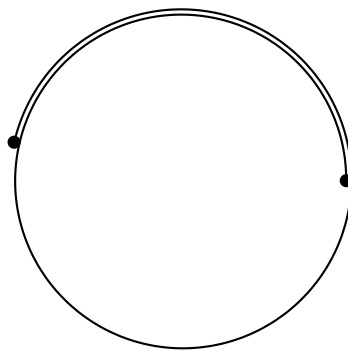


Figure 15:  $\lambda \in C_6$

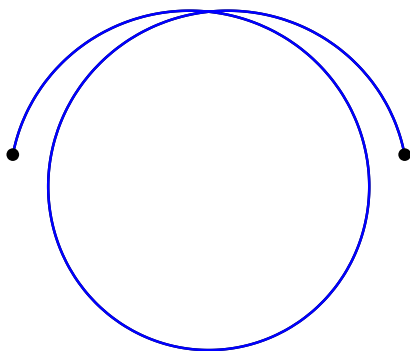


Figure 16:  $\lambda \in C_2, \operatorname{sn} \tau = 0$

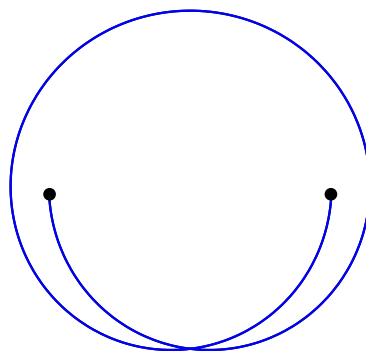


Figure 17:  $\lambda \in C_2, \operatorname{cn} \tau = 0$

- 1 •  $\lambda \in C_2$ ,  $\text{cn } \tau = 0$  (Fig. 17),
- 2 •  $\lambda \in C_6$  (Fig. 15).

3 **Remark 7.** Notice that despite the equality  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$  in Proposition 2,  
 4 items (2), (3) and in Propositions 3, 4, the corresponding point  $\text{Exp}(\lambda, t_{\text{MAX}}^1(\lambda))$   
 5 is not a Maxwell point but a limit of pairs of symmetric Maxwell points, see [36].  
 6 On the contrary, in Proposition 2, item (1), the point  $\text{Exp}(\lambda, t_{\text{MAX}}^1(\lambda))$  is a  
 7 Maxwell point, thus in Figs. 10, 11 we draw four symmetric elasticae corre-  
 8 sponding to the group of reflections  $\{\text{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ .

## 9 8 Continuity of $t_{\text{conj}}^1(\lambda)$ at infinite values

10 The following proposition states continuity of the first conjugate time near in-  
 11 finite values, when the limit geodesic is not abnormal.

12 **Proposition 5.** Let  $\lambda_n, \bar{\lambda} \in C$  and  $\lambda_n \rightarrow \bar{\lambda}$  as  $n \rightarrow \infty$ . We have  $t_{\text{conj}}^1(\lambda_n) \rightarrow$   
 13  $t_{\text{conj}}^1(\bar{\lambda}) = +\infty$  in the following cases:

- 14 (1)  $\lambda_n \in C_1$ ,  $\bar{\lambda} \in C_3 \cup C_5$ ,
- 15 (2)  $\lambda_n \in C_2$ ,  $\bar{\lambda} \in C_3 \cup C_5$ ,
- 16 (3)  $\bar{\lambda} \in C_7$ .

17 *Proof.* (1) We have  $\bar{\lambda} = (\bar{\theta}, \bar{c}, \bar{\alpha}, \bar{\beta}) \in C_3 \cup C_5$ , thus  $\bar{E} = \frac{\bar{c}^2}{2} - \bar{\alpha} \cos(\bar{\theta} - \bar{\beta}) = \bar{\alpha} > 0$ .  
 18 Since  $\lambda_n = (\theta_n, c_n, \alpha_n, \beta_n) \rightarrow \bar{\lambda}$ , then  $E_n + \alpha_n \rightarrow \bar{E} + \bar{\alpha} = 2\bar{\alpha} > 0$ , thus  
 19  $k_n = \sqrt{\frac{E_n + \alpha_n}{2\alpha_n}} \rightarrow 1$ . So  $t_{\text{MAX}}^1(\lambda_n) \rightarrow +\infty$ , thus  $t_{\text{conj}}^1(\lambda_n) \rightarrow +\infty = t_{\text{conj}}^1(\bar{\lambda})$ .

20 (2) Similarly to item (1), we have  $E_n + \alpha_n \rightarrow 2\bar{\alpha} > 0$ , thus  $k_n = \sqrt{\frac{2\alpha_n}{E_n + \alpha_n}} \rightarrow$   
 21 1. Then  $t_{\text{MAX}}^1(\lambda_n) \rightarrow +\infty$ , thus  $t_{\text{conj}}^1(\lambda_n) \rightarrow +\infty = t_{\text{conj}}^1(\bar{\lambda})$ .

22 (3) We have  $\bar{\lambda} = (\bar{\theta}, \bar{c}, \bar{\alpha}, \bar{\beta}) \in C_7$  with  $\bar{c} = \bar{\alpha} = 0$ . If  $\lambda_n = (\theta_n, c_n, \alpha_n, \beta_n) \rightarrow$   
 23  $\bar{\lambda}$ , then  $\alpha_n \rightarrow \bar{\alpha} = 0$ , thus  $t_{\text{MAX}}^1(\lambda_n) = \frac{2}{\sqrt{\alpha_n}} p_1(k_n) \rightarrow +\infty$  in the case  $\lambda_n \in C_1$ .  
 24 The cases  $\lambda_n \in \cup_{i=2}^7 C_i$  are considered similarly.  $\square$

**Remark 8.** Let  $\lambda = (\varphi, k, \alpha, \beta) \in C_1$  with  $\text{cn } \tau = 0$ ,  $\tau = \sqrt{\alpha}(\varphi + t_{\text{MAX}}^1(\lambda)/2)$ ,  
 $k \in (0, k_1)$ . Then  $t_{\text{conj}}^1(\lambda) = t_{\text{MAX}}^1(\lambda)$  by Propos. 2. Let  $\alpha = \text{const}$ ,  $k \rightarrow +0$ ,  
 then  $\lambda \rightarrow \bar{\lambda} \in C_4$ . We have

$$\lim_{k \rightarrow +0} t_{\text{MAX}}^1(\lambda) = \frac{2}{\sqrt{\alpha}} p_1^z(0), \quad p_1^z(0) \in (\pi, 3\pi/2)$$

by Propos. 2.1 [36]. Thus

$$\lim_{k \rightarrow +0} t_{\text{conj}}^1(\lambda) < +\infty = t_{\text{conj}}^1(\bar{\lambda}).$$

25 So  $t_{\text{conj}}^1(\lambda)$  is discontinuous at points  $\lambda \in C_4$ .

## 9 Two-sided bounds of $t_{\text{conj}}^1(\lambda)$

There is a numerical evidence of the following two-sided bounds of the first conjugate time:

$$\lambda \in C_1 \Rightarrow t_{\text{MAX}}^1(\lambda) \leq t_{\text{conj}}^1(\lambda) \leq \frac{2}{\sqrt{\alpha}} \max(p_1^z(k), p_1^V(k)), \quad (9.1)$$

$$\lambda \in C_2 \Rightarrow t_{\text{MAX}}^1(\lambda) \leq t_{\text{conj}}^1(\lambda) \leq \frac{4}{\sqrt{\alpha}} kK. \quad (9.2)$$

- 2 We believe that the upper bounds can be proved by methods of this paper.  
 3 See the plots justifying bounds (9.1) and (9.2) of the function  $k \mapsto t_{\text{conj}}^1(\lambda)$   
 4 at Figs. 18 and 19 respectively (for  $\alpha = 1$ ), for a fixed  $\varphi$ .

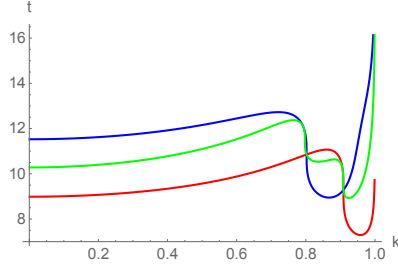


Figure 18: Bound (9.1):  $k \mapsto t_{\text{conj}}^1(\lambda)$ ,  $\varphi = \text{const}$

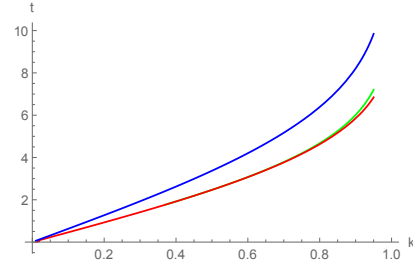


Figure 19: Bound (9.2):  $k \mapsto t_{\text{conj}}^1(\lambda)$ ,  $\varphi = \text{const}$

- 5 Figure 20 presents a plot of the function  $\varphi \mapsto t_{\text{conj}}^1(\lambda)$  for  $\lambda \in C_1$ , for a  
 6 fixed  $k$ . It shows that the first conjugate time  $t_{\text{conj}}^1(\lambda)$  is not preserved by the  
 7 flow of pendulum (1.20) (i.e., the vertical part of the Hamiltonian vector field  
 8  $\vec{H}$ ), unlike the first Maxwell time  $t_{\text{MAX}}^1(\lambda)$ . Figure 20 confirms periodicity of  
 9 the function  $\varphi \mapsto t_{\text{conj}}^1(\lambda)$ , which is a manifestation of invariance of the first  
 10 conjugate time w.r.t. the reflection  $\varepsilon^3$ , see Cor. 1.

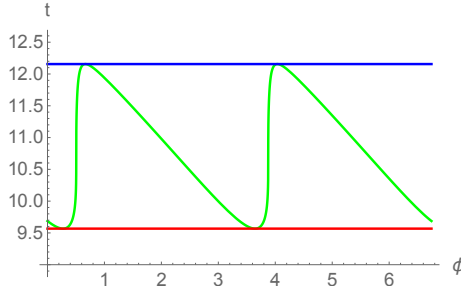


Figure 20: Bound (9.1):  $\varphi \mapsto t_{\text{conj}}^1(\lambda)$ ,  
 $k = \text{const}$

- 11 There is a numerical evidence that bound (9.1) is exact. Moreover, the both



1 bounds (9.1) and (9.2) contain just one value of  $t_{\text{conj}}^1(\lambda)$ , so they can be used  
 2 for reliable numerical evaluation of the first conjugate time.

## 3 10 Conclusion

4 In this work we used several methods for bounding conjugate points:

- 5 • direct estimate of Jacobian of the exponential mapping via comparison  
 6 functions (for  $\lambda \in C_1$ ),
- 7 • homotopy invariance of the index of the second variation (for  $\lambda \in C_2 \cup C_6$ ),
- 8 • projection to lower-dimensional SR minimizers (for  $\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7$ ).

9 We believe that these methods can be useful for bounding conjugate points in  
 10 other SR problems.

11 Using the estimate of cut time obtained in work [36] (Theorem 1) and the  
 12 estimate of conjugate time proved in this work (Theorem 2), one can prove  
 13 Conjecture 1 and get a description of global structure of the exponential map in  
 14 the SR problem on the Cartan group. So we can reduce this problem to solving  
 15 a system of algebraic equations. This will be the subject of forthcoming works.

## 16 A Appendix: 17 Explicit formulas for coefficients of Jacobian

18 In this appendix we present explicit formulas for the coefficients  $a_{01}$ ,  $a_{12}$  of the  
 19 Jacobian  $J_1$  (5.2) in the domain  $C_2$ .

$$\begin{aligned}
 a_{01} &= \sum_{j=0}^3 \sum_{i=0}^j c_{i,j-i} F^i(u_1) E^{j-i}(u_1), & (A.1) \\
 c_{00} &= -2k^2 \sin^2 u_1 \cos^2 u_1 \sqrt{1 - k^2 \sin^2 u_1}, \\
 c_{01} &= \cos u_1 \sin u_1 (4 + k^2(1 - 6 \sin^2 u_1)), \\
 c_{02} &= -3\sqrt{1 - k^2 \sin^2 u_1} (1 - 2 \sin^2 u_1), \\
 c_{11} &= (2 - k^2) \sqrt{1 - k^2 \sin^2 u_1}, \\
 c_{20} &= (1 - k^2) \sqrt{1 - k^2 \sin^2 u_1} (1 - 2 \sin^2 u_1), \\
 c_{03} &= -2 \cos u_1 \sin u_1, \\
 c_{12} &= (2 - k^2) \cos u_1 \sin u_1, \\
 c_{21} &= 2(1 - k^2) \cos u_1 \sin u_1, \\
 c_{30} &= -(1 - k^2)(2 - k^2) \cos u_1 \sin u_1,
 \end{aligned}$$

$$\begin{aligned}
a_{21} &= \sum_{j=0}^5 \sum_{i=0}^j d_{i,j-i} F^i(u_1) E^{j-i}(u_1), \\
d_{00} &= -6k^7 \sin^3 u_1 \cos^3 u_1, \\
d_{01} &= 20k^5 \sin^2 u_1 \cos^2 u_1 \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{10} &= -6k^5 (2 - k^2) \sin^2 u_1 \cos^2 u_1 \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{02} &= -2k^3 \sin u_1 \cos u_1 (12 - k^2 (1 + 10 \sin^2 u_1)), \\
d_{11} &= \frac{k^3}{2} \sin u_1 \cos u_1 (32 - 8k^2 (1 + 6 \sin^2 u_1) + 3k^4 (1 + 8 \sin^2 u_1)), \\
d_{20} &= \frac{k^3}{2} \sin u_1 \cos u_1 (16 + 3k^6 \sin^2 u_1 + k^4 (9 - 8 \sin^2 u_1) - 4k^2 (7 - 2 \sin^2 u_1)), \\
d_{03} &= 8k(2 - k^2) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{12} &= -\frac{k}{2} (32 - 32k^2 + 15k^4) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{21} &= -\frac{k}{2} (32 - 48k^2 + 10k^4 + 3k^6) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{30} &= \frac{k}{2} (32 - 64k^2 + 41k^4 - 9k^6) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{04} &= -10k^3 \sin u_1 \cos u_1, \\
d_{13} &= 12k^3 (2 - k^2) \sin u_1 \cos u_1, \\
d_{22} &= -\frac{3}{2} k^3 (8 - 8k^2 + 3k^4) \sin u_1 \cos u_1, \\
d_{31} &= -\frac{k^3}{2} (16 - 24k^2 + 6k^4 + k^6) \sin u_1 \cos u_1, \\
d_{40} &= \frac{3}{2} k^3 (1 - k^2) (2 - k^2)^2 \sin u_1 \cos u_1, \\
d_{05} &= 4k \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{14} &= -6k(2 - k^2) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{23} &= k(8 - 8k^2 + 3k^4) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{32} &= \frac{k}{2} (16 - 24k^2 + 6k^4 + k^6) \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{41} &= -3k(1 - k^2) (2 - k^2)^2 \sqrt{1 - k^2 \sin^2 u_1}, \\
d_{50} &= \frac{k}{2} (1 - k^2) (2 - k^2)^3 \sqrt{1 - k^2 \sin^2 u_1}.
\end{aligned} \tag{A.2}$$

<sup>1</sup> Here  $F(u_1)$  and  $E(u_1)$  are elliptic integrals of the first and second kinds [42].

## 1 Acknowledgement

2 The author is grateful to the reviewer whose comments allowed to improve  
3 presentation of the paper.

## 4 References

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