Extremal Controls for the Duits Car

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Abstract. We study a time minimization problem for a model of a car that can move forward on a plane and turn in place. Trajectories of this system are used in image processing for the detection of salient lines. The problem is a modification of a well-known sub-Riemannian problem in the roto-translation group, where one of the controls is restricted to be non-negative. The problem is of interest in geometric control theory as a model example in which the set of admissible controls contains zero on the boundary. We apply a necessary optimality condition—Pontryagin maximum principle to obtain a Hamiltonian system for normal extremals. By analyzing the Hamiltonian system we show a technique to obtain a single explicit formula for extremal controls. We derive the extremal controls and express the extremal trajectories in quadratures.

Keywords: Sub-Riemannian · Geometric control · Reeds-Shepp car

1 Introduction

Consider a model of an idealized car moving on a plane, see Fig. 1. The car has two parallel wheels, equidistant from the axle of the wheelset. Both wheels have independent drives that can rotate forward and backward so that the corresponding rolling of the wheels occurs without slipping. The configuration of the system is described by the triple $q = (x, y, \theta) \in \mathbb{M} = \mathbb{R}^2 \times S^1$, where $(x, y) \in \mathbb{R}^2$ is the central point, and $\theta \in S^1$ is the orientation angle of the car. Note that the configuration space $\mathbb{M}$ forms the Lie group of roto-translations $\mathrm{SE}(2) \simeq \mathbb{M} = \mathbb{R}^2 \times S^1$.

From the driver's point of view, the car has two controls: the accelerator u_1 and the steering wheel u_2. Consider the configuration $e = (0, 0, 0)$, which corresponds to the car located in the origin and oriented along the positive direction of abscissa. An infinitesimal translation is generated by the vector ∂_x and rotation—by ∂_θ. They are possible motions controlled by u_1 and u_2 respectively. The remaining direction ∂_y is forbidden since the immediate motion of the car in direction perpendicular to its wheels is not possible. Thus, the dynamics of the car in the origin is given by $\dot{x} = u_1$, $\dot{y} = 0$, $\dot{\theta} = u_2$.

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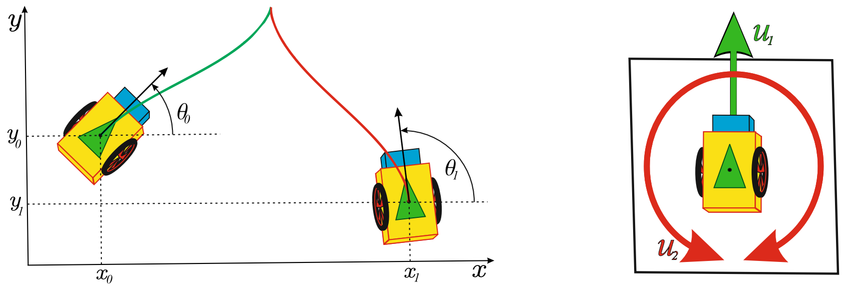

Fig. 1. Left: classical model of a car that can move forward and backward and rotate in place. Its trajectory represents a cusp when the car switches the direction of movement to opposite. Arcs of the trajectory where the car is moving forward/backward are depicted in green/red correspondingly. **Right:** control u_1 is responsible for translations and u_2 for rotations of the car. For the Duits car motion backward is forbidden $u_1 \geq 0$. (Color figure online)

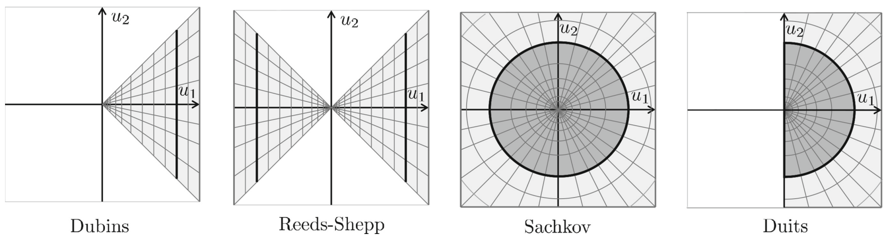

Fig. 2. Set of admissible controls for various models of a car moving on a plane.

The origin e is unit element of the group SE(2). Any other element $q \in \mathrm{SE}(2)$ is generated by left multiplication $L_q e = q \cdot e$. Dynamics in a configuration q is

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \tag{1}$$

where the vector fields X_i are obtained via push-forward of left multiplication $X_1(q) = L_{q*}\partial_x$, $X_2 = L_{q*}\partial_\theta$, and the forbidden direction is $X_3(q) = L_{q*}\partial_y$:

$$X_1(q) = \cos\theta\,\partial_x + \sin\theta\,\partial_y, \quad X_2(q) = \partial_\theta, \quad X_3(q) = \sin\theta\,\partial_x - \cos\theta\,\partial_y.$$

Various sets of admissible controls $U \ni (u_1, u_2)$ lead to different models of the car, see [1]. e.g., see Fig. 2, the time minimization problem for

- $u_1 = 1$, $|u_2| \leq \kappa$, $\kappa > 0$ leads to Dubins car [2];
- $|u_1| = 1$, $|u_2| \leq \kappa$, $\kappa > 0$ leads to Reeds-Shepp car [3];
- $u_1^2 + u_2^2 \leq 1$ leads to the model of a car, which trajectories are given by sub-Riemannian length minimizes, studied by Sachkov [4];
- $u_1 \geq 0$, $u_1^2 + u_2^2 \leq 1$ leads to the model of a car moving forward and turning in place, proposed by Duits et al. [5].

System (1) appears in robotics as a model of a car-like robot. The system also arises in the modelling of the human visual system and image processing. A mathematical model of the primary visual cortex of the brain as a sub-Riemannian structure in the space of positions and orientations is developed by Petitot [6], Citti and Sarti [7]. According to this model, contour completion occurs by minimizing the excitation energy of neurons responsible for the area of the visual field, where the contour is hidden from observation.

The principles of biological visual systems are actively used in computer vision. Based on these principles, effective methods of image processing are created, e.g.: image reconstruction [8,9], detection of salient lines in images [10].

In particular, the problem of salient curves detection arises in the analysis of medical images of the human retina when searching for blood vessels. In [10], the set of admissible controls is the disk $u_1^2+u_2^2 \leq 1$. A disadvantage of this model is the presence of cusps, see Fig. 2. Such curves are not desirable for vessel tracking. To eliminate this drawback, the restriction of the set of admissible control to a half-disc was proposed in [5]. The results of vessel tracking via the minimal paths in this model, which we call the Duits car, can be found in [11].

The problem of optimal trajectories of the Duits car with a given external cost is studied in [5]. In particular, the authors develop a numerical method for finding optimal trajectories using Fast-Marching algorithm [12]. They also study the special case of uniform external cost, which we consider in this paper, and formulate a statement that cusp points are replaced by so-called key points, which are points of in-place rotations, see [5, Theorem 3].

In this paper, we study the time-minimization problem for the Duits car. By direct application of Pontryagin maximum principle (PMP) and analysis of the Hamiltonian system of PMP, we show a formal proof of the statement regarding the replacement of cusps by key points. We also present a technique of obtaining the explicit form of extremal controls by reducing the system to the second-order ODE and solving it in Jacobi elliptic functions.

The problem under consideration is of interest in geometric control theory [13], as a model example of an optimal control problem in which zero control is located on the boundary of the set of admissible controls. A general approach [14] to similar problems is based on convex trigonometry. The approach covers the class of optimal control problems with two-dimensional control belonging to an arbitrary convex compact set containing zero in its interior. However, this approach does not admit immediate generalization to the case when zero lies on the boundary. For systems of this type, the development of new methods is required. This article examines in detail a particular case of such a system.

2 Problem Formulation

We consider the following control system:

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \begin{array}{l} (x, y, \theta) = q \in \mathrm{SE}(2) = \mathbb{M}, \\ u_1^2 + u_2^2 \leq 1, \; u_1 \geq 0. \end{array} \tag{2}$$

We study a time minimization problem, where for given boundary conditions q_0, $q_1 \in \mathbb{M}$, one aims to find the controls $u_1(t)$, $u_2(t) \in L^\infty([0,T],\mathbb{R})$, such that the corresponding trajectory $\gamma : [0,T] \to \mathbb{M}$ transfers the system from the initial state q_0 to the final state q_1 by the minimal time:

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min. \tag{3}$$

System (2) is invariant under action of SE(2). Thus, w.l.o.g., we set $q_0 = (0,0,0)$.

3 Pontryagin Maximum Principle

It can be shown that, in non-trivial case, problem on the half-disc is equivalent to the problem on the arc $u_1^2 + u_2^2 = 1$. Denote $u = u_2$, then $u_1 = \sqrt{1-u^2}$.

A necessary optimality condition is given by PMP [13]. Denote $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T^*\mathbb{M}$. It can be shown that abnormal extremals are given by in-place rotations $\dot{x} = \dot{y} = 0$. Application of PMP in normal case gives the expression of extremal control $u = h_2$ and leads to the Hamiltonian system

$$\begin{cases} \dot{x} = \sqrt{1-h_2^2}\cos\theta, \\ \dot{y} = \sqrt{1-h_2^2}\sin\theta, \\ \dot{\theta} = h_2, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = \sqrt{1-h_2^2}h_3, \\ \dot{h}_3 = h_2 h_1, \end{cases} \tag{4}$$

with the (maximized) Hamiltonian

$$H = 1 = \begin{cases} |h_2|, & \text{for } h_1 \le 0, \\ \sqrt{h_1^2 + h_2^2}, & \text{for } h_1 > 0, \end{cases} \tag{5}$$

The subsystem for state variables x, y, θ is called the *horizontal* part, and the subsystem for conjugate variables h_1, h_2, h_3 is called the *vertical* part of the Hamiltonian system. An extremal control is determined by a solution of the vertical part, while an extremal trajectory is a solution to the horizontal part.

The vertical part has the first integrals: the Hamiltonian H and the Casimir

$$E = h_1^2 + h_3^2. \tag{6}$$

Remark 1. Casimir functions are universal conservation laws on Lie groups. Connected joint level surfaces of all Casimir functions are coadjoint orbits (see [15]).

In Fig. 3 we show variants of the mutual arrangement of the level surface of the Hamiltonian $H = 1$, which consists of two half-planes glued with half of the cylinder, and the level surface of the Casimir $E \geq 0$, which is a cylinder. In Fig. 4 we show the phase portrait on the surface $H = 1$.

Depending on the sign of h_1, we have two different dynamics. When h_1 switches its sign, the dynamics switches from one to another. We denote by $t_0 \in \{t_0^0 = 0, t_0^1, t_0^2, \ldots\}$ the instance of time when the switching occurs. Note, that at the instances t_0 the extremal trajectory $(x(t), y(t), \theta(t))$ intersects the so-called 'cusp-surface' in SE(2), analytically computed and analysed in [16].

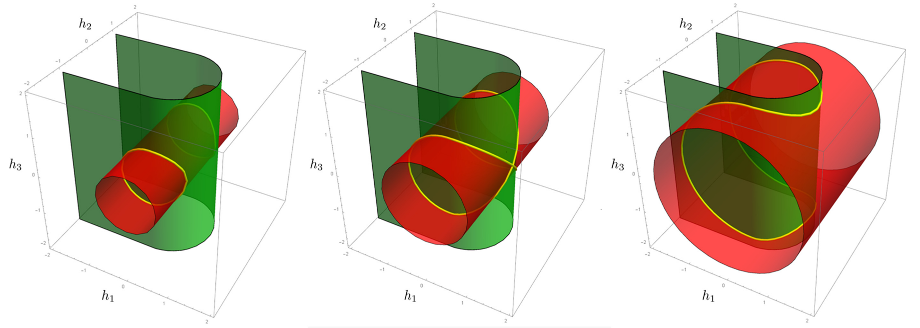

Fig. 3. Level surfaces of the Hamiltonian H (in green) and the Casimir E (in red). The intersection line is highlighted in yellow. **Left:** $E < 1$. **Center:** $E = 1$. **Right:** $E > 1$. (Color figure online)

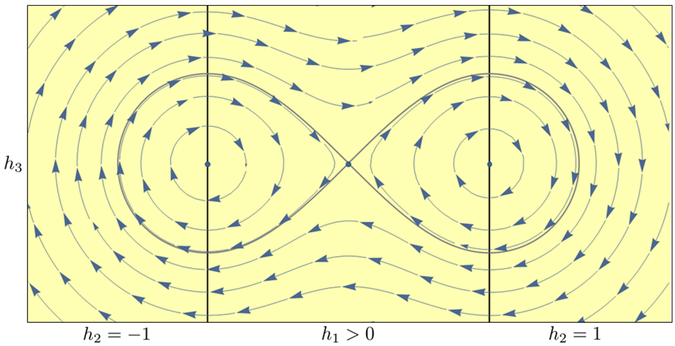

Fig. 4. Phase portrait on the level surface $H = 1$ of the Hamiltonian.

3.1 The Case $h_1 < 0$.

Due to (5), we have $h_2 = h_{20} = \pm 1$. Denote $s_2 = h_2$. We study the system

$$\begin{cases} \dot{x} = 0, & x(t_0) = x_0, \\ \dot{y} = 0, & y(t_0) = y_0, \\ \dot{\theta} = s_2, & \theta(t_0) = \theta_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -s_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_3 = s_2 h_1, & h_3(t_0) = h_{30}. \end{cases} \tag{7}$$

We immediately see that solutions to the horizontal part (extremal trajectories) are rotations around the fixed point (x_0, y_0) with constant speed $s_2 = \pm 1$.

Solutions to the vertical part are given by arcs of the circles. The motion is clockwise, when $s_2 = -1$, and counterclockwise, when $s_2 = 1$:

$$h_1(t) = h_{10} \cos(t - t_0) - s_2\, h_{30} \sin(t - t_0), \quad h_3(t) = h_{30} \cos(t - t_0) + s_2\, h_{10} \sin(t - t_0).$$

It remains to find the first instance of time $t_1 > t_0$, at which the dynamics switches. That is the moment when the condition $h_1 < 0$ ceases to be met:

$$t_1 - t_0 = \arg\left(-s_2 h_{30} - \boldsymbol{i} h_{10}\right) \in (0, \pi]. \tag{8}$$

Note that in the case $t_0 > 0$, that is, when at least one switch has already occurred, the formula (8) is reduced to $t_1 - t_0 = \pi$.

3.2 The Case $h_1 > 0$

By (5) we have $h_1 = \sqrt{1 - h_2^2}$. The Hamiltonian system of PMP has the form

$$\begin{cases} \dot{x} = h_1 \cos\theta, & x(t_0) = x_0, \\ \dot{y} = h_1 \sin\theta, & y(t_0) = y_0, \\ \dot{\theta} = h_2, & \theta(t_0) = \theta_0, \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_2 h_3, & h_1(t_0) = h_{10}, \\ \dot{h}_2 = h_1 h_3, & h_2(t_0) = h_{20}, \\ \dot{h}_3 = h_2 h_1, & h_3(t_0) = h_{30}. \end{cases} \tag{9}$$

This system is a model example in geometric control theory [13]. An explicit solution in Jacobi elliptic functions was obtained in [4], where the authors reduced the vertical part to the equation of mathematical pendulum. The solution is given by different formulas in different areas of the phase portrait. The specific form is determined by the nature of the movement of the pendulum: oscillation, rotation, movement along the separatrix, stable or unstable equilibrium.

We propose another technique that leads to an explicit parameterization of the solutions by a single formula. First, we derive the ODE on the function h_2. Then we find its explicit solution. Finally, we express the remaining components h_1, h_3 in terms of the already found function h_2 and initial conditions.

Denote $M = E - 2$. By virtue of (9), we have

$$\ddot{h}_2 + M h_2 + 2h_2^3 = 0, \tag{10}$$

with initial conditions $h_2(t_0) = h_{20} =: a$, $\dot{h}_2(t_0) = h_{10} h_{30} =: b$.

An explicit solution of this Cauchy problem, see [18, Appendix A], is given by

$$\begin{aligned} &h_2(t) = s\,\mathrm{cn}\left(\tfrac{t-t_0}{k} + sF(\alpha, k), k\right), \quad k = \tfrac{1}{\sqrt{E}}, \\ &\alpha = \arg\left(s\,a + \boldsymbol{i}\,s\sqrt{1-a^2}\right) \in (-\pi, \pi], \quad s = -\mathrm{sign}(b), \end{aligned}$$

where F denotes the elliptic integral of the first kind in the Legendre form, and cn denotes the elliptic cosine [17].

Next, we express h_1 and h_3 via h_2 and initial conditions.

Since $h_2 \in C(\mathbb{R})$ is bounded, there exists an integral

$$H_2(t) = \int_{t_0}^{t} h_2(\tau)\mathrm{d}\,\tau = s \arccos\left(\mathrm{dn}\left(\frac{t - t_0}{k} + sF(\alpha, k), k\right)\right)\Bigg|_{\tau = t_0}^{\tau = t},$$

where dn denotes the delta amplitude [17].

It can be shown that the Cauchy problem on (h_1, h_3) has a unique solution

$$h_1(t) = h_{10} \cos H_2(t) - h_{30} \sin H_2(t), \quad h_3(t) = h_{30} \cos H_2(t) + h_{10} \sin H_2(t). \tag{11}$$

The extremal trajectories are found by integration of the horizontal part:

$$x(t) = x_0 + \int_{t_0}^{t} h_1(\tau) \cos\theta(\tau)\,\mathrm{d}\tau, \quad y(t) = y_0 + \int_{t_0}^{t} h_1(\tau) \sin\theta(\tau)\,\mathrm{d}\tau, \quad \theta(t) = \theta_0 + H_2(t).$$

4 Extremal Controls and Trajectories

We summarize the previous section by formulating the following theorem

Theorem 1. *The extremal control $u(t)$ in problem (2)–(3) is determined by the parameters* $h_{20} \in [-1, 1]$, $h_{10} \in (-\infty, 0] \cup \left\{ \sqrt{1 - h_{20}^2} \right\}$, $h_{30} \in \mathbb{R}$.
Let $s_1 = \operatorname{sign}(h_{10})$ and $\sigma = \frac{s_1+1}{2}$. The function $u(t)$ is defined on time intervals formed by splitting the ray $t \geq 0$ by instances $t_0 \in \{0 = t_0^0, t_0^1, t_0^2, t_0^3, \ldots\}$ as

$$u(t) = \begin{cases} u\left(t_0^{i-\sigma}\right) \in \{-1, 1\}, & \textit{for } t \in [t_0^{i-\sigma+s_1}, t_0^{i-\sigma+s_1+1}), \\ \operatorname{scn}\left(\frac{t-t_0}{k} + sF(\alpha, k), k\right), & \textit{for } t \in [t_0^{i-\sigma}, t_0^{i-\sigma+1}), \end{cases} \tag{12}$$

where $i \in \{2n - 1 \mid n \in \mathbb{N}\}$,

$$k = \frac{1}{\sqrt{h_{10}^2 + h_{30}^2}}, \quad s = -\operatorname{sign}(h_{30}), \quad \alpha = \arg\left(s\, h_{20} + \boldsymbol{i}\, s\sqrt{1 - h_{20}^2}\right).$$

The corresponding extremal trajectory has the form $\theta(t) = \int\limits_0^t u(\tau)\,\mathrm{d}\tau$,

$$x(t) = \int\limits_0^t \sqrt{1 - u^2(\tau)} \cos\theta(\tau)\,\mathrm{d}\tau, \quad y(t) = \int\limits_0^t \sqrt{1 - u^2(\tau)} \sin\theta(\tau)\,\mathrm{d}\tau.$$

Proof relies on the expression of the extremal controls $u_1 = \sqrt{1 - u^2}$, $u_2 = h_2 = u$, which follows from maximum condition of PMP, see Sect. 3. The index i together with the parameters s_1 and σ specify the dynamics on the corresponding time interval. The dynamics switches, when h_1 changes its sign. Explicit formula (12) for the extremal control is obtained in Sect. 3.1 and Sect. 3.2. The extremal trajectory is obtained by integration of the horizontal part of (4).

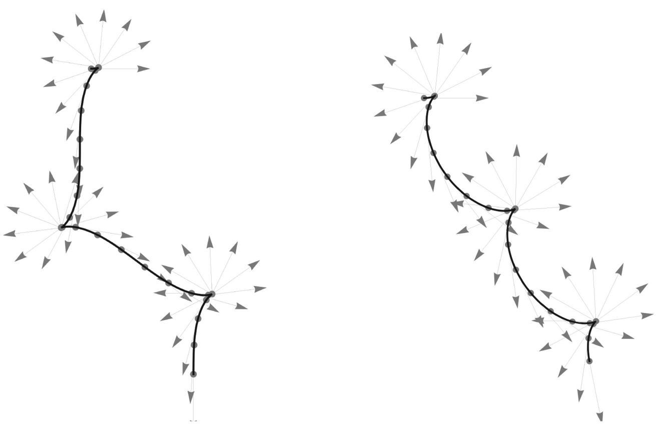

Fig. 5. Projection to the plane (x, y) of two different extremal trajectories. The gray arrow indicates the orientation angle θ at the instances of time $t \in \{0, 0.5, 1, \ldots, 20\}$. **Left:** $h_1^0 = 0.5$, $h_2^0 = \sqrt{3}/2$, $h_3^0 = 1$ **Right:** $h_1^0 = 0.5$, $h_2^0 = \sqrt{3}/2$, $h_3^0 = 0.7$.

By analyzing the solution, we note that there is a relation between the extremal trajectories of the Duits car and the sub-Riemannian geodesics in SE(2). Projection to the plane (x, y) of the trajectories in two models coincide, while the dynamics of the orientation angle θ differs: in Duits model, the angle θ uniformly increases/decreases by π radians at a cusp point. See Fig. 5. Note that an optimal motion of the Duits car can not have internal in-place rotations, see [5]. The in-place rotations may occur at the initial and the final time intervals.

References

1. Laumond, J.-P.: Feasible Trajectories for Mobile Robots with Kinematic and Environment Constraints. IAS (1986)
2. Dubins, L.E.: On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. Am. J. Math. **79**(3), 497–516 (1957)
3. Reeds, J.A., Shepp, L.A.: Optimal paths for a car that goes both forwards and backwards. Pacific J. Math. **145**(2), 367–393 (1990)
4. Sachkov, Y.L.: Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane. In: ESAIM: COCV, vol. 17, pp. 293–321 (2011)
5. Duits, R., Meesters, S.P.L., Mirebeau, J.-M., Portegies, J.M.: Optimal paths for variants of the 2D and 3D Reeds-Shepp car with applications in image analysis. J. Math. Imaging Vis. **60**, 816–848 (2018)
6. Petitot, J.: The neurogeometry of pinwheels as a sub-Riemannian contact structure. J. Physiol. Paris **97**(2–3), 265–309 (2003)
7. Citti, G., Sarti, A.: A cortical based model of perceptual completion in the roto-translation space. JMIV **24**(3), 307–326 (2006)
8. Mashtakov, A.P., Ardentov, A.A., Sachkov, Y.L.: Parallel algorithm and software for image inpainting via sub-Riemannian minimizers on the group of roto translations. NMTMA **6**(1), 95–115 (2013)
9. Boscain, U., Gauthier, J., Prandi, D., Remizov, A.: Image reconstruction via non-isotropic diffusion in Dubins/Reed-Shepp-like control systems. In: 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, pp. 4278–4283 (2014)
10. Bekkers, E.J., Duits, R., Mashtakov, A., Sanguinetti, G.R.: A PDE approach to data-driven sub-Riemannian geodesics in SE(2). SIAM JIS **8**(4), 2740–2770 (2015)
11. Scharpach, W.L.J.: Optimal paths for the Reeds-Shepp car with monotone spatial control and vessel tracking in medical image analysis. Ms. Thesis (2018)
12. Mirebeau, J.-M.: Anisotropic Fast-Marching on cartesian grids using lattice basis reduction. SIAM J. Num. Anal. **52**(4), 1573–1599 (2014)
13. Agrachev, A.A., Sachkov, Y.L.: Control theory from the geometric viewpoint. In: Agrachev, A.A. (ed.) Encyclopaedia of Mathematical Sciences. Springer, Berlin (2004). https://doi.org/10.1007/978-3-662-06404-7
14. Ardentov, A., Lokutsievskiy, L., Sachkov, Yu.L.: Explicit solutions for a series of classical optimization problems with 2-dimensional control via convex trigonometry. arXiv: Optimization and Control (2020)
15. Laurent-Gengoux, C., Pichereau, A., Vanhaecke, P.: Poisson structures. In: Pichereau, A. (ed.) Grundlehren der mathematischen Wissenschaften, p. 347. Springer-Verlag, Berlin (2013). https://doi.org/10.1007/978-3-642-31090-4

16. Duits, R., Boscain, U., Rossi, F., Sachkov, Y.: Association fields via cuspless sub-Riemannian geodesics in SE(2). J. Math. Imaging Vis. **49**(2), 384–417 (2013). https://doi.org/10.1007/s10851-013-0475-y
17. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1962)
18. M. Lakshmanan, S. Rajasekar, Nonlinear Dynamics: Integrability Chaos and Patterns. Advanced Texts in Physics. Springer-Verlag, Berlin p. 619 (2003)