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CONTROL THEORY

# Integrability of Left-Invariant Sub-Riemannian Structures on the Special Linear Group $SL_2(R)$

**A. P. Mashtakov and Yu. L. Sachkov**
*Program Systems Institute, Russian Academy of Sciences, Pereslavl-Zalessky, Yaroslavl Region, Russia*
*e-mail: alexey.mashtakov@gmail.com, sachkov@sys.botik.ru*
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**Abstract**— We show that the Hamiltonian system of ordinary differential equations of the Pontryagin maximum principle for left-invariant sub-Riemannian structures of elliptic type on the Lie group $SL_2(\mathbb{R})$ is Liouville integrable.

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## 1. INTRODUCTION

Let $G$ be a connected three-dimensional unimodular Lie group, and let $L$ be the Lie algebra of left-invariant vector fields on $G$. A left-invariant sub-Riemannian (SR) structure on $G$ is defined as a left-invariant rank 2 subbundle $\Delta$ of the tangent bundle $TG$, $\Delta + [\Delta, \Delta] = TG$, equipped with a left-invariant inner product $g$ on $\Delta$ [1]. An SR structure can be defined with the use of an orthonormal frame $f_1, f_2 \in L$,

$$\Delta_q = \operatorname{span}(f_1(q), f_2(q)), \qquad g(f_i(q), f_j(q)) = \delta_{ij}, \qquad i, j = 1, 2, \tag{1}$$

where $q \in G$ and $\delta_{ij}$ is the Kronecker delta.

Finding minimizers for SR structures is the main problem of SR geometry, which is studied in the present paper. An *SR minimizer* is defined as a Lipschitz curve $q : [0, t_1] \to G$ such that $\dot{q}(t) \in \Delta_{q(t)}$ for almost all $t \in [0, t_1]$ and the curve length

$$l(q(\cdot)) = \int_0^{t_1} \sqrt{g(\dot{q}(t), \dot{q}(t))}\, dt$$

is minimal on the set of all such curves joining two given points $q(0) = q_0$ and $q(t_1) = q_1$. An *SR geodesic* is defined as a curve on $G$ whose sufficiently small arcs are SR minimizers.

In other words, an SR minimizer is a solution of the optimal control problem [2]

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \qquad q \in G, \qquad (u_1, u_2) \in \mathbb{R}^2, \tag{2}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{4}$$

The most efficient approach to the problem of finding SR minimizers is based on optimal control theory and consists of the following stages [2].

1. Proof of the existence of SR minimizers. (In problems of SR geometry in the general case, this stage is a standard consequence of the Rashevsky–Chow theorem and the Filippov theorem [2].)

2. Parametrization of SR geodesics with the use of the Pontryagin maximum principle [3, p. 164].

3. Choice of SR minimizers among SR geodesics with the use of second-order optimality conditions and a detailed analysis of the structure of the family of SR geodesics.

Note that the search of a parametrization of SR geodesics can be a nontrivial problem for left-invariant SR structures on Lie groups.

One can ask whether such a parametrization is theoretically possible in some natural sense; this is the question concerning the integrability of the system of ordinary differential equations (ODE) that defines the SR geodesics. An example of a six-dimensional Lie group with a nonintegrable system of ODE for the SR geodesics can be found in [4].

The aim of the present paper is to construct a complete system of first integrals for left-invariant SR structures of elliptic type (see below) on the Lie group $SL_2(\mathbb{R})$ (that is, the special linear group of degree two over the field of real numbers). The Lie group $SL_2(\mathbb{R})$ is defined as the group of linear area-preserving transformations of the plane $\mathbb{R}^2$. Throughout the following, we mean the group over the field $\mathbb{R}$ of real numbers and use the notation $SL(2) = SL_2(\mathbb{R})$. Left-invariant SR structures on three-dimensional Lie groups were classified in [5] to within local isometries; in accordance with the results of that paper, two families of nonequivalent SR structures can be defined on the group $SL(2)$. The difference is determined by the restriction of the Killing form to the distribution. (It can be positive definite or indefinite.) We say that a structure of elliptic type $SL_e(2)$ is defined on the group $SL(2)$ in the first case and a structure of hyperbolic type $SL_h(2)$, in the second case.

In the present paper, we show that the Hamiltonian system of the Pontryagin maximum principle (the system of ODE for the SR geodesics) is Liouville integrable for left-invariant SR structures of elliptic type $SL_e(2)$.

## 2. SR STRUCTURES AND PONTRYAGIN MAXIMUM PRINCIPLE

Left-invariant contact SR structures on three-dimensional Lie groups were classified in [5] to within local isometries. In particular, it was shown that if $G$ is a unimodular group, i.e., if its Lie algebra is one of the Lie algebras $h_3$, $so(3)$, $sl(2)$, $se(2)$, and $sh(2)$, then there exists an orthonormal frame of the form (1) such that $L = \operatorname{span}(f_0, f_1, f_2)$ and the commutation relations

$$[f_2, f_1] = f_0, \tag{5}$$
$$[f_1, f_0] = (\chi + \varkappa) f_2, \tag{6}$$
$$[f_2, f_0] = (\chi - \varkappa) f_1 \tag{7}$$

hold for some constants $\chi \geq 0$ and $\varkappa \in \mathbb{R}$.

By [2], the SR geodesics parametrized by the arc length for contact left-invariant structures on Lie groups are the projections $q(t) = \pi(\lambda(t))$, $\pi : T^*G \to G$, of trajectories of the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, $\lambda \in T^*G$, where the Hamiltonian function has the form

$$H(\lambda) = (h_1^2(\lambda) + h_2^2(\lambda))/2, \qquad h_i(\lambda) = \langle \lambda, f_i(q) \rangle, \qquad q = \pi(\lambda).$$

Here the Hamiltonian vector field $\vec{H}$ on the cotangent bundle $T^*G$ is given by the equation

$$\sigma_\lambda(\cdot\,, \vec{H}) = d_\lambda H, \qquad \lambda \in T^*G,$$

where $\sigma = ds$ and $s_\lambda = \lambda \circ \pi_*$.

In the present paper, we study the problem of Liouville integrability of the Hamiltonian vector field $\vec{H}$.

By virtue of relations (5)–(7), we have the following relations for the Poisson brackets:

$$\{H, h_1\} = h_2\{h_2, h_1\} = h_2 h_0,$$
$$\{H, h_2\} = h_1\{h_1, h_2\} = -h_1 h_0,$$
$$\{H, h_0\} = h_1\{h_1, h_0\} + h_2\{h_2, h_0\} = 2\chi h_1 h_2.$$

Therefore, the Hamiltonian system with Hamiltonian function $H$ has the form

$$\dot h_1 = h_2 h_0, \tag{8}$$
$$\dot h_2 = -h_1 h_0, \tag{9}$$
$$\dot h_0 = 2\chi h_1 h_2, \tag{10}$$
$$\dot q = h_1 f_1 + h_2 f_2. \tag{11}$$

It is well known that the Hamiltonian function $H$ is a first integral of the Hamiltonian system (8)–(11). In the polar coordinates $h_1 = r\cos\theta$, $h_2 = r\sin\theta$, the vertical subsystem of the Hamiltonian system (8)–(11) (for the adjoint variables $h_i$) acquires the form

$$\dot r = 0, \qquad \dot\theta = -h_0, \qquad \dot h_0 = \chi r^2 \sin(2\theta).$$

By performing yet another change of variables $\gamma = 2\theta$, $c = -2h_0$, we obtain the classical system describing the mathematical pendulum,

$$\dot r = 0, \qquad \dot\gamma = c, \qquad \dot c = -2\chi r^2 \sin\gamma.$$

It is well known that this system has a first integral, namely, the total energy of the pendulum,

$$E = c^2/2 - 2\chi r^2 \cos\gamma = 2h_0^2 - 2\chi(h_1^2 - h_2^2). \tag{12}$$

To prove the Liouville integrability of the Hamiltonian system, one should construct a complete system of first integrals, i.e., indicate three functionally independent first integrals in involution [6, p. 121]. In this section, we have shown that the Hamiltonian system (8)–(11) has two left-invariant first integrals, the Hamiltonian function $H$ and the total energy $E$ of the pendulum. The lacking third first integral for the Hamiltonian system (8)–(11) can be constructed on the basis of right-invariant vector fields on the group.

## 3. LIE GROUP $SL(2)$. SR STRUCTURES OF ELLIPTIC TYPE $SL_e(2)$

In the most frequently used representation, the Lie group $SL(2)$ and its Lie algebra $sl(2)$ are given by the $2\times 2$ matrices [7, p. 13 of the Russian translation]

$$SL(2) = \{X \in \mathbb{R}^{2\times 2} |\ \det X = 1\}, \tag{13}$$
$$sl(2) = \{A \in \mathbb{R}^{2\times 2} |\ \operatorname{tr} A = 0\} = \operatorname{span}(E_{11} - E_{22}, E_{12}, E_{21}), \tag{14}$$

where $E_{ij}$ stands for the matrix whose unique nonzero entry is at position $(i, j)$ and is equal to unity.

In what follows, we use a different representation of the group $SL(2)$; more precisely, we define an element of the group by a $3\times 3$ matrix of the form

$$SL(2) = \left\{ G = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \det G = 1 \right\}. \tag{15}$$

For the Lie algebra $sl(2)$, we use the basis

$$sl(2) = \left\{ A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid A_{11} + A_{22} = 0 \right\} = \operatorname{span}(A_1, A_2, A_3), \tag{16}$$

$$A_1 = E_{12} + E_{21}, \qquad A_2 = E_{11} - E_{22}, \qquad A_3 = E_{12} - E_{21}, \tag{17}$$
$$[A_2, A_1] = 2A_3, \qquad [A_1, A_3] = -2A_2, \qquad [A_2, A_3] = 2A_1. \tag{18}$$

For the parametrization of the group $SL(2)$ with the use of three parameters $\nu$, $\alpha$, and $\varphi$, we use the factorization

$$SL(2) \ni G = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = A(\alpha)N(\nu)K(\varphi),$$

$$A(\alpha) = \begin{pmatrix} e^{\alpha} & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N(\nu) = \begin{pmatrix} \cosh\nu & \sinh\nu & 0 \\ \sinh\nu & \cosh\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $v \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $\varphi \in S^1$. The explicit expressions for $G_{ij}$ are as follows:

$$G_{11} = e^{\alpha}(\cos\varphi\cosh\nu - \sin\varphi\sinh\nu), \qquad G_{12} = e^{\alpha}(\sin\varphi\cosh\nu + \cos\varphi\sinh\nu),$$
$$G_{21} = e^{-\alpha}(-\sin\varphi\cosh\nu + \cos\varphi\sinh\nu), \qquad G_{22} = e^{-\alpha}(\cos\varphi\cosh\nu + \sin\varphi\sinh\nu).$$

Set $q = (\nu, \alpha, \varphi)^T$. We use the following isomorphism of sets:

$$SL(2) \cong \mathbb{R}_\nu \times \mathbb{R}_\alpha \times S_\varphi, \qquad G = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \nu \\ \alpha \\ \varphi \end{pmatrix} = q.$$

The canonical frame for sub-Riemannian structures of elliptic type $SL_e(2)$ is defined as follows:

$$f_1 = \frac{1}{2}GA_1, \qquad f_2 = \frac{\xi}{2}GA_2, \qquad f_0 = \frac{\xi}{2}GA_3, \qquad G \in SL(2), \qquad \xi > 1, \tag{19}$$

$$[f_1, f_0] = -f_2, \qquad [f_2, f_0] = \xi^2 f_1, \qquad [f_2, f_1] = f_0. \tag{20}$$

Here and in the following, we proceed from the two parameters $(\chi, \varkappa)$ introduced in [5] to a single parameter $\xi$ by using the relations $\chi = (-1+\xi^2)/2$ and $\varkappa = (-1-\xi^2)/2$. The explicit expressions for $f_1$, $f_2$, and $f_0$ are as follows:

$$f_1 = \frac{1}{2}\begin{pmatrix} G_{12} & G_{11} & 0 \\ G_{22} & G_{21} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \frac{\xi}{2}\begin{pmatrix} G_{11} & -G_{12} & 0 \\ G_{21} & -G_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_0 = \frac{\xi}{2}\begin{pmatrix} -G_{12} & G_{11} & 0 \\ -G_{22} & G_{21} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next, consider the optimal control problem

$$\dot{G} = u_1 f_1 + u_2 f_2, \qquad G \in SL(2), \qquad (u_1, u_2) \in \mathbb{R}^2, \tag{21}$$

$$G(0) = \mathrm{Id}, \qquad G(t_1) = G_1, \tag{22}$$

$$l = \int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min. \tag{23}$$

Note that the integral (23) in the considered problem differs from the original one (4) by the absence of the root in the integrand. By using the Cauchy–Schwarz inequality, one can show that these problems are equivalent (see [1, p. 6]). Note also that, by virtue of the left invariance of the problem, one can fix the initial condition at the unit of the group, $G(0) = \mathrm{Id}$, without loss of generality.

In problem (21)–(23), we use the Pontryagin maximum principle, which is a necessary optimality condition. In the next section, we write out the Hamiltonian system of the Pontryagin maximum principle via left-invariant Hamiltonians linear on the fibers of the cotangent bundle, and then we prove the Liouville integrability of the resulting system.

## 4. INTEGRABILITY OF SR STRUCTURES OF ELLIPTIC TYPE $SL_e(2)$

In this section, we study problem (21)–(23), where the left-invariant vector fields $f_i$ are given by formulas (19) and (20).

Consider a smooth curve $q(\cdot) = (\nu(\cdot), \alpha(\cdot), \varphi(\cdot))^T \in C^\infty(\mathbb{R} \to \mathbb{R}_\nu \times \mathbb{R}_\alpha \times S^1_\varphi)$ issuing from the point $\{\nu = 0,\ \alpha = 0,\ \varphi = 0\}$. In Section 3, we have parametrized the group $SL(2)$ by three parameters $(\nu, \alpha, \varphi)$. The smooth curve $q(\cdot)$ determines a smooth curve $G(\cdot)$ on $SL(2)$ given by the one-parameter family of matrices $G(t) = \{G(\nu(t), \alpha(t), \varphi(t)) |\ t \in \mathbb{R}\}$ smoothly depending on the parameter $t$ and satisfying the condition $G(0) = \mathrm{Id}$. The velocity vector of the curve $G(t)$ can be represented as

$$\dot{G} = \frac{\partial G}{\partial \nu}\dot{\nu} + \frac{\partial G}{\partial \alpha}\dot{\alpha} + \frac{\partial G}{\partial \varphi}\dot{\varphi}.$$

By combining the coefficients multiplying $\dfrac{\partial G}{\partial \nu}$, $\dfrac{\partial G}{\partial \alpha}$, and $\dfrac{\partial G}{\partial \varphi}$, one can represent the control system (21) in the vector form

$$\dot{q} = \begin{pmatrix} \dot{\nu} \\ \dot{\alpha} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) \\ \sin(2\varphi)/\cosh(2\nu) \\ -\sin(2\varphi)\tanh(2\nu) \end{pmatrix} \frac{u_1}{2} + \begin{pmatrix} -\sin(2\varphi) \\ \cos(2\varphi)/\cosh(2\nu) \\ -\cos(2\varphi)\tanh(2\nu) \end{pmatrix} \frac{\xi u_2}{2}.$$

In a similar way, one can obtain the correspondence between left-invariant vector fields $X_i \sim f_i$ $(i = \{1, 2, 0\})$ in the matrix and vector form,

$$f_1 = \frac{1}{2}GA_1 \sim \frac{1}{2}\begin{pmatrix} \cos(2\varphi) \\ \sin(2\varphi)/\cosh(2\nu) \\ -\sin(2\varphi)\tanh(2\nu) \end{pmatrix} = X_1, \quad f_2 = \frac{\xi}{2}GA_2 \sim \frac{\xi}{2}\begin{pmatrix} -\sin(2\varphi) \\ \cos(2\varphi)/\cosh(2\nu) \\ -\cos(2\varphi)\tanh(2\nu) \end{pmatrix} = X_2,$$

$$f_0 = \frac{\xi}{2}GA_3 \sim \frac{\xi}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = X_0.$$

The right-invariant vector fields $Y_i = -A_iG$ can be expressed as follows:

$$-\frac{1}{2}A_1G = \frac{1}{2}\begin{pmatrix} -G_{21} & -G_{22} & 0 \\ -G_{11} & -G_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \frac{1}{2}\begin{pmatrix} -\cosh(2\alpha) \\ \sinh(2\alpha)\tanh(2\nu) \\ \sinh(2\alpha)/\cosh(2\nu) \end{pmatrix} = Y_1,$$

$$-\frac{\xi}{2}A_2G = \frac{\xi}{2}\begin{pmatrix} -G_{11} & -G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \frac{\xi}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = Y_2,$$

$$-\frac{\xi}{2}A_3G = \frac{\xi}{2}\begin{pmatrix} -G_{21} & -G_{22} & 0 \\ G_{11} & G_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \frac{\xi}{2}\begin{pmatrix} \sinh(2\alpha) \\ -\cosh(2\alpha)\tanh(2\nu) \\ -\cosh(2\alpha)/\cosh(2\nu) \end{pmatrix} = Y_0.$$

By $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in T^*_q(SL(2))$ we denote an element of the cotangent space of the group $SL(2)$ at the point $q$. We rewrite the left-invariant Hamiltonians $h_i(\lambda, q) = \langle \lambda, X_i(q) \rangle$, which are linear on the fibers of the cotangent bundle $T^*(SL(2))$, in the form

$$h_1 = \frac{1}{2}\left(\lambda_1\cos(2\varphi) + \frac{(\lambda_2 - \lambda_3\sinh(2\nu))}{\cosh(2\nu)}\sin(2\varphi)\right),$$

$$h_2 = \frac{\xi}{2}\left(-\lambda_1\sin(2\varphi) + \frac{(\lambda_2 - \lambda_3\sinh(2\nu))}{\cosh(2\nu)}\cos(2\varphi)\right), \qquad h_0 = \frac{\xi\lambda_3}{2}.$$

In a similar way, the right-invariant Hamiltonians $g_i(\lambda) = \langle \lambda, Y_i \rangle$ can be represented as

$$g_1 = \frac{1}{2}\left(-\lambda_1 \cosh(2\alpha) + \frac{(\lambda_3 + \lambda_2 \sinh(2\nu))}{\cosh(2\nu)} \sinh(2\alpha)\right), \qquad g_2 = -\frac{\xi \lambda_2}{2},$$
$$g_0 = -\frac{\xi}{2}\left(-\lambda_1 \sinh(2\alpha) + \frac{(\lambda_3 + \lambda_2 \sinh(2\nu))}{\cosh(2\nu)} \cosh(2\alpha)\right).$$

Note that the mapping $\Omega : (\lambda_1, \lambda_2, \lambda_3) \to (h_1, h_2, h_0)$ is nonsingular and the right-invariant Hamiltonians $g_i$ can be expressed via $h_i$. In what follows, we use only the coordinates $h_i$ but do not present a explicit expression of $g_i$ via $h_i$, because it is quite cumbersome.

The Hamiltonian system (8)–(11) for SR structures of elliptic type $SL_e(2)$ acquires the form

$$\begin{aligned} &\dot h_1 = h_2 h_0, \quad \dot h_2 = -h_1 h_0, \qquad \dot h_0 = (-1 + \xi^2) h_1 h_2, \\ &\dot\nu = \frac{1}{2}(h_1 \cos(2\varphi) - \xi h_2 \sin(2\varphi)), \\ &\dot\alpha = \frac{1}{2}\frac{h_1 \sin(2\varphi) + \xi h_2 \cos(2\varphi)}{\cosh(2\nu)}, \\ &\dot\varphi = \frac{1}{2}(-h_1 \sin(2\varphi) - \xi h_2 \cos(2\varphi)) \tanh(2\nu). \end{aligned} \tag{24}$$

**Theorem.** *The Hamiltonian system* (24) *of the Pontryagin maximum principle for sub-Riemannian structures of elliptic type on the Lie group $SL(2)$ is Liouville integrable.*

**Proof.** We should indicate three functionally independent first integrals in involution for the Hamiltonian system (24).

Since the left translations on a group commute with the right ones, it follows that the right-invariant Hamiltonians $g_i$ are first integrals of the left-invariant Hamiltonian system (24). We have thereby obtained three first integrals of the Hamiltonian system in question.

As was mentioned in Section 1, the vertical subsystem in (24) for the dual variables $h_i$ is independent of the state variables $(\nu, \alpha, \varphi)$ and can be reduced to the equation of the mathematical pendulum. The Hamiltonian function

$$H = (h_1^2 + h_2^2)/2$$

and the total energy

$$E = 2h_0^2 - (\xi^2 - 1)(h_1^2 - h_2^2)$$

of the pendulum are first integrals of system (24). Let us take one of the right-invariant Hamiltonians $g_i$ and show that we have obtained a complete set of first integrals. To this end, one should show that these three first integrals are functionally independent and are in involution. To be definite, we take $g_2$, because this integral has the simplest form,

$$g_2 = -(h_2 \cos(2\varphi) + h_1 \xi \sin(2\varphi)) \cosh(2\nu) - h_3 \sinh(2\nu).$$

The right-invariant Hamiltonian $g_2$ is Poisson commuting with the left-invariant first integrals $H$ and $E$; i.e.,

$$\{H, E\} = \{H, g_2\} = \{E, g_2\} = 0.$$

It follows that the three integrals are in involution. It remains to show that they are functionally independent. To this end, it suffices to show that their gradients $\nabla H$, $\nabla E$, and $\nabla g_2$ are linearly independent on an open dense set $U \subseteq T^* SL(2)$. Since the functions $H$, $E$, and $g_2$ are analytic, it follows that this condition is satisfied provided that there exists at least one point at which the gradients $\nabla H$, $\nabla E$, and $\nabla g_2$ are linearly independent. Let us write out the Jacobian matrix

$$\mathrm{J} = \begin{pmatrix} \nabla H \\ \nabla E \\ \nabla g_2 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & 0 & 0 & 0 & 0 \\ -2(-1+\xi^2) h_1 & 2(-1+\xi^2) h_2 & 4h_0 & 0 & 0 & 0 \\ -\xi \cosh(2\nu)\sin(2\varphi) & -\cosh(2\nu)\cos(2\varphi) & \sinh(2\nu) & \dfrac{\partial g_2}{\partial \nu} & \dfrac{\partial g_2}{\partial \alpha} & \dfrac{\partial g_2}{\partial \varphi} \end{pmatrix}.$$

By $\tilde{\mathrm{J}}$ we denote the third-order minor of the matrix J consisting of the first three columns. The condition

$$\tilde{\mathrm{J}} = 4h_0 \cosh(2\nu)(h_1 \cos(2\varphi) - h_2\xi \sin(2\varphi)) - 4h_1 h_2(-1 + \xi^2)\sinh(2\nu) \not\equiv 0$$

implies that the matrix J has rank 3 at least at one point. The proof of the theorem is complete.

We have thereby shown that the Hamilton system (24) for SR structures of elliptic type $SL_e(2)$ is Liouville integrable, and the Liouville integrability of SR structures of hyperbolic type $SL_h(2)$ can be proved in a similar way.

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