Sbornik: Mathematics **202**:9 1–25

Matematicheskiĭ Sbornik **202**:9 97–120 DOI 10.1070/SM2011v202n09ABEH004190

Extremal trajectories and the asymptotics of the Maxwell time in the problem of the optimal rolling of a sphere on a plane

A. P. Mashtakov and Yu. L. Sachkov

Abstract. The problem of a sphere rolling on a plane without twisting or slipping is considered. It is required to roll the sphere from one contact configuration to another so that the length of the curve described by the contact point is minimal. A parametrization of extremal trajectories is obtained. The asymptotics of extremal trajectories and the behaviour of the Maxwell time for the rolling of a sphere over sinusoids of small amplitude are studied; for such trajectories estimates for the so-called cut time are obtained.

Bibliography: 21 titles.

Keywords: optimal control, geometric methods, symmetries of the exponential map, rolling of surfaces, Euler elastics.

§ 1. Introduction

The paper is devoted to studying the optimal rolling for a sphere on a plane without twisting or slipping. A state of the system is described by the contact point between the sphere and the plane and the orientation of the sphere in three-dimensional space. It is required to roll the sphere from a given initial state to a given terminal state so that the curve described by the contact point on the plane has minimal length. The control is the velocity of the centre of the sphere.

This problem is of great significance in robotics when modelling the motion of a sphere in a hand of a robot-manipulator. Problems involving surfaces rolling are of great interest in mechanics, robotics, and control theory (see, for example, [1]–[4]).

The problem of the optimal rolling for a sphere on a plane was posed by Hammersley [5]. Arthurs and Walsh [6] proved that the equations for extremal trajectories in this problem are integrable in terms of elliptic functions. Jurdjevic [7], [8] showed that in an optimal rolling the contact point between the sphere and the plane moves over Euler elastics (stationary configurations of an elastic rod on a plane; see [9], [10]) and described the possible types of rolling for a sphere. However, an explicit parametrization of extremal trajectories was not obtained.

The important question of when extremal trajectories are optimal remains open. Small arcs of extremal trajectories are optimal, but large arcs, generally speaking,

This research was supported by the Russian Foundation for Basic Research (grant no. 09-01-00246-a).

AMS 2010 *Mathematics Subject Classification*. Primary 49K15; Secondary 70B10, 93B27.

are not. A point where an extremal trajectory ceases to be optimal is called a cut point. The study of these points on extremal trajectories was started in [11]. Continuous and discrete symmetries of the problem were described, the corresponding Maxwell points were characterized (intersection points of extremal trajectories with the same values of the functional and time). It is known (see [12]) that an extremal trajectory cannot be optimal after a Maxwell point. In [11] the Maxwell points corresponding to continuous and discrete symmetries of the problem were described by algebraic equations in the state space.

This paper is a direct continuation of [11]. Results in two directions are obtained. First, we give an explicit parametrization of extremal trajectories by elliptic functions and integrals; to obtain this parametrization we introduce natural elliptic coordinates in the space of conjugate variables of Pontryagin's maximum principle. Second, we analyse the asymptotics of extremal trajectories when the sphere rolls along elastics close to a straight line (that is, along sinusoids of small amplitude); we study the behaviour of Maxwell points for these trajectories and obtain explicit estimates for the cut time.

Recall the statement of the optimal control problem and some well-known results. Let $(x, y) \in \mathbb{R}^2$ be the tangency point between the sphere and the plane, and let $R \in \mathrm{SO}(3)$ be the rotation of three-dimensional space taking the current orientation of the sphere to the initial one. The problem of optimal rolling for a sphere of unit radius on the plane is stated as follows (see [7], [8]):

$$\dot{x} = u_1, \qquad \dot{y} = u_2, \tag{1.1}$$

$$\dot{R} = R(u_2 A_1 - u_1 A_2), \tag{1.2}$$

$$Q = (x, y, R) \in M = \mathbb{R}^2 \times \mathrm{SO}(3), \qquad u = (u_1, u_2) \in \mathbb{R}^2, \tag{1.3}$$

$$Q(0) = Q_0 = (0, 0, \mathrm{Id}), \qquad Q(t_1) = Q_1, \tag{1.4}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{1.5}$$

Henceforth we use the basis matrices in the Lie algebra so(3)

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Admissible controls are assumed to be measurable and essentially bounded, and admissible trajectories to be Lipschitzian.

The problem (1.1)–(1.5) is a left-invariant sub-Riemannian problem on the Lie group $M = \mathbb{R}^2 \times \mathrm{SO}(3)$. We introduce the following frame on this Lie group: $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $V_i(R) = RA_i$, $i = 1, 2, 3$. In terms of the left-invariant fields $X_1 = e_1 - V_2$, $X_2 = e_2 + V_1$, the control system (1.1)–(1.3) takes the form

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \qquad Q \in M = \mathbb{R}^2 \times \mathrm{SO}(3), \quad (u_1, u_2) \in \mathbb{R}^2. \tag{1.7}$$

The functional (1.5) is the functional of sub-Riemannian length for the left-invariant sub-Riemannian structure defined by the fields X_1, X_2 as an orthonormal basis:

$$l = \int_0^{t_1} \langle \dot{Q}, \dot{Q} \rangle^{1/2}\, dt \to \min, \qquad \langle X_i, X_j \rangle = \delta_{ij}, \quad i, j = 1, 2. \tag{1.8}$$

By the Chow-Rashevskiĭ theorem (see [4]), the system (1.7) is completely controllable, that is, any two points of the state space M can be connected by a trajectory of the system. Filippov's theorem (see [4]) implies the existence of optimal controls in the problem (1.1)–(1.5). The Pontryagin maximum principle is applied to analyse optimal controls (see [13], [4]). In the abnormal case, the sphere rolls in a straight line along the (x, y)-plane. For the normal case, the Hamiltonian system of Pontryagin's maximum principle (for trajectories of unit velocity $\langle \dot{Q}, \dot{Q} \rangle \equiv 1$) was obtained in Jurdjevic's paper [7] in the following form:

$$\dot{\theta} = c, \qquad \dot{c} = -r \sin\theta, \qquad \dot{\alpha} = \dot{r} = 0, \tag{1.9}$$
$$\dot{x} = \cos(\theta + \alpha), \qquad \dot{y} = \sin(\theta + \alpha), \tag{1.10}$$
$$\dot{R} = R\Omega, \qquad \Omega = \sin(\theta + \alpha) A_1 - \cos(\theta + \alpha) A_2, \tag{1.11}$$
$$\theta \in S^1, \qquad c \in \mathbb{R}, \qquad r \geqslant 0, \qquad \alpha \in S^1, \qquad Q = (x, y, R) \in M, \tag{1.12}$$
$$Q(0) = Q_0 = (0, 0, \mathrm{Id}). \tag{1.13}$$

Equations (1.9)–(1.11) give a coordinate representation of the Hamiltonian system on the level surface $\{H = 1/2\}$ in the cotangent bundle for the Hamiltonian $H = ((h_1 - H_2)^2 + (h_2 + H_1)^2)/2$, where $h_i(\lambda) = \langle \lambda, e_i \rangle$, $H_i(\lambda) = \langle \lambda, V_i \rangle$, $\lambda \in T^*M$ (see [7], [11] for details). The subsystem (1.9) of the Hamiltonian system for the conjugate variables (θ, c, r, α) is the equation for a pendulum, and the projections of extremal curves onto the (x, y)-plane are Euler elastics — stationary configurations of an elastic rod on the plane with fixed ends and fixed tangents at the ends (see [10]). Jurdjevic [7] described various qualitative types of rolling for a sphere along elastics of various forms (inflectional, non-inflectional, a circle, a straight line), and also obtained algebraic and differential equations for the Euler angles along extremal curves (we present these equations and use some of them in § 3).

Continuous and discrete symmetries of the exponential map parametrizing the solutions of the Hamiltonian system were described in [11]:

$$\mathrm{Exp} : (\lambda, t) \mapsto Q_t, \qquad (\lambda, t) \in N = C \times \mathbb{R}_+, \quad Q_t \in M,$$
$$C = \left\{ \lambda \in T^*_{Q_0} M \;\middle|\; H(\lambda) = \frac{1}{2} \right\} = \bigl\{ (\theta, c, \alpha, r) \mid \theta \in S^1,\ c \in \mathbb{R},\ r \geqslant 0,\ \alpha \in S^1 \bigr\}.$$

The continuous symmetries $\{\Phi^\beta \mid \beta \in S^1\}$ are rotations through an angle β in the (x, y)-plane, and the discrete symmetries ε^1, ε^2, ε^3 are the reflections of the trajectories of the pendulum (1.9) with respect to the coordinate axes $\{\theta = 0\}$, $\{c = 0\}$ and the origin $(\theta, c) = (0, 0)$, respectively. The action of symmetries was determined both in the inverse image N and the image M of the exponential map. A description of the Maxwell sets corresponding to the symmetries ε^i, $i = 1, 2, 3$, was obtained:

$$\begin{aligned} \mathrm{MAX}^i = \bigl\{ &(\lambda, t) \in N \mid \exists\, \beta \in S^1 : \\ &(\widetilde{\lambda}, t) = \varepsilon^i \circ \Phi^\beta(\lambda, t),\ \mathrm{Exp}(\lambda, s) \not\equiv \mathrm{Exp}(\widetilde{\lambda}, s),\ \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(\widetilde{\lambda}, t) \bigr\}. \end{aligned}$$

In particular, the following assertion was proved for the symmetry ε^1.

Theorem 1.1 (see [11], Theorem 2). *Suppose that* $t > 0$ *and* $Q_s = (x_s, y_s, R_s) = \operatorname{Exp}(\lambda, s)$ *is an extremal trajectory such that*

(i) $q_3(t) = 0$;

(ii) *the elastic* $\{(x_s, y_s) \mid s \in [0, t]\}$ *is not degenerate and not centred at a point of inflection.*

Then $(\lambda, t) \in \mathrm{MAX}^1$; *therefore the trajectory* Q_s, $s \in [0, t_1]$, *is not optimal for any* $t_1 > t$.

Here, $q = q_0 + iq_1 + jq_2 + kq_3$ is the quaternion of unit length corresponding to the matrix $R \in \mathrm{SO}(3)$ (see details in § 4, as well as in [11]). In § 6 the behaviour of the Maxwell points $(\lambda, t) \in \mathrm{MAX}^1$ near the stable equilibrium position of the pendulum equations (1.9) is analysed. In [11], propositions similar to Theorem 1.1 were obtained for the Maxwell sets MAX^2, MAX^3.

This paper has the following structure. In § 2, we construct a partition of the cylinder C into subsets corresponding to motions of the pendulum (1.9) of the same type. Using this partition, elliptic coordinates rectifying the phase flow of the pendulum are introduced on a subset of full measure of the cylinder C. In § 3, these coordinates are used to obtain a parametrization of extremal trajectories.

In §§ 4–6 the asymptotics of extremal curves and the behaviour of the Maxwell set MAX^1 near the stable equilibrium position of the pendulum are analysed. Using the fact that this set is described in terms of quaternions, in § 4 a control system is derived describing the variation of the quaternion of unit length q corresponding to the rotation matrix R. In § 5, asymptotic expansions of the trajectories of this system near the stable equilibrium position of the pendulum are calculated. In § 6 the behaviour of the Maxwell set MAX^1 for the corresponding extremal trajectories is analysed; the asymptotics of the function $q_3(t)$ defining this set are analysed; estimates for the cut time along extremal trajectories corresponding to small oscillations of the pendulum (1.9) are obtained.

§ 2. Elliptic coordinates in the inverse image of the exponential map

The pendulum equations (1.9) have the energy integral

$$E = \frac{c^2}{2} - r\cos\theta \in [-r, +\infty).$$

The cylinder C is partitioned into the following invariant subsets of this equation:

$$C = \bigcup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \varnothing, \quad i \neq j,$$

$$C_1 = \{\lambda \in C \mid E \in (-r, r),\ r > 0\}, \qquad C_2 = \{\lambda \in C \mid E \in (r, +\infty),\ r > 0\},$$
$$C_3 = \{\lambda \in C \mid E = r > 0,\ c \neq 0\}, \qquad C_4 = \{\lambda \in C \mid E = -r,\ r > 0\},$$
$$C_5 = \{\lambda \in C \mid E = r > 0,\ c = 0\}, \qquad C_6 = \{\lambda \in C \mid r = 0,\ c \neq 0\},$$
$$C_7 = \{\lambda \in C \mid r = 0,\ c = 0\}.$$

The pendulum equations (1.9) can easily be integrated for $\lambda \in \bigcup_{i=4}^{7} C_i$. However, for $\lambda \in \bigcup_{i=1}^{3} C_i$, to integrate these equations and the complete Hamiltonian system (1.9)–(1.11) we shall need special elliptic coordinates. Similar coordinates were used to study several optimal control problems for which the conjugate system of Pontryagin's maximum principle reduces to the pendulum equation (see [12], [14], [15]).

In the domain $\widehat{C} = \bigcup_{i=1}^{3} C_i$, the elliptic coordinates (φ, k, α, r) are introduced as follows (henceforth, the Jacobi functions sn, cn, dn, E and the complete elliptic integral of the first kind K are used; see [16], [17]).

If $\lambda = (\theta, c, \alpha, r) \in C_1$, then

$$\sin\biggl(\frac{\theta}{2}\biggr) = k\,\mathrm{sn}(\sqrt{r}\varphi, k), \qquad \cos\biggl(\frac{\theta}{2}\biggr) = \mathrm{dn}(\sqrt{r}\varphi, k), \qquad \frac{c}{2} = k\sqrt{r}\,\mathrm{cn}(\sqrt{r}\varphi, k);$$

here, $k = \sqrt{(E + r)/(2r)} \in (0, 1)$ and $\sqrt{r}\varphi \pmod{4K} \in [0, 4K]$.

If $\lambda = (\theta, c, \alpha, r) \in C_2$, then

$$\sin\biggl(\frac{\theta}{2}\biggr) = \pm\,\mathrm{sn}\biggl(\frac{\sqrt{r}\varphi}{k}, k\biggr), \qquad \cos\biggl(\frac{\theta}{2}\biggr) = \mathrm{cn}\biggl(\frac{\sqrt{r}\varphi}{k}, k\biggr),$$
$$\frac{c}{2} = \frac{\pm\sqrt{r}}{k}\,\mathrm{dn}\biggl(\frac{\sqrt{r}\varphi}{k}, k\biggr),$$

where $\pm = \operatorname{sgn} c$; here, $k = \sqrt{2r/(E + r)} \in (0, 1)$ and $\sqrt{r}\varphi \pmod{2kK} \in [0, 2kK]$.

If $\lambda \in C_3$, then

$$\sin\biggl(\frac{\theta}{2}\biggr) = \pm\tanh(\sqrt{r}\varphi), \qquad \cos\biggl(\frac{\theta}{2}\biggr) = \frac{1}{\cosh(\sqrt{r}\varphi)}, \qquad \frac{c}{2} = \frac{\pm\sqrt{r}}{\cosh(\sqrt{r}\varphi)},$$

where $\pm = \operatorname{sgn} c$; here, $k = 1$ and $\varphi \in (-\infty, +\infty)$.

The coordinate k is the re-parametrized energy of the pendulum. Direct differentiation shows that the pendulum equations (1.9) are rectified in the coordinates (φ, k, α, r):

$$\dot{\varphi} = 1, \qquad \dot{k} = 0, \qquad \dot{\alpha} = 0, \qquad \dot{r} = 0, \tag{2.1}$$

that is, the coordinate φ is the time of motion of the pendulum.

§ 3. Parametrization of extremals

3.1. Integrating the subsystem for the conjugate variables. If

$$\lambda = (\varphi, k, \alpha, r) \in \widehat{C},$$

then by equations (2.1) in elliptic coordinates the trajectories of the pendulum (1.9) have the form $\varphi_t = \varphi + t$, $k, \alpha, r = \mathrm{const}$. Taking the expressions for the elliptic coordinates in the domain $\widehat{C}$ (see § 2) into account, we obtain the following parametrization of the solutions (θ_t, c_t) of the system (1.9).

If $\lambda \in C_1$, then

$$\sin\biggl(\frac{\theta_t}{2}\biggr) = k\,\mathrm{sn}(\sqrt{r}\varphi_t, k), \qquad \cos\biggl(\frac{\theta_t}{2}\biggr) = \mathrm{dn}(\sqrt{r}\varphi_t, k), \qquad \frac{c_t}{2} = k\sqrt{r}\,\mathrm{cn}(\sqrt{r}\varphi_t, k).$$

If $\lambda \in C_2$, then

$$\sin\left(\frac{\theta_t}{2}\right) = \pm\,\mathrm{sn}\left(\frac{\sqrt{r}\varphi_t}{k}, k\right), \qquad \cos\left(\frac{\theta_t}{2}\right) = \mathrm{cn}\left(\frac{\sqrt{r}\varphi_t}{k}, k\right),$$
$$\frac{c_t}{2} = \frac{\pm\sqrt{r}}{k}\,\mathrm{dn}\left(\frac{\sqrt{r}\varphi_t}{k}, k\right), \qquad \pm = \operatorname{sgn} c.$$

If $\lambda \in C_3$, then

$$\sin\left(\frac{\theta_t}{2}\right) = \pm\tanh(\sqrt{r}\varphi_t), \qquad \cos\left(\frac{\theta_t}{2}\right) = \frac{1}{\cosh(\sqrt{r}\varphi_t)},$$
$$\frac{c_t}{2} = \frac{\pm\sqrt{r}}{\cosh(\sqrt{r}\varphi_t)}, \qquad \pm = \operatorname{sgn} c.$$

For the cases $\lambda \in \bigcup_{i=4}^{7} C_i$, the system (1.9) can be integrated directly:

$$\begin{aligned}
&\theta_t \equiv 0,\ c_t \equiv 0 && \text{for } \lambda \in C_4;\\
&\theta_t \equiv \pi,\ c_t \equiv 0 && \text{for } \lambda \in C_5;\\
&\theta_t = ct + \theta,\ c_t \equiv c \neq 0 && \text{for } \lambda \in C_6;\\
&\theta_t \equiv \theta,\ c_t \equiv 0 && \text{for } \lambda \in C_7.
\end{aligned}$$

3.2. Integrating the equations for x, y. To integrate equations (1.10) with the initial conditions $x_0 = y_0 = 0$, we use the symmetry of the problem — the rotation

$$\overline{x} = x\cos\alpha + y\sin\alpha, \qquad \overline{y} = -x\sin\alpha + y\cos\alpha. \tag{3.1}$$

In the new variables we obtain the Cauchy problem

$$\dot{\overline{x}}_t = \cos\theta_t, \qquad \dot{\overline{y}}_t = \sin\theta_t, \qquad \overline{x}_0 = \overline{y}_0 = 0. \tag{3.2}$$

Using the expressions for $\sin(\theta_t/2)$, $\cos(\theta_t/2)$ obtained in § 3.1, we integrate equations (3.2) for $\lambda \in \widehat{C}$ and obtain the following parametrization of the Euler elastics $(\overline{x}_t, \overline{y}_t)$.

If $\lambda \in C_1$, then

$$\overline{x}_t = \frac{2(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)) - \sqrt{r}t}{\sqrt{r}}, \qquad \overline{y}_t = \frac{2k(\mathrm{cn}(\sqrt{r}\varphi) - \mathrm{cn}(\sqrt{r}\varphi_t))}{\sqrt{r}}.$$

If $\lambda \in C_2$, then

$$\overline{x}_t = \frac{2(\mathrm{E}(\sqrt{r}\varphi_t/k) - \mathrm{E}(\sqrt{r}\varphi/k) - (2-k^2)\sqrt{r}t/(2k))}{k\sqrt{r}},$$
$$\overline{y}_t = \frac{\pm 2(\mathrm{dn}(\sqrt{r}\varphi/k) - \mathrm{dn}(\sqrt{r}\varphi_t/k))}{k\sqrt{r}}, \qquad \pm = \operatorname{sgn} c.$$

If $\lambda \in C_3$, then

$$\overline{x}_t = \frac{2(\tanh(\sqrt{r}\varphi_t) - \tanh(\sqrt{r}\varphi)) - \sqrt{r}t}{\sqrt{r}},$$
$$\overline{y}_t = \frac{\pm 2(1/\cosh(\sqrt{r}\varphi) - 1/\cosh(\sqrt{r}\varphi_t))}{\sqrt{r}}, \qquad \pm = \operatorname{sgn} c.$$

For $\lambda \in \bigcup_{i=4}^{7} C_i$, equations (3.2) can be integrated directly:

$$
\begin{gathered}
\overline{x}_t = t, \quad \overline{y}_t = 0 \qquad \text{for } \lambda \in C_4; \\
\overline{x}_t = -t, \quad \overline{y}_t = 0 \qquad \text{for } \lambda \in C_5; \\
\overline{x}_t = \frac{\sin(ct+\theta) - \sin\theta}{c}, \quad \overline{y}_t = \frac{\cos\theta - \cos(ct+\theta)}{c} \qquad \text{for } \lambda \in C_6; \\
\overline{x}_t = t\cos\theta, \quad \overline{y}_t = t\sin\theta \qquad \text{for } \lambda \in C_7.
\end{gathered}
$$

In terms of the elliptic coordinates, it is natural to give conditions for an elastic $\{(x_s, y_s) \mid s \in [0,t]\}$ to be centred at a point of inflection (at a vertex) that is, conditions for the midpoint of the elastic $(x_{t/2}, y_{t/2})$ to be a point of inflection (respectively, a vertex). We define a variable τ as follows:

$$
\begin{aligned}
\tau &= \sqrt{r}\left(\varphi + \frac{t}{2}\right) \quad \text{for } \lambda \in C_1 \cup C_3, \\
\tau &= \frac{\sqrt{r}(\varphi + \frac{t}{2})}{k} \quad \text{for } \lambda \in C_2.
\end{aligned}
$$

Lemma 3.1. *Let* $\lambda \in \widehat{C}$, $\operatorname{Exp}(\lambda, s) = (x_s, y_s, R_s)$, $t > 0$, *and let* $\gamma = \{(x_s, y_s) \mid s \in [0,t]\}$.

(i) *The elastic* γ *is centred at a point of inflection if and only if* $\lambda \in C_1$ *and* $\operatorname{cn}\tau = 0$.

(ii) *The elastic* γ *is centred at a vertex if and only if* $\operatorname{sn}\tau = 0$ *for* $\lambda \in C_1$, $\operatorname{sn}\tau\operatorname{cn}\tau = 0$ *for* $\lambda \in C_2$, $\tau = 0$ *for* $\lambda \in C_3$.

Proof. The curvature of the elastic γ at a point (x_s, y_s) is equal to $\dot{\theta}_s = c_s$. For $\lambda \in \widehat{C}$ the elastic γ is not degenerate (is not a rectilinear segment or an arc of a circle); therefore the points of inflection of γ are determined by the condition $c_s = 0$, and the vertices by the condition $\dot{c}_s = -r\sin\theta_s = 0$.

(i) The midpoint of the elastic $(x_{t/2}, y_{t/2})$ is a point of inflection if and only if $c_{t/2} = 0$. We use the expressions for the component c_s of extremals in elliptic coordinates (§ 3.1). If $\lambda \in C_1$, then the equation $c_{t/2} = 0$ is equivalent to $\operatorname{cn}\tau = 0$; if $\lambda \in C_2$, then $c_{t/2} \neq 0$, since $\operatorname{dn}\tau \neq 0$; if $\lambda \in C_3$, then $c_{t/2} \neq 0$, since $1/\cosh\tau \neq 0$.

Part (ii) is proved in similar fashion.

3.3. Integrating the equation for *R*. To integrate equation (1.11) we use the following results of Jurdjevic [7]. Along extremal trajectories, the matrix R and the vector

$$
\widetilde{P} = (H_1, H_2, H_3)^T = \big(\sin(\theta+\alpha) - r\sin\alpha, r\cos\alpha - \cos(\theta+\alpha), c\big)^T
$$

satisfy the identity $R\widetilde{P} \equiv \text{const} \in \mathbb{R}^3$, $|R\widetilde{P}|^2 = 1 + r^2 + 2E =: M$. Suppose that $M > 0$. If $R\widetilde{P} = (0, 0, \sqrt{M})^T$, then the rotation matrix has the factorization $R(t) = e^{\varphi_1(t)A_3} e^{\varphi_2(t)A_2} e^{(\varphi_3(t)-\alpha)A_3}$, where the Euler angles φ_1, φ_2, φ_3 satisfy the equations

$$\cos\varphi_2 = \frac{c}{\sqrt{M}}, \qquad \sin\varphi_2 = \frac{\pm\sqrt{M-c^2}}{\sqrt{M}}, \tag{3.3}$$

$$\cos\varphi_3 = \frac{\mp\sin\theta}{\sqrt{M-c^2}}, \qquad \sin\varphi_3 = \frac{\pm(r-\cos\theta)}{\sqrt{M-c^2}}, \tag{3.4}$$

$$\dot\varphi_1 = \frac{\sqrt{M}(1-r\cos\theta)}{M-c^2}. \tag{3.5}$$

Equations (3.3)–(3.5) are well defined if $M-c^2>0$. Using the equation

$$M-c^2 = (1-r)^2 + 4r\sin^2\left(\frac{\theta}{2}\right),$$

for $r\neq 1$ we have $M-c^2>0$. If $r=1$, then by continuity equations (3.3)–(3.5) turn into the following equations:

$$\cos\varphi_2 = \frac{c}{\sqrt{M}}, \qquad \sin\varphi_2 = \frac{\pm 2\sin\frac{\theta}{2}}{\sqrt{M}}, \tag{3.6}$$

$$\cos\varphi_3 = \mp\cos\left(\frac{\theta}{2}\right), \qquad \sin\varphi_3 = \pm\sin\left(\frac{\theta}{2}\right), \tag{3.7}$$

$$\dot\varphi_1 = \frac{\sqrt{M}}{2}. \tag{3.8}$$

If $R\widetilde{P}$ is an arbitrary vector in $\mathbb{R}^3$ of length $\sqrt{M}$, then a suitable rotation takes it to the vector $(0,0,\sqrt{M})^T$. Using the invariance of the problem under left translations on SO(3), we obtain the following expression for the rotation matrix:

$$R(t) = e^{(\alpha-\varphi_3^0)A_3} e^{-\varphi_2^0 A_2} e^{\varphi_1(t)A_3} e^{\varphi_2(t)A_2} e^{(\varphi_3(t)-\alpha)A_3}, \tag{3.9}$$

where the angles φ_i are determined from relations (3.3)–(3.5) for $r\neq 1$, and (3.6)–(3.8) for $r=1$, and the angle φ_1 satisfies the initial condition $\varphi_1^0=0$.

The matrix exponentials containing φ_2, φ_3 that occur in the factorization (3.9) are expressed in terms of the functions $\cos\varphi_2$, $\sin\varphi_2$, $\cos\varphi_3$ and $\sin\varphi_3$, which by relations (3.3), (3.4), (3.6) and (3.7) are expressed in terms of the variables c, $\cos(\theta/2)$, $\sin(\theta/2)$, which are in turn represented in § 3.1 as functions of the elliptic coordinates, or directly. For $r=1$ we have $\varphi_1(t)=\sqrt{M}t/2$. We put off dealing with integrating equation (3.5) for $r\neq 1$ until § 3.4.

In the case $M=0$ we have $r=1$, $c=0$, $\theta=0$, whence, $u_1=\cos\alpha$, $u_2=\sin\alpha$. Therefore, $\Omega = u_2A_1 - u_1A_2 \equiv \text{const}$ and $R(t)=e^{t\Omega}$.

3.4. Integrating the equation for $\boldsymbol{\varphi_1}$. To integrate equation (3.5) with the initial condition $\varphi_1(0)=0$ we transform the right-hand side of this equation:

$$\sqrt{M}\frac{1-r\cos\theta}{M-c^2} = \sqrt{M}\left(\frac{1}{2} + \frac{1-r^2}{2(M-c^2)}\right).$$

1) Let $\lambda \in C_1$; then from § 3.1 we obtain $c_s = 2k\sqrt{r}\,\mathrm{cn}(\sqrt{r}(\varphi+s))$, and from § 2 we obtain $E = 2k^2r - r$, whence $M = (1-r)^2 + 4k^2r$. We transform the integral:

$$\int_0^t \frac{ds}{M - c_s^2} = \int_0^t \frac{ds}{(1-r)^2 + 4k^2r\,\mathrm{sn}^2(\sqrt{r}(\varphi+s))}$$
$$= \frac{1}{\sqrt{r}(1-r)^2} \int_{\sqrt{r}\varphi}^{\sqrt{r}(\varphi+t)} \frac{dp}{1 - l\,\mathrm{sn}^2 p}, \qquad l = -\frac{4k^2r}{(1-r)^2}.$$

We introduce into consideration the elliptic integral of the third kind in the following form:

$$\Pi(n,u,k) = \int_0^u \frac{dt}{(1-n\sin^2 t)\sqrt{1-k^2\sin^2 t}} = \int_0^{F(u,k)} \frac{dv}{1-n\,\mathrm{sn}^2 v}.$$

Then

$$\int_0^t \frac{ds}{M-c_s^2} = \frac{1}{\sqrt{r}(1-r)^2}\big(\Pi(l,\mathrm{am}(\sqrt{r}(\varphi+t)),k) - \Pi(l,\mathrm{am}(\sqrt{r}\varphi),k)\big),$$

and so

$$\varphi_1(t) = \frac{\sqrt{M}}{2}t + \frac{\sqrt{M}(1+r)}{2\sqrt{r}(1-r)}\big(\Pi(l,\mathrm{am}(\sqrt{r}(\varphi+t)),k) - \Pi(l,\mathrm{am}(\sqrt{r}\varphi),k)\big),$$

where $l = -4k^2r/(1-r)^2$. Henceforth we shall use the elliptic integral of the first kind F and the Jacobi amplitude am (see [16]).

2) Let $\lambda \in C_2$; then $c_s = \pm 2\sqrt{r}/k\,\mathrm{dn}(\sqrt{r}(\varphi+t)/k)$, $M = (1-r)^2 + 4r/k^2$, and a similar calculation gives

$$\varphi_1(t) = \frac{\sqrt{M}}{2}t + \frac{\sqrt{M}k(1+r)}{2\sqrt{r}(1-r)}\left(\Pi\left(l,\mathrm{am}\left(\frac{\sqrt{r}(\varphi+t)}{k}\right),k\right) - \Pi\left(l,\mathrm{am}\left(\frac{\sqrt{r}\varphi}{k}\right),k\right)\right),$$

where $l = -4r/(1-r)^2$.

3) Let $\lambda \in C_3$; then $c_s = \pm 2\sqrt{r}/\cosh(\sqrt{r}(\varphi+s))$, $M = (1+r)^2$, and the angle φ_1 is expressed in terms of elementary functions:

$$\varphi_1(t) = \frac{\sqrt{M}}{2}t + \frac{\sqrt{M}k(1-r^2)}{8r^{3/2}}\big(I(\sqrt{r}(\varphi+t),a) - I(\sqrt{r}\varphi,a)\big),$$
$$I(v,a) = \int_0^v \frac{dt}{a^2+\tanh^2 t} = \frac{at - \arctan a + \arctan(e^t(a^2\cosh t + \sinh t)/a)}{a+a^3},$$

where $a = (1-r)/(2\sqrt{r})$.

4) Let $\lambda \in C_6$; then $r = 0$, $c \equiv \mathrm{const} \neq 0$, and $M = 1 + c^2$, and so $\dot\varphi_1 = \sqrt{M} = \sqrt{1+c^2}$ and $\varphi_1(t) = \sqrt{1+c^2}t$.

5) When $\lambda \in C_4 \cup C_5 \cup C_7$ the expression for the rotation matrix is calculated directly:

$$\theta_t \equiv \mathrm{const} = \theta, \qquad \Omega = \sin(\alpha+\theta)A_1 - \cos(\alpha+\theta)A_2 \equiv \mathrm{const}, \qquad R(t) = e^{t\Omega}.$$

Thus, for the problem of optimal rolling for a sphere on a plane we have obtained a parametrization of the normal extremals, that is, the trajectories of the Hamiltonian system (1.9)–(1.13). The equations for the conjugate variables θ, c are integrated in § 3.1; the equations for the contact point between the sphere and the plane $(x, y) \in \mathbb{R}^2$ are integrated in § 3.2; finally, the equation for the rotation matrix of the sphere $R \in \mathrm{SO}(3)$ is integrated in §§ 3.3, 3.4. By contrast with related optimal control problems in which the subsystem for the conjugate variables of the maximum principle is reduced to the pendulum equation — the sub-Riemannian problem in the Martinet case (see [18]), the nilpotent sub-Riemannian problem with growth vector $(2, 3, 5)$ (see [17]), the Euler elastic problem (see [14]), the sub-Riemannian problem on the group of motions of the plane (see [15]), — where the extremals are parametrized by the Jacobi functions cn, sn, dn, E, in the problem of a sphere rolling on a plane the elliptic integral of the third kind Π appears as well. This presents a substantial new difficulty in the analysis of the problem of the rolling of a sphere on a plane and indicates that it has a more complicated nature by comparison with the other problems mentioned above. Other characteristics of this kind will be pointed out in § 6 after our investigation of the limiting behaviour of the Maxwell set and cut time.

§ 4. A control system in terms of quaternions

To describe the variation in the orientation of a rolling sphere it is convenient to use quaternions along with the rotation matrix R. In [11] quaternions were used to obtain the equations of the Maxwell sets. In this section we derive the equations of the control system describing the rolling of a sphere on a plane in terms of quaternions.

Let

$$\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, \dots, q_3 \in \mathbb{R}\}$$

be the quaternion algebra,

$$S^3 = \{q \in \mathbb{H} \mid |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$$

the unit sphere, and

$$I = \{q \in \mathbb{H} \mid \operatorname{Re} q = q_0 = 0\}$$

the subspace of purely imaginary quaternions. Every quaternion $q \in S^3$ defines a rotation in Euclidean space I (see [19], [20]):

$$q \in S^3 \quad \Longrightarrow \quad R_q(a) = qaq^{-1}, \quad a \in I, \quad R_q \in \mathrm{SO}(3) \cong \mathrm{SO}(I).$$

The map $p\colon q \mapsto R_q$ is a double cover S^3 over $\mathrm{SO}(3)$: $R_q = R_{\widehat{q}}$ if and only if $q = \pm\widehat{q}$. Because the projection $p\colon S^3 \to \mathrm{SO}(3)$ is a local diffeomorphism, any vector field V on $\mathrm{SO}(3)$ has a unique lift to S^3, that is, a vector field W on S^3 such that $p_*W = V$. Therefore the control system $\dot{R} = R(u_2A_1 - u_1A_2)$ also has a unique lift to S^3. To calculate this lift we use the expression for the matrix $R_q = p(q) \in \mathrm{SO}(3)$, $q = q_0 + iq_1 + jq_2 + kq_3 \in S^3$, given in [21]:

$$R = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\, q_1\, q_2 - 2\, q_0\, q_3 & 2\, q_0\, q_2 + 2\, q_1\, q_3 \\ 2\, q_1\, q_2 + 2\, q_0\, q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2\, q_0\, q_1 + 2\, q_2\, q_3 \\ -2\, q_0\, q_2 + 2\, q_1\, q_3 & 2\, q_0\, q_1 + 2\, q_2\, q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}. \tag{4.1}$$

Differentiating this matrix, in view of the system $\dot{R} = R(u_2 A_1 - u_1 A_2)$, we obtain a system of equations with respect to $\dot{q}_0$, $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$. By solving this system we obtain the following control system on S^3:

$$\begin{cases} \dot{q}_0 = \dfrac{1}{2}(q_2 u_1 - q_1 u_2), \\ \dot{q}_1 = \dfrac{1}{2}(q_3 u_1 + q_0 u_2), \\ \dot{q}_2 = \dfrac{1}{2}(-q_0 u_1 + q_3 u_2), \\ \dot{q}_3 = \dfrac{1}{2}(-q_1 u_1 - q_2 u_2), \end{cases} \qquad q = q_0 + iq_1 + jq_2 + kq_3 \in S^3, \quad (u_1, u_2) \in \mathbb{R}^2. \tag{4.2}$$

The matrix $R(t)$ satisfying the system $\dot{R} = R(u_2 A_1 - u_1 A_2)$ and the initial condition $R(0) = \mathrm{Id}$ is uniquely associated with the quaternion $q(t)$ satisfying the system (4.2) and the initial condition $q(0) = 1$. Thus, the control system (1.1), (1.2) on $\mathbb{R}^2 \times \mathrm{SO}(3)$ has a lift to $\mathbb{R}^2 \times S^3$ of the form (1.1), (4.2) with the initial conditions $(x, y)(0) = (0, 0)$, $q(0) = 1$.

§ 5. Asymptotics of extremal trajectories

The study of extremal trajectories in the problem under consideration is of great interest. In view of the complexity of the parameter equations of these trajectories, it is quite difficult to conduct this study in full. In this section and the next we begin this study with the asymptotic case corresponding to small oscillations of the pendulum (1.9); in these asymptotics, the elastics (x, y) along which the sphere is rolling are represented by sinusoids of small amplitude.

In this section we derive the asymptotics of extremal trajectories mentioned above. To represent the orientation of the sphere in space we do not use the rotation matrix $R \in \mathrm{SO}(3)$ but the quaternion $q \in S^3$, since the Maxwell sets used to determine the optimality of extremal trajectories were described in [11] in terms of quaternions. Therefore we use the control system (4.2) introduced in § 4. The corresponding normal Hamiltonian system of the maximum principle has the form

$$\dot{\theta} = c, \qquad \dot{c} = -r \sin\theta, \qquad \dot{\alpha} = \dot{r} = 0, \tag{5.1}$$

$$\dot{x} = u_1, \qquad \dot{y} = u_2, \tag{5.2}$$

$$\dot{q}_0 = \frac{1}{2}(q_2 u_1 - q_1 u_2), \tag{5.3}$$

$$\dot{q}_1 = \frac{1}{2}(q_3 u_1 + q_0 u_2), \tag{5.4}$$

$$\dot{q}_2 = \frac{1}{2}(-q_0 u_1 + q_3 u_2), \tag{5.5}$$

$$\dot{q}_3 = \frac{1}{2}(-q_1 u_1 - q_2 u_2), \tag{5.6}$$

$$u_1 = \cos(\theta + \alpha), \qquad u_2 = \sin(\theta + \alpha), \tag{5.7}$$

$$(x, y)(0) = (0, 0), \qquad (q_0, q_1, q_2, q_3)(0) = (1, 0, 0, 0). \tag{5.8}$$

Suppose that $\lambda = (\theta, c, \alpha, r) \in C_1$, so that $r > 0$. We assume that

$$r \in [r_{\min}, r_{\max}], \qquad r_{\max} > r_{\min} > 0, \tag{5.9}$$

and derive the asymptotics of solutions of the system (5.1)–(5.8) as $\theta_0^2 + c_0^2 \to 0$.

In the system (5.1)–(5.8) we go over to new variables:

$$(t, \theta, c, \alpha, r, x, y, u_1, u_2, q_0, q_1, q_2, q_3) \to (s, \theta, d, \alpha, m, \overline{x}, \overline{y}, \overline{u}_1, \overline{u}_2, \overline{q}_0, \overline{q}_1, \overline{q}_2, \overline{q}_3),$$
$$s = mt, \qquad d = \frac{c}{m}, \qquad m = \sqrt{r},$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A(\alpha) \begin{pmatrix} \overline{u}_1 \\ \overline{u}_2 \end{pmatrix}, \quad \text{where } A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = A(\alpha) \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A(\alpha) \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \end{pmatrix}, \quad \begin{cases} q_0 = \overline{q}_0, \\ q_3 = \overline{q}_3. \end{cases}$$

We denote differentiation with respect to the new time $\frac{\mathrm{d}}{\mathrm{d}s}$ by $'$. In the new variables the system (5.1)–(5.8) takes the form

$$\theta' = d, \qquad d' = -\sin\theta, \qquad \alpha' = m' = 0, \tag{5.10}$$
$$\overline{x}' = \frac{\overline{u}_1}{m}, \qquad \overline{y}' = \frac{\overline{u}_2}{m}, \qquad \overline{u}_1 = \cos\theta, \qquad \overline{u}_2 = \sin\theta, \tag{5.11}$$
$$\overline{q}_0' = \frac{1}{2m}(\overline{q}_2\overline{u}_1 - \overline{q}_1\overline{u}_2), \qquad \overline{q}_1' = \frac{1}{2m}(\overline{q}_3\overline{u}_1 + \overline{q}_0\overline{u}_2), \tag{5.12}$$
$$\overline{q}_2' = \frac{1}{2m}(-\overline{q}_0\overline{u}_1 + \overline{q}_3\overline{u}_2), \qquad \overline{q}_3' = \frac{1}{2m}(-\overline{q}_1\overline{u}_1 - \overline{q}_2\overline{u}_2), \tag{5.13}$$
$$(\overline{x}, \overline{y})(0) = (0, 0), \qquad (\overline{q}_0, \overline{q}_1, \overline{q}_2, \overline{q}_3)(0) = (1, 0, 0, 0). \tag{5.14}$$

By condition (5.9), $O(\theta_0^2 + c_0^2)$ is $O(\theta_0^2 + d_0^2)$, and conversely. We now calculate the asymptotics of solutions of the system (5.10)–(5.14) with the initial condition $\theta(0) = \theta_0$, $d(0) = d_0$ as $\theta_0^2 + d_0^2 \to 0$ to within $O(\theta_0^2 + d_0^2)$. We set $\rho_0 = \sqrt{\theta_0^2 + d_0^2}$.

The asymptotics of solutions of the pendulum equations (5.10) are well known — these are small oscillations:

$$\theta(s) = \theta_0 \cos s + d_0 \sin s + O(\rho_0^2), \qquad d(s) = -\theta_0 \sin s + d_0 \cos s + O(\rho_0^2).$$

Hence,

$$\overline{u}_1(s) = 1 + O(\rho_0^2), \qquad \overline{u}_2(s) = \theta_0 \cos s + d_0 \sin s + O(\rho_0^2). \tag{5.15}$$

Integrating the ordinary differential equations (5.11) with the initial conditions (5.14), we obtain

$$\overline{x}(s) = \frac{s}{m} + O(\rho_0^2), \tag{5.16}$$
$$\overline{y}(s) = \frac{1}{m}(\theta_0 \sin s + d_0(1 - \cos s)) + O(\rho_0^2). \tag{5.17}$$

Thus, to within $O(\rho_0^2)$, the curve $(\overline{x}, \overline{y})$, and therefore also the original curve (x, y), is sinusoidal, of small amplitude ρ_0/m.

We will now calculate the asymptotics for the components of the quaternion $\overline{q}_0$, $\overline{q}_1$, $\overline{q}_2$, $\overline{q}_3$. From equations (5.12), (5.13) and the expansions (5.15) we obtain the equations

$$\overline{q}_0'(s) = \frac{1}{2m}(\overline{q}_2 - \overline{q}_1(\theta_0 \cos s + d_0 \sin s)) + O(\rho_0^2), \tag{5.18}$$

$$\overline{q}_1'(s) = \frac{1}{2m}(\overline{q}_3 + \overline{q}_0(\theta_0 \cos s + d_0 \sin s)) + O(\rho_0^2), \tag{5.19}$$

$$\overline{q}_2'(s) = \frac{1}{2m}(-\overline{q}_0 + \overline{q}_3(\theta_0 \cos s + d_0 \sin s)) + O(\rho_0^2), \tag{5.20}$$

$$\overline{q}_3'(s) = \frac{1}{2m}(-\overline{q}_1 - \overline{q}_2(\theta_0 \cos s + d_0 \sin s)) + O(\rho_0^2). \tag{5.21}$$

Let $\overline{q}_i(s) = \alpha_{i0}(s) + \alpha_{i1}(s)\theta_0 + \alpha_{i2}(s)d_0 + O(\rho_0^2)$, $i = 0, \dots, 3$. Then from the expansions (5.18)–(5.21) we obtain

$$\alpha_{00}' = \frac{\alpha_{20}}{2m}, \qquad \alpha_{10}' = \frac{\alpha_{30}}{2m}, \qquad \alpha_{20}' = -\frac{\alpha_{00}}{2m}, \qquad \alpha_{30}' = -\frac{\alpha_{10}}{2m},$$

and from the initial conditions (5.14) we obtain

$$\alpha_{00}(0) = 1, \qquad \alpha_{10}(0) = \alpha_{20}(0) = \alpha_{30}(0) = 0.$$

Therefore,

$$\alpha_{00} = \cos\frac{s}{2m}, \qquad \alpha_{10} = 0, \qquad \alpha_{20} = -\sin\frac{s}{2m}, \qquad \alpha_{30} = 0.$$

From (5.18)–(5.21) we obtain the differential equations

$$\begin{aligned} \alpha_{01}' &= \frac{\alpha_{21}}{2m}, & \alpha_{02}' &= \frac{\alpha_{22}}{2m}, \\ \alpha_{21}' &= -\frac{\alpha_{01}}{2m}, & \alpha_{22}' &= -\frac{\alpha_{02}}{2m}, \\ \alpha_{11}' &= \frac{1}{2m}\left(\alpha_{31} + \cos\frac{s}{2m}\cos s\right), & \alpha_{12}' &= \frac{1}{2m}\left(\alpha_{32} + \cos\frac{s}{2m}\sin s\right), \\ \alpha_{31}' &= \frac{1}{2m}\left(-\alpha_{11} + \sin\frac{s}{2m}\cos s\right), & \alpha_{32}' &= \frac{1}{2m}\left(-\alpha_{12} + \sin\frac{s}{2m}\sin s\right) \end{aligned}$$

for the coefficients α_{i1}, α_{i2}, with the initial conditions $\alpha_{ij}(0) = 0$, $i = 0, \dots, 3$, $k = 1, 2$. The solution of these differential equations has the form

$$\begin{aligned} &\alpha_{01}(s) \equiv 0, \quad \alpha_{02}(s) \equiv 0, \\ &\alpha_{11}(s) = \frac{1}{2(m^2-1)}\left(m\cos\frac{s}{2m}\sin s - (1+\cos s)\sin\frac{s}{2m}\right), \\ &\alpha_{12}(s) = \frac{1}{2(m^2-1)}\left(m(1-\cos s)\cos\frac{s}{2m} - \sin s\sin\frac{s}{2m}\right), \\ &\alpha_{21}(s) \equiv 0, \quad \alpha_{22}(s) \equiv 0, \\ &\alpha_{31}(s) = \frac{1}{2(m^2-1)}\left((-1+\cos s)\cos\frac{s}{2m} + m\sin s\sin\frac{s}{2m}\right), \\ &\alpha_{32}(s) = \frac{1}{2(m^2-1)}\left(\sin s\cos\frac{s}{2m} - m(1+\cos s)\sin\frac{s}{2m}\right). \end{aligned}$$

Thus, we obtain the following asymptotics for the components of the quaternion $\overline{q}_i$:

$$\overline{q}_0(s) = \cos\frac{s}{2m} + O(\rho_0^2), \tag{5.22}$$

$$\overline{q}_1(s) = \frac{1}{2(m^2-1)}\left(m\cos\frac{s}{2m}\sin s - (1+\cos s)\sin\frac{s}{2m}\right)\theta_0 + \frac{1}{2(m^2-1)}\left(m(1-\cos s)\cos\frac{s}{2m} - \sin s\sin\frac{s}{2m}\right)d_0 + O(\rho_0^2), \tag{5.23}$$

$$\overline{q}_2(s) = -\sin\frac{s}{2m} + O(\rho_0^2), \tag{5.24}$$

$$\overline{q}_3(s) = \frac{1}{2(m^2-1)}\left((-1+\cos s)\cos\frac{s}{2m} + m\sin s\sin\frac{s}{2m}\right)\theta_0 + \frac{1}{2(m^2-1)}\left(\sin s\cos\frac{s}{2m} - m(1+\cos s)\sin\frac{s}{2m}\right)d_0 + O(\rho_0^2). \tag{5.25}$$

Note that these expansions have a removable singularity for $m = 1$, since the numerators of all fractions with the denominator $m^2 - 1$ vanish for $m = 1$:

$$\alpha_{11}(s) = \frac{1}{4}\cos\frac{s}{2}(s+\sin s) + O(m-1),$$
$$\alpha_{12}(s) = \frac{1}{4}\sin\frac{s}{2}(s+\sin s) + O(m-1),$$
$$\alpha_{31}(s) = \frac{1}{4}\sin\frac{s}{2}(-s+\sin s) + O(m-1),$$
$$\alpha_{32}(s) = \frac{1}{4}\cos\frac{s}{2}(s-\sin s) + O(m-1),$$
$$\overline{q}_1 = \alpha_{11}\theta_0 + \alpha_{12}d_0 + O(\rho_0^2), \qquad \overline{q}_3 = \alpha_{31}\theta_0 + \alpha_{32}d_0 + O(\rho_0^2).$$

Thus, in (5.16), (5.17) and (5.22)–(5.25) we have obtained asymptotic expansions for the variables $\overline{x}$, $\overline{y}$ and $\overline{q}_i$, respectively. The expansions for the original variables x, y, q_i are expressed in terms of the expansions we have obtained using formulae (5.10).

In the next section we use the asymptotics we have obtained to analyse the Maxwell points and cut time as $\rho_0 \to 0$.

§ 6. The limiting behaviour of the Maxwell set and cut time

In [11], the equation $q_3(t) = 0$ defining the Maxwell set MAX^1 for nondegenerate elastics not centred at a point of inflection was obtained (see Theorem 1.1 or [11], Theorem 2). In this section we study the asymptotics of the roots of this equation as $\rho_0 \to 0$.

We use equation (5.25). We set $p = s/2$ and return to the original equation $q_3(t) = \overline{q}_3(t)$; then

$$q_3(p, m, \theta_0, d_0) = \frac{d_0\cos p - \theta_0\sin p}{m^2-1}\left(\cos\frac{p}{m}\sin p - m\cos p\sin\frac{p}{m}\right) + O(\rho_0^2). \tag{6.1}$$

The roots of the factor $d_0\cos p - \theta_0\sin p$ have a simple geometric meaning for the sinusoid $(\overline{x}^0(s), \overline{y}^0(s)) = (s/m, (\theta_0\sin s + d_0(1-\cos s))/m)$, which is the principal term of the asymptotics of the elastic $(\overline{x}(s), \overline{y}(s))$, and therefore also for the sinusoid $(x^0(s), y^0(s)) = (\cos\alpha\overline{x}^0(s) + \sin\alpha\overline{y}^0(s), -\sin\alpha\overline{x}^0(s) + \cos\alpha\overline{y}^0(s))$, which is the principal term of the asymptotics of the elastic $(x(s), y(s))$ as $\rho_0 \to 0$. It is easy to see that the sinusoid $\{(x^0(\sigma), y^0(\sigma)) \mid \sigma \in [0, s]\}$ is centred at a point of inflection if and only if $d_0 \cos p - \theta_0 \sin p = 0$.

By Theorem 1.1 if the function q_3 vanishes, this means a Maxwell point exists for an elastic not centred at a point of inflection. Therefore we analyse the roots of the factor $(\cos\frac{p}{m}\sin p - m\cos p\sin\frac{p}{m}))/(m^2-1)$. For $m = 0$ this factor has a non-removable singularity, and for $m = 1$ it does not vanish for $p \neq 0$. Therefore in what follows we consider the function

$$g_1(p, m) = \cos\frac{p}{m}\sin p - m\cos p\sin\frac{p}{m}, \qquad m \in (0,1) \cup (1, +\infty), \qquad p > 0, \tag{6.2}$$

and analyse its first positive root

$$p_1(m) = \min\{p > 0 \mid g_1(p, m) = 0\}. \tag{6.3}$$

We will show that as $\rho_0 \to 0$ the trajectories $Q_t = \mathrm{Exp}(\lambda, t)$, $\lambda = (\theta_0, d_0, m, \alpha) \in C_1$, contain a cut point on the intervals $t \in [0, t_1 + \varepsilon]$, $t_1 = 2p_1(m)/m$, that is, they are not optimal.

6.1. An analysis of the function $p_1(m)$. In Theorem 6.1 we prove that the function $p_1(m)$ is finite for $m > 0$, $m \neq 1$, give two-sided estimates for it, and prove the properties of monotonicity and regularity.

We define the following functions:

$$\widehat{p}(m) = \begin{cases} m\rho & \text{for } m \in \left(0, \dfrac{1}{2}\right], \\ m\pi\left(\left[\dfrac{m}{1-m}\right] + 1\right) & \text{for } m \in \left(\dfrac{1}{2}, 1\right), \\ \pi\left(\left[\dfrac{1}{m-1}\right] + 1\right) & \text{for } m \in (1, 2], \\ \rho & \text{for } m > 2, \end{cases} \quad \text{where } \rho = \tan\rho,\ \rho \in \left(\pi, \frac{3\pi}{2}\right),$$

$$\widetilde{p}(m) = \begin{cases} m\pi\left(\left[\dfrac{m}{1-m}\right] + 2\right) & \text{for } m \in (0,1), \\ \pi\left(\left[\dfrac{1}{m-1}\right] + 2\right) & \text{for } m \in (1, +\infty). \end{cases}$$

Theorem 6.1. *Let $m \in (0,1) \cup (1, +\infty)$. Then the function $g_1(p, m)$ has a minimal positive root $p_1(m)$ satisfying the following properties*:

a) $\widehat{p}(m) \leqslant p_1(m) < \widetilde{p}(m)$; *therefore*, $\lim_{m \to 1} p_1(m) = +\infty$;

b) $p_1(m)$ *is increasing for $m \in (0,1)$ and decreasing for $m \in (1, +\infty)$*;

c) *the function $p_1(m)$ is continuous for $m \in (0,1) \cup (1, +\infty)$*;

d) *the function $p_1(m)$ has the following particular values*:

- *if $m^* = 1 + \frac{1}{n}$, then $p_1(m^*) = \pi(n+1)$*,
- *if $m^* = \frac{n}{n+1}$, then $p_1(m^*) = \pi n$*,
- *if $m^* = 1 + \frac{2}{2n+1}$, then $p_1(m^*) = \pi\left(n + \frac{3}{2}\right)$*,
- *if $m^* = \frac{2n+1}{2n+3}$, then $p_1(m^*) = \pi(n + \frac{1}{2})$*;

e) *the function* $p_1(m)$ *is continuously differentiable for*

$$m \in (0,1) \cup (1,+\infty) \setminus \left(\left\{ \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ 1 + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} \right);$$

for any $n \in \mathbb{N}$ *we have* $p_1'(\frac{n}{n+1}) = +\infty$ *and* $p_1'(\frac{n+1}{n}) = -\infty$;

f) *the function* $p \mapsto g_1(p,m)$ *changes sign at the point* $p = p_1(m)$.

The proof of this theorem is based on Lemmas 6.1–6.9, which we give below. The graphs of the function $p_1(m)$ and its boundaries $\widehat{p}(m)$, $\widetilde{p}(m)$ are shown in Fig. 1.

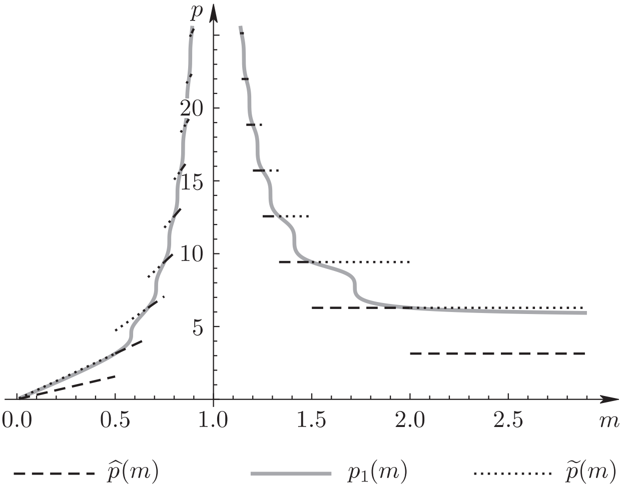

Figure 1. The graphs of the functions $\widehat{p}(m)$, $p_1(m)$, and $\widetilde{p}(m)$, where $\widehat{p}(m) \leqslant p_1(m) < \widetilde{p}(m)$.

Lemma 6.1. *If* $m > 1$ *and* $p \in (0,\pi)$, *then* $g_1(p,m) > 0$ *and therefore*, $p_1(m) > \pi$.

Proof. Using the expression for the derivative

$$\frac{\partial g_1}{\partial p} = \frac{m^2-1}{m} \sin p \sin \frac{p}{m}, \tag{6.4}$$

the function $p \mapsto g_1(p,m)$ is increasing for $p \in (0,\pi)$. Since $g_1(0,m) = 0$, we have $g_1(p,m) > 0$ for any $p \in (0,\pi)$.

Lemma 6.2. *If* $m > 1$, *then* $g_1(\widetilde{p}(m),m) < 0$ *and therefore* $p_1(m) < \widetilde{p}(m)$.

Proof. Suppose that m belongs to the interval $(1,2]$. This interval is covered by the half-open intervals $\widetilde{I}(n) = \left(1 + \frac{1}{n+1}, 1 + \frac{1}{n}\right]$, where $n \in \mathbb{N}$. If $m \in \widetilde{I}(n)$, then $\widetilde{p} = \widetilde{p}(m) = \pi(n+2)$ and $g_1(\widetilde{p},m) = (-1)^{n+1} m \sin\left(\frac{2+n}{m}\pi\right)$. Since $n < \frac{2+n}{m} < n+1$, the following hold:

- if $n = 2s$, then $\sin\left(\frac{2+n}{m}\pi\right) > 0$ and therefore, $g_1(\widetilde{p},m) < 0$;
- if $n = 2s-1$, then $\sin\left(\frac{2+n}{m}\pi\right) < 0$ and therefore, $g_1(\widetilde{p},m) < 0$.

For the case $m > 2$ we have $\widetilde{p}(m) = 2\pi$. By a straightforward substitution we verify that $g_1(2\pi,m) = -m \sin\frac{2\pi}{m} < 0$.

Lemma 6.3. *If $m = 1 + \frac{1}{n}$, where $n \in \mathbb{N}$, then $p_1(m) = \pi(n+1) = \widehat{p}(m)$.*

Proof. We set $m^* = 1 + \frac{1}{n}$ and $p^* = \pi(n+1)$, where $n \in \mathbb{N}$. Obviously, $g_1(p^*, m^*) = 0$. We claim that $g_1(p, m^*) > 0$ for all $p \in (0, p^*)$. Consider the values of the function $g_1(p, m^*)$ at the critical points with respect to the variable p, where $\frac{\partial g_1}{\partial p} = 0$. The function g_1 has two sets of such critical points: $p^1(k_1) = \pi k_1$ and $p^2(k_2) = \pi k_2 m$, where $k_1, k_2 \in \mathbb{N}$. By considering the critical points $p^1(k_1), p^2(k_2) \in (0, p^*)$ we conclude that $g_1(p, m^*) > 0$ at them. Therefore $g_1(p, m^*) > 0$ for all $p \in (0, p^*)$. Consequently, $p_1(m^*) = p^*$.

Lemma 6.4. *Let $\Omega = (1, +\infty) \setminus \{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$. Then $p_1 \in C^1(\Omega)$ and $p_1'(m) < 0$ for $m \in \Omega$.*

Proof. Lemma 6.3 gives the support points $m^* = 1 + \frac{1}{n}$ at which the value of the function $p_1(m^*) = p^* = \pi(n+1)$ is known. At these points we have $\frac{\partial g_1}{\partial p}\big|_{(p^*, m^*)} = 0$ and the graph of the function $p_1(m)$ has a vertical tangent. This lemma asserts that the function $p_1(m)$ is smooth for $m > 1$, $m \neq m^*$; we prove this inclusion based on the inequation $\frac{\partial g_1}{\partial p}\big|_{p = p_1(m)} \neq 0$ and the implicit function theorem.

Let $m \in \Omega \cap (1, 2]$; then $m \in I(n) = \left(1 + \frac{1}{n+1}, 1 + \frac{1}{n}\right)$ for some $n \in \mathbb{N}$. Consider the values $g_1(p^i(k_i), m)$ at the critical points $p^1(k_1), p^2(k_2)$, where $\frac{\partial g_1}{\partial p} = 0$. It is easy to verify that $g_1(p^i(k_i), m) \neq 0$, that is, $\frac{\partial g_1}{\partial p}\big|_{p = p_1(m)} \neq 0$ holds for all $m \in I(n)$. By the implicit function theorem, $p_1(m)$ is a continuously differentiable function on $I(n)$, and

$$p_1'(m) = \left.\frac{-\frac{\partial g_1(p,m)}{\partial m}}{\frac{\partial g_1(p,m)}{\partial p}}\right|_{p = p_1(m)} = -\left.\frac{p + m \cot p\left(-m + p \cot \frac{p}{m}\right)}{m(m^2 - 1)}\right|_{p = p_1(m)}.$$

From the condition $g_1(p, m)|_{p = p_1(m)} = 0$ we obtain that

$$m \cot p\big|_{p = p_1(m)} = \cot \frac{p}{m}\bigg|_{p = p_1(m)}.$$

Then $\operatorname{sgn}(p_1'(m)) = -\operatorname{sgn}(f(p_1(m), m))$, where $f(p, m) = p - m \cot \frac{p}{m} + p \cot^2 \frac{p}{m}$. Consider the function f as a polynomial of second degree with respect to $\cot \frac{p}{m}$. Since $m \leqslant 2$ and $p_1(m) > \pi > 1$ by Lemma 6.1, the discriminant of this polynomial satisfies $\mathrm{D}(f) = m^2 - 4p^2 < 0$; therefore, $f(p_1(m), m) > 0$ and $p_1'(m) < 0$.

Suppose that $m > 2$. Then $\frac{\partial g_1}{\partial p}\big|_{p = p_1(m)} \neq 0$. The function $f(p, m)$ is positive for $p = p_1(m)$, since either $\mathrm{D}(f) < 0$ and therefore, $f(p, m) > 0$; or $\mathrm{D}(f) \leqslant 0$, then $m \geqslant 2p$, while $\cot \frac{p}{m} > q_2$, where q_2 is the greater of the roots of the function f as a quadratic polynomial with respect to $\cot \frac{p}{m}$. Thus, $f(p_1(m), m) > 0$ and therefore, $p_1'(m) < 0$.

Lemma 6.5. *The function $p_1(m)$ is continuous for $m > 1$.*

Proof. In view of Lemmas 6.3 and 6.4 it is required to prove that the limits

$$\lim_{m \to 1 + 1/n \pm 0} p_1(m) = p_1\left(1 + \frac{1}{n}\right) = \pi(n+1)$$

exist. Since $p_1(m)$ is monotonic and bounded for $m \neq 1 + \frac{1}{n}$, the finite limits

$$p_\pm(n) = \lim_{m \to 1+1/n \pm 0} p_1(m)$$

exist. The inequalities $p_+(n) < \pi(n+1)$ and $p_-(n) < \pi(n+1)$ contradict the equation

$$p_1\left(1 + \frac{1}{n}\right) = \pi(n+1).$$

And the inequalities $p_+(n) > \pi(n+1)$ and $p_-(n) > \pi(n+1)$ contradict the continuity of the curve $\{(p,m) \mid g_1(p,m) = 0\}$ in a neighbourhood of the point $\left(\pi(n+1), 1 + \frac{1}{n}\right)$, which follows by the implicit function theorem from the inequation

$$\frac{\partial g_1}{\partial m}\left(\pi(n+1), 1 + \frac{1}{n}\right) = -m\left(1 + \frac{1}{n}\right) \neq 0.$$

Lemma 6.6. *If $m > 1$, then $p_1(m) \geqslant \widehat{p}(m)$ and therefore, $\lim_{m \to 1+0} p_1(m) = +\infty$.*

Proof. It follows from Lemmas 6.4, 6.5 that the function $p_1(m)$ is monotonically decreasing for $m \in (1, +\infty)$.

Let $m \in (1,2] = \bigcup_{n \in \mathbb{N}} \widetilde{I}(n)$. If $m \in \widetilde{I}(n)$, then $p_1(m) \geqslant \pi(n+1) = \pi([\frac{1}{m-1}]+1) = \widetilde{p}(m)$. Since $\lim_{m \to 1+0} \widetilde{p}(m) = +\infty$ and $p_1(m) \geqslant \widetilde{p}(m)$ for any $m \in (1,2]$, we have $\lim_{m \to 1+0} p_1(m) = +\infty$.

For $m > 2$ consider the limit $\lim_{m \to +\infty} g_1(p,m) = \sin p - p \cos p$. Since $p_1(m)$ is decreasing, we have $p_1(m) > \rho = \widehat{p}(m)$, where $\rho \in (\pi, \frac{3\pi}{2})$ is a root of the equation $p = \tan p$, which is equivalent to the equation $\sin p - p \cos p = 0$.

Lemma 6.7. *If $m = 1 + \frac{2}{2n+1}$, then $p_1(m) = \pi(n + \frac{3}{2})$.*

Proof. We set $m_* = 1 + \frac{2}{2n+1}$ and $p_* = \pi(n + \frac{3}{2})$. Obviously, $g_1(p_*, m_*) = 0$. Next, we verify that on the segment $p \in [\widehat{p}(m_*), p_*]$, where $\widehat{p}(m_*) = \pi(n+1)$, there are no roots of the function $g_1(p, m_*)$ other than p_*. This follows from the fact that $g_1(\widehat{p}(m_*), m_*) > 0$ and $\frac{\partial g_1}{\partial p}(p, m_*) \neq 0$ for $p \in [\widehat{p}(m_*), p_*)$.

Lemma 6.8. *If $m > 1$, then the function $p \mapsto g_1(p,m)$ changes sign from plus to minus when passing through the point $p = p_1(m)$.*

Proof. It follows from Lemma 6.4 that $\frac{\partial g_1}{\partial p}\big|_{p=p_1(m)} \neq 0$ for $m \in \Omega$; therefore $g_1(p,m)$ changes sign when passing through $p = p_1(m)$. It follows from Lemma 6.1 that $g_1(p,m) > 0$ for all $p \in (0, p_1(m))$. Therefore, when passing through $p = p_1(m)$, the sign changes from plus to minus.

Let $m = m^* = 1 + \frac{1}{n}$. It follows from Lemmas 6.1, 6.3 that $g_1(p, m^*) > 0$ for $p \in (0, p_1(m^*))$. Since $g_1(p,m)$ is an analytic function, $p_1(m^*)$ is an isolated root of the function $g_1(p, m^*)$. By continuity, the preceding paragraph implies that $g_1(p, m^*)$ changes sign from plus to minus when passing through $p_1(m^*)$.

Lemma 6.9. *For any $n \in \mathbb{N}$ we have $p_1'\left(1 + \frac{1}{n}\right) = -\infty$.*

Proof. By Lemma 6.3, for any $m^* = 1 + \frac{1}{n}$ the explicit value $p^* = p_1(m^*) = \pi(n+1)$ is known. We have

$$\frac{\partial g_1(p^*, m^*)}{\partial m} = \pi n, \qquad \frac{\partial g_1(p^*, m^*)}{\partial p} = 0;$$

therefore,

$$\lim_{m\to m^*} p_1'(m) = -\lim_{m\to m^*} \left.\frac{\frac{\partial g_1}{\partial m}(p,m)}{\frac{\partial g_1}{\partial p}(p,m)}\right|_{p=p_1(m)} = \infty.$$

Since $p_1(m)$ is continuous, Lagrange's finite increment theorem gives

$$p_1'(m^*) = \lim_{\Delta m\to 0} \frac{p_1(m^*+\Delta m) - p_1(m^*)}{\Delta m} = \lim_{\Delta m\to 0} p_1'(\widetilde{m}) = \lim_{m\to m^*} p_1'(m) = \infty,$$

where $\widetilde{m} \in (m^*, m^*+\Delta m)$. Since the function $p_1(m)$ is decreasing for $m > 1$ we conclude that $p_1'(m^*) = -\infty$.

Proof of Theorem 6.1. In the case $m > 1$, the theorem follows from Lemmas 6.2–6.9 above.

In the case $m \in (0,1)$ we use the change of variables $\overline{p} = \frac{p}{m}$, $\overline{m} = \frac{1}{m} > 1$, in which $g_1(\overline{p},\overline{m}) = \frac{1}{m} g_1(p,m)$. We conclude that $g_1(\overline{p},\overline{m}) = 0$ if and only if $g_1(p,m) = 0$. Thus, $p_1(m) = mp_1(\frac{1}{m})$ and for $m \in (0,1)$ properties a)–f) of the function $p_1(m)$ follow from the same properties for $m > 1$.

6.2. Maxwell time and cut time for a sphere rolling over sinusoids of small amplitude. Let $\lambda = (\theta, d, m, \alpha) \in C_1$. In this section we analyse the behaviour of the Maxwell set MAX^1 and the cut time

$$t_{\mathrm{cut}}(\lambda) = \sup\bigl\{t > 0 \mid Q_s = \mathrm{Exp}(\lambda, s) \text{ is optimal for } s \in [0,t]\bigr\}$$

as $(\theta, d) \to (0,0)$, that is, near the stable equilibrium position of the mathematical pendulum (1.9). In this case, the elastics (x_t, y_t) in the principal term are sinusoids of small amplitude $\frac{\rho}{m}$. In view of Lemma 3.1, Theorem 1.1 given in § 1 asserts the following. If $t > 0$ is such that $q_3(t) = 0$ and $\operatorname{cn}\tau = \operatorname{cn}\bigl(m(\frac{t}{2}+\varphi), k\bigr) \neq 0$, then $(\lambda, t) \in \mathrm{MAX}^1$ and the trajectory $Q_s = \mathrm{Exp}(\lambda, s)$, $s \in [0,t]$, is not optimal; therefore, $t_{\mathrm{cut}}(\lambda) \leqslant t$.

We introduce the polar coordinates (ρ, χ): $d = \rho\cos\chi$, $\theta = \rho\sin\chi$; then equation (6.1) can be rewritten in the form

$$q_3(\rho,\chi,m,t) = \rho\frac{\cos(\chi + \frac{tm}{2})}{m^2-1} g_1\left(\frac{tm}{2}, m\right) + \rho^2 h, \quad \text{where } |h| \leqslant C. \tag{6.5}$$

We fix $\overline{\chi} \in S^1$ and $\overline{m} > 0$, $\overline{m} \neq 1$. To study the behaviour of the Maxwell time as $\rho \to 0$, $\chi \to \overline{\chi}$, $m \to \overline{m}$ we introduce the sets

$$D_\delta = \bigl\{\lambda = (\rho,\chi,m,\alpha) \in C_1 \mid 0 < \rho < \delta,\ |\chi - \overline{\chi}| < \delta,\ |m - \overline{m}| < \delta\bigr\}.$$

We set $t_1 = t_1(\overline{m}) = \frac{2p_1(\overline{m})}{\overline{m}} > 0$, where $p_1(m)$ is the minimal positive root of the function $g_1(p,m)$ (see § 6.1). We also set

$$I_\varepsilon = \bigl\{t > 0 \mid t_1 - \varepsilon < t < t_1 + \varepsilon\bigr\}, \qquad \varepsilon > 0.$$

The following theorem describes the behaviour of the Maxwell set in a neighbourhood of the stable equilibrium position of the pendulum $(\theta, d) = (0,0)$.

Theorem 6.2. *Suppose that $\overline{\chi} \in S^1$, $\overline{m} > 0$, $\overline{m} \neq 1$, and $\cos(\overline{\chi} + t_1\overline{m}/2) \neq 0$. Then*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta = \delta(\varepsilon) > 0 \quad \forall\, \lambda \in D_\delta \quad \exists\, t \in I_\varepsilon \colon (\lambda, t) \in \mathrm{MAX}^1 .$$

In the proof of this theorem we use the following lemma, which guarantees that there are no elastics centred at a point of inflection for the values of the parameters λ, t under consideration.

Lemma 6.10. *Suppose that $\overline{m}$, t_1, $\overline{\chi}$ satisfy the hypotheses of Theorem 6.2. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$\mathrm{cn}\biggl(m\Bigl(\frac{t}{2} + \varphi\Bigr), k\biggr) \neq 0$$

for any $\lambda \in D_\delta$ and $t \in I_\varepsilon$.

Proof. Suppose the opposite. Suppose that there exist sequences $t^n \in \mathbb{R}_+$, $\lambda_n = (\rho_n, \chi_n, m_n, \alpha_n) \in C_1$ such that $\rho_n \to 0$, $m_n \to \overline{m}$, $\chi_n \to \overline{\chi}$, $t^n \to t_1$, and $\mathrm{cn}(m_n(t^n/2 + \varphi_n), k_n) = 0$. From the definitions of the elliptic coordinates (φ, k) (see § 2) and the polar coordinates (χ, ρ) we obtain

$$k^2 = \frac{d^2}{4} + \sin^2\frac{\theta}{2} = \frac{\rho^2}{4}\biggl(\cos^2\chi + \sin^2\chi\frac{\sin^2\frac{\theta}{2}}{(\frac{\theta}{2})^2}\biggr),$$

$$\mathrm{cn}(m\varphi, k) = \cos\chi\frac{\rho}{2k}, \qquad \mathrm{sn}(m\varphi, k) = \sin\chi\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\,\frac{\frac{\rho}{2}}{k}.$$

Consequently, $k_n \sim \rho_n/2$ and $m_n\varphi_n \to \overline{\chi}$ as $n \to \infty$. Therefore,

$$\mathrm{cn}\biggl(m_n\Bigl(\frac{t^n}{2} + \varphi_n\Bigr), k_n\biggr) \to \cos\Bigl(\overline{\chi} + \frac{t_1\overline{m}}{2}\Bigr) \neq 0,$$

a contradiction.

Proof of Theorem 6.2. Since $\cos(\overline{\chi} + t_1\overline{m}/2) \neq 0$ and $\overline{m} \neq 1$, there exists a neighbourhood $D_{\delta_0} \times I_{\varepsilon_0}$ in which $\cos(\chi + tm/2) \neq 0$ and $m^2 - 1 \neq 0$. Taking Lemma 6.10 into account and making δ_0, ε_0 smaller if necessary, we obtain that the inequation $\mathrm{cn}\,\tau = \mathrm{cn}(m(t/2 + \varphi), k) \neq 0$ holds for all $(\lambda, t) \in D_{\delta_0} \times I_{\varepsilon_0}$. Therefore, if the equation $q_3(\lambda, t) = q_3(\rho, \chi, m, t) = 0$ holds for some $(\lambda, t) \in D_{\delta_0} \times I_{\varepsilon_0}$, then $(\lambda, t) \in \mathrm{MAX}^1$ by Theorem 1.1 and Lemma 3.1.

For $\lambda \in D_{\delta_0}$ and $t \in I_{\varepsilon_0}$ we define the function

$$\widetilde{q}_3(\lambda, t) = \frac{q_3(\lambda, t)}{\rho\cos(\chi + \frac{tm}{2})}(m^2 - 1),$$

and by (6.5) we have the decomposition

$$\widetilde{q}_3(\lambda, t) = g_1\Bigl(\frac{tm}{2}, m\Bigr) + \rho\widetilde{h}, \quad \text{where } |\widetilde{h}| \leqslant C.$$

By Theorem 6.1 there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \varepsilon_1)$ we have $g_1(\overline{m}(t_1 - \varepsilon)/2, \overline{m}) > 0$ and $g_1(\overline{m}(t_1 + \varepsilon)/2, \overline{m}) < 0$ in the case $\overline{m} > 1$ (in the case $\overline{m} \in (0,1)$ the function g_1 has the opposite signs but the arguments do not change). As $g_1(p, m)$ is continuous there exist $\delta_1(\varepsilon) \in (0, \delta_0)$ and $\gamma > 0$ such that $g_1(m(t_1 - \varepsilon)/2, m) > \gamma$ and $g_1(m(t_1 + \varepsilon)/2, m) < -\gamma$ for all $m \in (\overline{m} - \delta_1, \overline{m} + \delta_1)$. Therefore there exists $\delta_2(\varepsilon) \in (0, \delta_1(\varepsilon)]$ such that

$$\widetilde{q}_3(\lambda, t_1 - \varepsilon) > \frac{\gamma}{2} \quad \text{and} \quad \widetilde{q}_3(\lambda, t_1 + \varepsilon) < -\frac{\gamma}{2}$$

for $\lambda \in D_{\delta_2}$. Consequently, there exists $t \in I_\varepsilon$ for which $\widetilde{q}_3(\lambda, t) = 0$ and therefore, $q_3(\lambda, t) = 0$.

Thus, for all $\varepsilon \in (0, \varepsilon_1)$ there exists $\delta = \delta_2(\varepsilon)$ such that for any $\lambda \in D_\delta$ there exists $t \in I_\varepsilon$ for which $q_3(\lambda, t) = 0$, $\operatorname{cn}\tau \neq 0$, and therefore, $(\lambda, t) \in \mathrm{MAX}^1$. The assertion of the theorem is proved for small ε. By increasing ε and leaving δ fixed, we obtain the assertion of the theorem for arbitrary $\varepsilon > 0$.

Corollary 6.1. *Suppose that a sequence $\lambda_n = (\rho_n, \chi_n, m_n, \alpha_n) \in C_1$ satisfies the conditions*

$$\rho_n \to 0, \qquad m_n \to \overline{m} > 0, \qquad \overline{m} \neq 1, \qquad \chi_n \to \overline{\chi}, \qquad \cos\left(\frac{\overline{m} t_1}{2} + \overline{\chi}\right) \neq 0.$$

Then

$$\varlimsup_{n\to\infty} t_{\mathrm{cut}}(\lambda_n) \leqslant t_1, \qquad t_1 = t_1(\overline{m}). \tag{6.6}$$

Proof. We fix any $\varepsilon > 0$. For sufficiently large n the element λ_n belongs to the domain D_δ indicated in Theorem 6.2. Then there exists $t^n \in (t_1 - \varepsilon, t_1 + \varepsilon)$ for which $(\lambda_n, t^n) \in \mathrm{MAX}^1$ and therefore, $t_{\mathrm{cut}}(\lambda_n) \leqslant t^n < t_1 + \varepsilon$. Since ε is arbitrary, we conclude that $\varlimsup_{n\to\infty} t_{\mathrm{cut}}(\lambda_n) \leqslant t_1$.

We claim that in the statement of Corollary 6.1 we can get rid of the conditions $\chi_n \to \overline{\chi}$ and $\cos(\overline{m}t_1/2 + \overline{\chi}) \neq 0$.

Theorem 6.3. *Suppose that a sequence $\lambda_n = (\rho_n, \chi_n, m_n, \alpha_n) \in C_1$ satisfies the conditions $\rho_n \to 0$, $m_n \to \overline{m} > 0$, $\overline{m} \neq 1$. Then inequality (6.6) holds.*

In other words, Theorem 6.3 asserts that

$$\varlimsup_{\rho\to 0,\, m\to\overline{m}} t_{\mathrm{cut}}(\lambda) \leqslant t_1(\overline{m}) \quad \text{for } \overline{m} > 0, \ \overline{m} \neq 1.$$

Proof. We begin with several remarks about the problem (1.1), (1.2), (1.3), (1.5) with general boundary conditions $Q(0) = Q'_0$, $Q(t_1) = Q'_1 \in \mathbb{R}^2 \times \mathrm{SO}(3)$. Since this problem is invariant under left translations on $\mathbb{R}^2 \times \mathrm{SO}(3)$, any solution with the general boundary condition is obtained from some solution with the particular boundary condition (1.4) by a left translation by the element $Q'_0 = (x'_0, y'_0, R'_0)$, that is, by a translation parallel to the vector (x'_0, y'_0) in the (x, y)-plane and multiplication of the matrix $R(t)$ on the left by the matrix R'_0.

From the Hamiltonian system (1.9)–(1.12) we obtain that the elastic $(x(t), y(t))$ and the matrix R'_0 determine the matrix $R(t)$ uniquely. We call the extremal curve $(x(t), y(t), R(t))$ the rolling of the sphere over the elastic $(x(t), y(t))$. If the rolling $(x(t), y(t), R(t))$, $t \in [0, \widehat{t}\,]$, is optimal, then due to the invariance of the problem under left translations on $\mathrm{SO}(3)$, any other rolling over the same elastic $(x(t), y(t), \widetilde{R}R(t))$, $t \in [0, \widehat{t}\,]$ is also optimal. In this case we say that the elastic $(x(t), y(t))$,

$t \in [0, \widehat{t}\,]$, is optimal. Because of the invariance of the problem under translations on $\mathbb{R}^2$, an elastic $(x(t), y(t))$, $t \in [0, \widehat{t}\,]$, is optimal if and only if any of its translates $(x(t) + \widetilde{x}, y(t) + \widetilde{y})$, $t \in [0, \widehat{t}\,]$, is optimal.

The elliptic coordinate φ (see § 2) is the time on the trajectories of the pendulum. Therefore, if we introduce the covectors $\lambda = (\varphi, k, m, \alpha)$, $\lambda = (\varphi + \sigma, k, m, \alpha) \in C_1$, $\sigma \in \mathbb{R}$, use elliptic coordinates, and denote the corresponding elastics as $(x(t), y(t))$, $(x'(t), y'(t))$, then we obtain

$$(x(t+\sigma), y(t+\sigma)) = (x'(t), y'(t)) + (x(\sigma), y(\sigma)).$$

We will now prove inequality (6.6). Arguing by contradiction, suppose that

$$\exists\, \varepsilon > 0 \ \ \forall\, N \in \mathbb{N} \ \ \exists\, n > N\colon \ Q_n(t) = \operatorname{Exp}(\lambda_n, t), \ t \in [0, t_1 + \varepsilon], \text{ is optimal.}$$

Choose any subsequence of the sequence λ_n on which the sequence $\chi_n \in S^1$ converges; let $\lim_{n\to\infty} \chi_n = \overline{\chi}$. We keep the notation λ_n for this subsequence. In view of Corollary 6.1, it is sufficient to consider the case $\cos(\overline{m} t_1/2 + \overline{\chi}) = 0$. We represent the covector λ_n using the elliptic coordinates: $\lambda_n = (\varphi_n, k_n, m_n, \alpha_n)$. In the same fashion as in Lemma 6.10 we conclude that $\varphi_n \to \overline{\varphi} = \overline{\chi}/\overline{m}$. For a small $\sigma > 0$ (which is chosen later), we define the covector $\lambda'_n = (\varphi_n + \sigma/m_n, k_n, m_n, \alpha_n) \in C_1$ and the corresponding extremal trajectory $Q'_n(t) = \operatorname{Exp}(\lambda'_n, t) = (x'_n(t), y'_n(t), R'_n(t))$. Since the parameters k and m coincide, the curves $(x_n(t), y_n(t))$ and $(x'_n(t), y'_n(t))$ belong to the same infinite elastic, up to motions of the plane, but they have different initial phases φ_n and $\varphi_n + \sigma/m_n$. We have $m_n(\varphi_n + \sigma/m_m) \to \overline{\chi} + \sigma = \overline{\chi}'$; therefore $\cos(\overline{m} t_1/2 + \overline{\chi}') \neq 0$ for sufficiently small σ. By Corollary 6.1 there exists $N_1 \in \mathbb{N}$ such that for $n > N_1$ the trajectory $Q'_n(t)$, $t \in [0, t_1 + \varepsilon/2]$, is not optimal, that is, the elastic $\gamma'_n = \{(x'_n(t), y'_n(t)) \mid t \in [0, t_1 + \varepsilon/2]\}$ is not optimal. But the arc γ'_n coincides, up to translations of the plane, with the arc $\gamma_n = \{(x_n(t), y_n(t)) \mid t \in [\sigma/m_n, \sigma/m_n + t_1 + \varepsilon/2]\}$; therefore the arc γ_n is not optimal. We choose $\sigma > 0$ so small that $[\sigma/m_n, \sigma/m_n + t_1 + \varepsilon/2]\} \subset [0, t_1]$. Then the non-optimal arc γ_n is contained in the optimal arc $\{(x_n(t), y_n(t)) \mid t \in [0, t_1 + \varepsilon]\}$, giving a contradiction.

We fix any compact set $K \subset \{m \in \mathbb{R} \mid m > 0,\ m \neq 1\}$ and define a subset of the cylinder C:

$$\Lambda_\delta = \bigl\{\lambda = (\rho, \chi, m, \alpha) \in C_1 \mid 0 < \rho < \delta,\ m \in K\bigr\}.$$

In the following theorem an estimate for the cut time $t_{\text{cut}}(\lambda)$ is obtained for covectors $\lambda \in \Lambda_\delta$ for sufficiently small δ, that is, for small oscillations of the pendulum (1.9).

Theorem 6.4. *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that the inequality $t_{\text{cut}}(\lambda) \leqslant \max_{m\in K} t_1(m) + \varepsilon$ holds for all $\lambda \in \Lambda_\delta$.*

In other words, Theorem 6.4 asserts that $\overline{\lim\limits_{\rho\to 0, m\in K}}\, t_{\text{cut}}(\lambda) \leqslant \max_{m\in K} t_1(m)$.

Proof. Suppose the opposite. Suppose that there exists $\varepsilon > 0$ such that for all $\delta_n = 1/n$, $n \in \mathbb{N}$, there exists $\lambda_n \in \Lambda_{1/n}$ for which $t_{\text{cut}}(\lambda_n) > \max_{m\in K} t_1(m) + \varepsilon$. Then for the sequence $\lambda_n = (\rho_n, \chi_n, m_n, \alpha_n)$ we have $\rho_n \to 0$. Since $(\chi_n, m_n) \in S^1 \times K$ and $S^1 \times K$ is a compact set, there exists a convergent subsequence

$$(\chi_{n_k}, m_{n_k}) \to (\overline{\chi}, \overline{m}), \qquad \overline{x} \in S^1, \quad \overline{m} \in K.$$

We conclude from Theorem 6.3 that $t_{\mathrm{cut}}(\lambda_{n_k}) \leqslant t_1(\overline{m}) + \varepsilon$ for sufficiently large k; therefore, $t_{\mathrm{cut}}(\lambda_{n_k}) \leqslant \max_{m\in K} t_1(m) + \varepsilon$. This contradicts the inequality

$$t_{\mathrm{cut}}(\lambda_n) > \max_{m\in K} t_1(m) + \varepsilon.$$

In §§ 5, 6 we studied the optimality properties of the extremal trajectories $Q_t = \operatorname{Exp}(\lambda, t)$, $\lambda = (\theta, d, m, \alpha)$, in a neighbourhood of the stable equilibrium position $(\theta, d) = (0, 0)$ of the pendulum equation (5.10). In § 5 we calculated the principal terms of the asymptotics of the extremal trajectories as $\rho = \sqrt{\theta^2 + d^2} \to 0$. In § 6 we studied the behaviour of the Maxwell set MAX^1 as $\rho \to 0$. Based on this, as $\rho \to 0$, $m \to \overline{m}$, for the extremal trajectories $Q_t = \operatorname{Exp}(\lambda, t)$ we obtained an upper estimate for the cut time of the form $t_{\mathrm{cut}}(\lambda) < t_1(\overline{m}) + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. The function $t_1(m) = 2p_1(m)/m$ defining this estimate is characterized by a quite complicated behaviour: its graph has vertical tangents at $m = (n+1)/n$ and $m = n/(n+1)$, $n \in \mathbb{N}$; furthermore, $\lim_{m\to 1} t_1(m) = +\infty$.

The instant of time $t = t_1(\overline{m})$ determines the asymptotics as $\rho \to 0$, $m \to \overline{m}$ for the trajectories Q_t of the Maxwell time corresponding to the reflection ε^1 of the phase portrait of the pendulum (1.9) in the coordinate axis θ. A similar time instant $t = t_2(\overline{m})$ can be defined for the reflection ε^2 of the phase portrait of the pendulum (1.9) in the coordinate axis c (that is, for the Maxwell stratum MAX^2). The behaviour of the function $t_2(m)$ is similar to the behaviour of the function $t_1(m)$; one can show that the graphs of these functions have infinitely many points of intersection. Approximate calculations also show that the functions $t_1(m)$ and $t_2(m)$ are boundaries of the first conjugate time $t_{\mathrm{conj}}(m)$ along the extremal trajectories (Fig. 2).

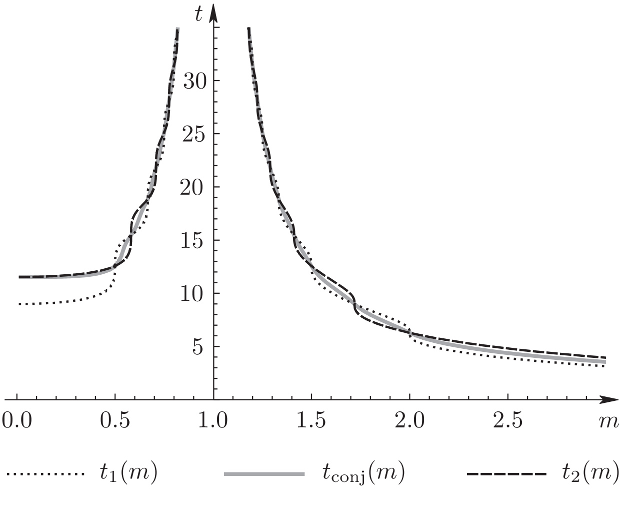

Figure 2. Graphs of the functions $t_1(m)$, $t_{\mathrm{conj}}(m)$, $t_2(m)$.

Note that in related optimal control problems — the sub-Riemannian problem in the Martinet case (see [18]), the nilpotent sub-Riemannian problem with growth vector $(2,3,5)$ (see [17]), the Euler elastic problem (see [14]), the sub-Riemannian problem on the group of motions of the plane (see [15]) — the global behaviour of the analogous Maxwell times $t_1(\lambda)$, $t_2(\lambda)$, $\lambda \in C$, is much simpler than the asymptotics of $t_1(\overline{m})$, $t_2(\overline{m})$ in the problem of the rolling of a sphere on a plane. This reflects the more complicated nature of this problem by comparison with the related problems mentioned above. And in view of the complexity of the parametrization of the extremal trajectories in this problem it seems it would be difficult to obtain its exact solution. However, based on the results we have obtained it is possible to develop an algorithm and a program for solving the problem of the rolling of a sphere on a plane approximately. This will be the subject of a future paper.

Bibliography

[1] Z. Li and J. Canny, "Motion of two rigid bodies with rolling constraint", *IEEE Trans. Robotics Automation* **6**:1 (1990), 62–72.

[2] A. Bicchi, D. Prattichizzo and S. S. Sastry, "Planning motions of rolling surfaces", *Proc. 34th IEEE Conf. on Decision and Control* (New Orleans, LA 1995), pp. 2812–2817.

[3] A. Marigo and A. Bicchi, "Rolling bodies with regular surface: the holonomic case", *Differential geometry and control* (Boulder, CO 1997), Proc. Sympos. Pure Math., vol. 64, Amer. Math. Soc., Providence, RI 1999, pp. 241–256.

[4] A. A. Agrachev and Yu. L. Sachkov, *Geometric control theory*, Fizmatlit, Moscow 2004. (Russian)

[5] J. M. Hammersley, "Oxford commemoration ball", *Probability, statistics and analysis*, London Math. Soc. Lecture Note Ser., vol. 79, Cambridge Univ. Press, Cambridge–New York 1983, pp. 112–142.

[6] A. M. Arthurs and G. R. Walsh, "On Hammersley's minimum problem for a rolling sphere", *Math. Proc. Cambridge Philos. Soc.* **99**:3 (1986), 529–534.

[7] V. Jurdjevic, "The geometry of the plate-ball problem", *Arch. Rational Mech. Anal.* **124**:4 (1993), 305–328.

[8] V. Jurdjevic, *Geometric control theory*, Cambridge Stud. Adv. Math., vol. 52, Cambridge Univ. Press, Cambridge 1997.

[9] L. Euler, "Supplement I: On elastic curves", *A method for finding curved lines that have properties of being a maximum or minimum, or a solution of the isoperimeter problem considered in the broadest sense*, GTTI, Moscow–Leningrad 1934, pp. 447–572. (Russian)

[10] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambridge Univ. Press, Cambridge 1927.

[11] Yu. L. Sachkov, "Maxwell strata and symmetries in the problem of optimal rolling of a sphere over a plane", *Mat. Sb.* **201**:7 (2010), 99–120; English transl. in *Sb. Math.* **201**:7 (2010), 1029–1051.

[12] Yu. L. Sachkov, "Complete description of the Maxwell strata in the generalized Dido problem", *Mat. Sb.* **197**:6 (2006), 111–160; English transl. in *Sb. Math.* **197**:6 (2006), 901–950.

[13] L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes*, Nauka, Moscow 1976; English transl., *Selected works. Vol.* 4., Classics Soviet Math., Gordon & Breach, New York 1986.

[14] Yu. L. Sachkov, "Maxwell strata in the Euler elastic problem", *J. Dyn. Control Syst.* **14**:2 (2008), 169–234.

[15] I. Moiseev and Yu. L. Sachkov, "Maxwell strata in sub-Riemannian problem on the group of motions of a plane", *ESAIM Control Optim. Calc. Var.* **16**:2 (2010), 380–399.

[16] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Univ. Press, New York 1962.

[17] Yu. L. Sachkov, "Exponential map in the generalized Dido problem", *Mat. Sb.* **194**:9 (2003), 63–90; English transl. in *Sb. Math.* **194**:9 (2003), 1331–1359.

[18] A. Agrachev, B. Bonnard, M. Chyba and I. Kupka, "Sub-Riemannian sphere in Martinet flat case", *ESAIM Control Optim. Calc. Var.* **2** (1997), 377–448.

[19] V. I. Arnol'd, *Geometry of complex numbers, quaternions, and spins*, MTsNMO, Moscow 2002. (Russian)

[20] L. S. Pontryagin, *Generalizations of numbers*, Library "Quantum", vol. 54, Nauka, Moscow 1986; German transl., L. S. Pontrjagin, *Verallgemeinerungen der Zahlen*, Akademie-Verlag, Berlin 1991.

[21] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge 1988.

A. P. Mashtakov
Institute of Program Systems of Russian Academy of Sciences, Pereslavl'-Zalesskiĭ
E-mail: alexey.mashtakov@gmail.com

Received 24/JUN/10
Translated by E. KHUKHRO

Yu. L. Sachkov
Institute of Program Systems of Russian Academy of Sciences, Pereslavl'-Zalesskiĭ
E-mail: sachkov@sys.botik.ru