Abnormal trajectories and abnormal set for the (2, 3, 5, 8)-distribution

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Plan of talk

1. 3 statements of the variational problem
2. 3 definitions of abnormal trajectories
3. Normal trajectories
4. Abnormal trajectories
5. Nonsmooth abnormal trajectories
6. Nice abnormal trajectories
7. Abnormal set
8. Sard's conjecture in sub-Riemannian geometry
9. Subanalytic sets, stratifications
10. Properties of the abnormal set
11. Optimality of abnormal trajectories
12. Observations and conjectures

1. Geometric statement of the problem

Given:

- points a_0, $a_1 \in \mathbb{R}^2$, connected by a curve $\gamma_0 \subset \mathbb{R}^2$,
- a number $S \in \mathbb{R}$,
- a point $c \in \mathbb{R}^2$,
- an ellipse $E \subset \mathbb{R}^2$ with the center a_0.

Find:
the shortest curve $\gamma \subset \mathbb{R}^2$, connecting a_0 and a_1, such that the domain bounded by the curves γ_0 and γ, has:

- algebraic area S,
- center of mass c,
- ellipse of inertia E.

2. Optimal control problem

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x), \quad x = (x_1, \dots, x_8) \in \mathbb{R}^8,\ u = (u_1, u_2) \in \mathbb{R}^2,$$
$$x(0) = x^0 = (0, \dots, 0), \quad x(t_1) = x^1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$

where the vector fields in the right-hand side are

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8},$$
$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4}\frac{\partial}{\partial x_7}.$$

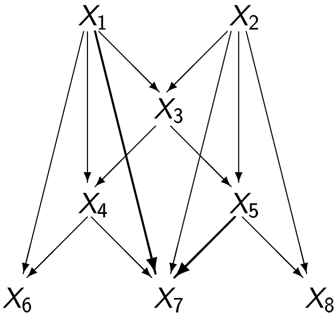
3. Sub-Riemannian problem

- G — connected simply connected free nilpotent Lie group of rank 2, step 4.
- its Lie algebra:

$$\mathfrak{g} = \mathrm{span}(X_1, \dots, X_8),$$
$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad [X_1, X_4] = X_6,$$
$$[X_2, X_4] = [X_1, X_5] = X_7, \quad [X_2, X_5] = X_8.$$

- X_i are left-invariant vector fields on G.
- Sub-Riemannian structure (D, g) on G:
  - $D = \mathrm{span}(X_1, X_2)$,
  - $g(X_i, X_j) = \delta_{ij}, \ i, j = 1, 2$.

Product table in the Lie algebra $\mathfrak{g}$



Growth vector of the distribution D:

$$(\dim D_x, \dim D_x^2, \dim D_x^3, \dim D_x^4) = (2, 3, 5, 8).$$

- $h_i(\lambda) = \langle \lambda, X_i \rangle$, $\lambda \in T^*G$,
- $\vec{h}_i \in \mathrm{Vec}(T^*G)$ Hamiltonian vector fields
- $h_u^\nu(\lambda) = u_1 h_1 + u_2 h_2 + \frac{\nu}{2}(u_1^2 + u_2^2)$, $u = (u_1, u_2) \in \mathbb{R}^2$, $\nu \in \mathbb{R}$

Theorem (PMP)

If $x(t)$, $u(t)$ are optimal, then there exists a Lipschitzian curve $\lambda_t \in T^*_{x(t)}G$ and a number $\nu \in \{-1, 0\}$ such that:

(1) $\dot{\lambda}_t = u_1(t)\vec{h}_1(\lambda_t) + u_2(t)\vec{h}_2(\lambda_t)$,

(2) $h^\nu_{u(t)}(\lambda_t) = \max_{v \in \mathbb{R}^2} h^\nu_v(\lambda_t)$,

(3) $(\lambda_t, \nu) \neq (0, 0)$.

- $\nu = -1 \quad \Rightarrow \quad x(t)$ normal trajectory,
- $\nu = 0 \quad \Rightarrow \quad x(t)$ abnormal trajectory.

2. Endpoint mapping

- $\mathrm{End} : L^2([0, t_1], \mathbb{R}^2) \to G,\ u(\cdot) \mapsto x(t_1),$
  $\dot{x} = u_1 X_1(x) + u_2 X_2(x),\ x(0) = x^0,$
- $D_u \mathrm{End} : L^2 \to T_{x(t_1)} G$

A control $u(\cdot)$ is called abnormal if it is a critical point of the mapping End:

$$\mathrm{Im}\, D_u \mathrm{End} \neq T_{x(t_1)} G.$$

The corresponding trajectory $x(\cdot)$ is called abnormal trajectory.

3. Characteristic curve

- $\sigma = dp \wedge dq \in \Lambda^2(T^*G)$ nondegenerate:

$$\operatorname{Ker} \sigma_\lambda = \{v \in T_\lambda(T^*G) \mid \sigma(v, \cdot\,) = 0\} = \{0\}.$$

- But for a submanifold $S \subset T^*G$ one may have $\operatorname{Ker}(\sigma_\lambda|_S) \neq \{0\}$.
- $S = \{\lambda \in T^*G \mid h_1(\lambda) = h_2(\lambda) = 0\}$ a smooth submanifold.
- A Lipschitzian curve $\lambda_t \in S$ is called an abnormal extremal if it is a characteristic curve of $\sigma|_S$:

$$\dot\lambda_t \in \operatorname{Ker}(\sigma_{\lambda_t}|_S).$$

  The corresponding trajectory $x(\cdot)$ is called abnormal trajectory.

Def. 1 $\quad\Leftrightarrow\quad$ Def. 2 $\quad\Leftrightarrow\quad$ Def. 3
(A. Agrachev, D. Barilari, U. Boscain, A comprehensive introduction to sub-Riemannian geometry, 2019)

Abnormal set

Abnormal set of a distribution $D = \mathrm{span}(X_1, X_2)$, corresponding to the identity element $\mathrm{Id} \in G$:

$$\mathrm{Abn} = \{x(t) \mid x(\cdot) \text{ — abnormal trajectory of the distribution } D, \quad x(0) = \mathrm{Id},\ t > 0\}.$$

A sub-Riemannian metric has the most complicated singularities near abnormal trajectories.

- How small (big) can be an abnormal set?
- What is its structure?
- Sard's conjecture in sub-Riemannian geometry:

$$\mathrm{mes}(\mathrm{Abn}) = 0.$$

Example 1: Heisenberg group.

- $G \cong \mathbb{R}^3$, $X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}$,
- $\mathrm{Abn} = \{\mathrm{Id}\}$.

Example 2: flat Martinet case.

- $M = \mathbb{R}^3$, $X_1 = \frac{\partial}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2}$,
- $\mathrm{Abn}(0) = \{x_2 = x_3 = 0\}$.

Example 3: Engel group.

- $G \cong \mathbb{R}^4$, $X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_4}$,
- $\mathrm{Abn} = \{x_1 = x_3 = 0,\ x_4 = \frac{x_2^3}{6}\}$.

Example 4: Cartan group.

- $G \cong \mathbb{R}^5$, $X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_5}$, $X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_4}$,
- $\mathrm{Abn} = \{x_3 = 0,\ x_4 = \frac{(x_1^2+x_2^2)x_2}{6},\ x_5 = -\frac{(x_1^2+x_2^2)x_1}{6}\}$.

Experimental facts

In all known examples of rank 2 distributions (step ≤ 3):

- abnormal set is a smooth algebraic manifold of codimension ≥ 2,
- for left-invariant distributions the abnormal set has codimension 3,
- Sard's conjecture is verified.

What happens at step 4?

Normal trajectories for the $(2, 3, 5, 8)$ sub-Riemannian problem

- $\nu = -1 \quad \Rightarrow \quad \dot{\lambda}_t = \vec{H}(\lambda_t), \ H = (h_1^2 + h_2^2)/2$
- Hamiltonian system on the Lie coalgebra $\mathfrak{g}^*$:

$$\dot{h}_i = \{H, h_i\}, \qquad i = 1, \dots, 8. \tag{1}$$

- Symplectic leaves (coadjoint orbits) in $\mathfrak{g}^*$ have maximal dimension 4.

Theorem (L.Lokutsievskiy, Yu.S.)

There exists an open set of 4-dimensional leaves such that the Hamiltonian system (1) *is not Liouville integrable on these leaves.*

Abnormal trajectories for the $(2,3,5,8)$ distribution

Theorem (PMP for the abnormal case)

A Lipschitzian curve $\lambda \in \mathrm{Lip}([0,t_1], T^*G)$ is an abnormal extremal if and only if there exists a control $u \in L^\infty([0,t_1], \mathbb{R}^2)$ such that:

$$\lambda_t \neq 0, \quad h_1 = h_2 = h_3 = 0,$$
$$u_1 h_4 + u_2 h_5 = 0,$$
$$\begin{pmatrix} \dot h_4 \\ \dot h_5 \end{pmatrix} = \begin{pmatrix} h_6 & h_7 \\ h_7 & h_8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \dot h_6 = \dot h_7 = \dot h_8 = 0,$$

the corresponding abnormal trajectory $x(t) = \pi(\lambda_t)$ satisfies the equation

$$\dot x = u_1 X_1 + u_2 X_2.$$

Nice abnormal trajectories

- In the case
$$(h_4^2 + h_5^2)(\lambda_t) \neq 0 \quad \Leftrightarrow \quad \lambda_t \in (D^2)^\perp \setminus (D^3)^\perp$$
the abnormal extremal λ_t is called *nice*.
- In this case $u_1 = -h_5, \ u_2 = h_4$ up to time reparameterization.
- The corresponding extremals are trajectories of the abnormal vector field
$$\vec{A} = -h_5 \vec{h}_1 + h_4 \vec{h}_2 .$$
- The corresponding trajectories are determined by the abnormal exponential mapping $\mathrm{Exp}(\lambda, t) = \pi \circ e^{t\vec{A}}(\lambda)$; in particular, they are smooth.

Nonsmooth abnormal trajectories

- If $(h_4^2 + h_5^2)(\lambda_t)$ vanishes for some t, then abnormal trajectories may be nonsmooth.
- Example:

$$x(t) = \begin{cases} e^{t(u_1 X_1 + u_2 X_2)}, & t \in [0, \bar{t}], \quad \bar{t} > 0, \\ e^{(t-\bar{t})(v_1 X_1 + v_2 X_2)} \circ e^{\bar{t}(u_1 X_1 + u_2 X_2)}, & t \geq \bar{t}. \end{cases}$$

- Such trajectories have corner point for $t = \bar{t}$, thus they are not optimal (E. Le Donne, E. Hakavuori).
- We call such trajectories asymptotic.

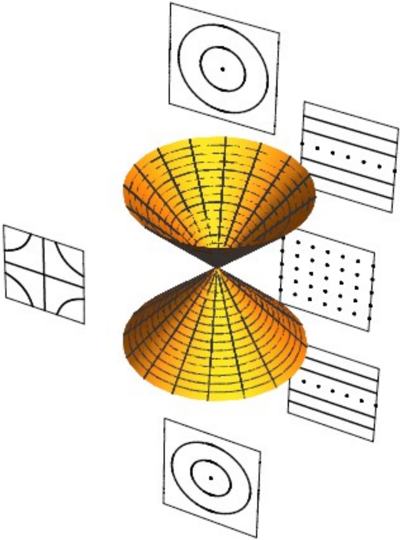
Hamiltonian system of PMP for nice trajectories

$$
\begin{aligned}
&h_1 = h_2 = h_3 = 0, \\
&\begin{pmatrix} \dot{h}_4 \\ \dot{h}_5 \end{pmatrix} = C \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}, \qquad C = \begin{pmatrix} h_7 & -h_6 \\ h_8 & -h_7 \end{pmatrix} \\
&\dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0, \\
&\dot{x} = -h_5 X_1(x) + h_4 X_2(x).
\end{aligned}
$$

Types of trajectories $(x_1(t), x_2(t))$

- $\Delta = \det C$,
- $\Delta > 0 \quad \Rightarrow \quad$ elliptic case,
  $(x_1(t), x_2(t))$ are ellipses,
- $\Delta < 0 \quad \Rightarrow \quad$ hyperbolic case,
  $(x_1(t), x_2(t))$ are hyperbolas and lines (asymptotes),
- $\Delta = 0 \quad \Rightarrow \quad$ parabolic case,
  $(x_1(t), x_2(t))$ are parabolas and lines.

Foliation of $(D^2)^\perp$ by abnormal extremals



Abnormal trajectories in the elliptic case $\Delta > 0$

$$
\begin{aligned}
x_i(t) &= P_i(t), \qquad i = 1, 2, \\
x_i(t) &= L_i t + P_i(t), \qquad i = 3, \dots, 8, \\
P_i(t) &= \sum_{j=0}^{N_i} (c_{ij} \cos(j\delta t) + s_{ij} \sin(j\delta t)), \\
L_i&,\ c_{ij},\ s_{ij} \in \mathbb{R}, \\
\delta &= \sqrt{\Delta}.
\end{aligned}
$$

Abnormal trajectories in the hyperbolic case $\Delta < 0$

If a vector (h_4^0, h_5^0) is not an eigenvector of the matrix C, then

$$
\begin{aligned}
& x_i(t) = P_i(t), \qquad i = 1, 2, \\
& x_i(t) = L_i t + P_i(t), \qquad i = 3, \dots, 8, \\
& P_i(t) = \sum_{j=0}^{N_i} (c_{ij} \operatorname{ch}(j\delta t) + s_{ij} \operatorname{sh}(j\delta t)), \\
& L_i, \ c_{ij}, \ s_{ij} \in \mathbb{R}, \\
& \delta = \sqrt{|\Delta|}.
\end{aligned}
$$

If a vector (h_4^0, h_5^0) is an eigenvector of the matrix C, then $x(t) = e^{(u_1 X_1 + u_2 X_2)t}$, and $(x_1(t), x_2(t))$ is a line.

Abnormal trajectories in the parabolic case $\Delta = 0$

$$x_i(t) = P_i(t), \qquad i = 1, \dots, 8,$$

$$P_i(t) = \sum_{j=0}^{N_i} a_{ij} t^j,$$

$$a_{ij} \in \mathbb{R}.$$

- One-parameter subgroups of the Lie group G tangent to the distribution D are abnormal trajectories (parabolic).
- We call such trajectories degenerate.

Subsets of the abnormal set

- Denote the sets of points in G filled by nice, asymptotic and degenerate trajectories starting at Id by $\mathrm{Abn}_{\mathrm{nice}}$, $\mathrm{Abn}_{\mathrm{as}}$ and $\mathrm{Abn}_{\mathrm{deg}}$ respectively.
- Then

$$\mathrm{Abn}_{\mathrm{nice}} = \mathrm{Exp}(\Pi \times \mathbb{R}_+), \qquad \Pi = (D^2)^{\perp} \cap \mathfrak{g}^*,$$
$$\mathrm{Abn}_{\mathrm{as}} = \{e^{v_1 X_1 + v_2 X_2} \circ e^{u_1 X_1 + u_2 X_2} \mid (u_1, u_2), (v_1, v_2) \in \mathbb{R}^2\},$$
$$\mathrm{Abn}_{\mathrm{deg}} = \{e^{u_1 X_1 + u_2 X_2} \mid (u_1, u_2) \in \mathbb{R}^2\}.$$

Properties of subsets of the abnormal set

Proposition

There hold the equalities

$$\mathsf{Abn} = \mathsf{Abn}_{\mathrm{as}} \cup \mathsf{Abn}_{\mathrm{nice}},$$
$$\mathsf{Abn}_{\mathrm{deg}} = \mathsf{Abn}_{\mathrm{as}} \cap \mathsf{Abn}_{\mathrm{nice}}.$$

Degenerate abnormal surface

- Degenerate abnormal trajectories are one-parameter subgroups tangent to Δ.
- Their projections to the plane (x, y) are lines.

Proposition

$\mathrm{Abn}_{\mathrm{deg}}$ *is a 2-dimensional smooth algebraic manifold, diffeomorphic to* $\mathbb{R}^2$*, a graph of a mapping* $(x_1, x_2) \mapsto (x_1, \dots, x_8)$*:*

$$\mathrm{Abn}_{\mathrm{deg}} = \Bigg\{ x \in \mathbb{R}^8 \mid x_3 = 0, \quad x_4 = \frac{(x_1^2 + x_2^2)\, x_2}{6}, x_5 = -\frac{(x_1^2 + x_2^2)\, x_1}{6}, \quad x_6 = \frac{x_1^3 x_2}{24}, \quad x_7 = 0, \quad x_8 = -\frac{x_1 x_2^3}{24} \Bigg\}.$$

Properties of the asymptotic abnormal set

- Asymptotic abnormal trajectories can be reduced to product of 2 one-parameter subgroups.
- Their projections to the plane (x, y) are broken lines with 2 edges.

Theorem

(1) $\mathrm{Abn}_{\mathrm{as}}$ *is a semialgebraic set and is not an algebraic manifold.*

(2) *The set* $\mathrm{Abn}_{\mathrm{as}}$ *is not closed,*

(3) $\mathrm{Abn}_{\mathrm{as}} \cap \{x_3 \neq 0\}$ *is a smooth 4-dimensional manifold, diffeomorphic to* $\mathbb{R}^4 \times \{\pm 1\}$*.*

(4) $\mathrm{Abn}_{\mathrm{as}} \cap \{x_3 = 0\} = \mathrm{Abn}_{\mathrm{deg}}$ *is a smooth 2-dimensional manifold, diffeomorphic to* $\mathbb{R}^2$*.*

Properties of the abnormal set

Theorem

(1) *The set* Abn *is subanalytic, thus Whitney stratifiable.*

(2) $\dim \mathrm{Abn} = 5$.

(3) $\mathrm{mes}(\mathrm{Abn}) = 0$.

(4) *The set* Abn *is not closed.*

(5) *The set* Abn *is not a smooth manifold and is not semianalytic.*

(6) $e^{\mathbb{R}X_0}(\mathrm{Abn}) = e^{\mathbb{R}Y}(\mathrm{Abn}) = \mathrm{Abn}$.

Remark

(1) proved earlier by F. Boarotto, D. Vittone (2020).

(3) proved earlier by E. Le Donne, G. P. Leonardi, R. Monti, D. Vittone (2018),

Symmetries of the abnormal set

- rotations

$$X_0 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_5 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_5} + P \frac{\partial}{\partial x_6} + Q \frac{\partial}{\partial x_7} + R \frac{\partial}{\partial x_8},$$

$$P = \frac{x_1^4}{24} - \frac{x_1^2 x_2^2}{8} - x_7, \quad Q = -\frac{x_1 x_2^3}{12} - \frac{x_1^3 x_2}{12} + 2x_6 - 2x_8,$$

$$R = -\frac{x_1^2 x_2^2}{8} + \frac{x_2^4}{24} + x_7,$$

- dilations

$$Y = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3} + 3x_4 \frac{\partial}{\partial x_4} + 3x_5 \frac{\partial}{\partial x_5} + 4x_6 \frac{\partial}{\partial x_6} + 4x_7 \frac{\partial}{\partial x_7} + 4x_8 \frac{\partial}{\partial x_8}.$$

Circular trajectories

The subset of Abn corresponding to circles $(x_1(t), x_2(t))$, and to circles up to the first loop:

$$\mathrm{Abn}_{\mathrm{circ}} = \{\mathrm{Exp}(\lambda, t) \mid h_6 = h_8 \neq 0,\ h_7 = 0,\ (h_4^0)^2 + (h_5^0)^2 \neq 0,\ t \in \mathbb{R}\},$$
$$\mathrm{Abn}^1_{\mathrm{circ}} = \{\mathrm{Exp}(\lambda, t) \mid h_6 = h_8 \neq 0,\ h_7 = 0,\ (h_4^0)^2 + (h_5^0)^2 \neq 0,\ t \in (0, 2\pi/\delta)\}.$$

Proposition

(1) $\mathrm{Abn}_{\mathrm{circ}}$ *is a 3-dimensional subanalytic set.*

(2) $\mathrm{Abn}^1_{\mathrm{circ}}$ *is a smooth 3-dimensional manifold, a graph of a smooth non-algebraic mapping* $(x_1, x_2, x_3) \mapsto (x_1, \ldots, x_8)$, *expressed in algebraic functions and the inverse function to*

$$t \mapsto \frac{t - \sin t}{1 - \cos t}.$$

Equilateral-hyperbolic trajectories

The subset of Abn corresponding to equilateral hyperbolas $(x_1(t), x_2(t))$ with asymptotes parallel to coordinate axes:

$$\mathrm{Abn}_{\mathrm{hyp}} = \{\mathrm{Exp}(\lambda, t) \mid h_6 = h_8 = 0,\ h_7 = 1,\ (h_4^0, h_5^0) \in \mathbb{R}^2,\ t \in \mathbb{R}\}.$$

Proposition

$\mathrm{Abn}_{\mathrm{hyp}}$ *is a smooth* 3*-dimensional manifold, a graph of a smooth nonalgebraic mapping* $(x_1, x_2, x_3) \mapsto (x_1, \dots, x_8)$ *expressed in algebraic functions and the inverse function to*

$$t \mapsto \frac{t - \operatorname{sh} t}{1 - \operatorname{ch} t}.$$

Hausdorff measure and metric dimension of the abnormal set

Theorem

(1) $\mathcal{H}^\mu(\mathsf{Abn}) = \begin{cases} \infty, & 0 \leq \mu \leq 5, \\ 0, & \mu > 5. \end{cases}$

(2) $\mathrm{met\,dim}(\mathsf{Abn}) = 5.$

Observations, questions and conjecture

New properties of the abnormal set:

- nonsmoothness,
- nonclosedness,
- nonsemianalyticity.

Questions:

1. What is the maximal codimension d of an algebraic manifold that contains Abn? ($d \geq 1$: E. Le Donne, G. P. Leonardi, R. Monti, D. Vittone (2018))
2. Is it possible to define Abn by smooth equalities and inequalities? (By analytic not possible: the set Abn is not semianalytic).

Conjecture

For left-invariant distributions of rank 2 *the abnormal set is a subanalytic set of codimension* 3.

Optimality of abnormal trajectories

- Degenerate trajectories (projecting to lines) are optimal.
- (Abnormal and normal) trajectories projecting to optimal trajectories on the Heisenberg, Engel, and Cartan group are optimal.
- Short arcs of nice abnormal trajectories are optimal.
- Asymptotic trajectories containing a corner point inside them are not optimal.
- Nondegenerate nice abnormal trajectories (projecting to 2-nd order curves) contain a cut point.
- Nice abnormal trajectories projecting to circles are locally optimal up to infinity (do not contain conjugate points).

Conjecture

All nice abnormal trajectories projecting to 2-nd order curves are locally optimal up to infinity (do not contain conjugate points).

Publications

1. L.V.Lokutsievsky, Yu.S., On Liouville integrability of sub-Riemannian problems on Carnot groups of step 4 and more, *Sbornik mathematics* 209:5 (2018), 74–119
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3. Yu. S., E. Sachkova, Symmetries and Parameterization of Abnormal Extremals in the Sub-Riemannian Problem with the Growth Vector $(2, 3, 5, 8)$, Russian Journal of Nonlinear Dynamics, 15 (2019), 4: 577 – 585.
4. Yu.S., E. Sachkova, The structure of abnormal extremals in a sub-Riemannian problem with growth vector $(2, 3, 5, 8)$, Sb. Math., 211:10 (2020), 1460–1485.
5. Yu.S., E. Sachkova, An Abnormal Set for the $(2, 3, 5, 8)$-Distribution, Math. Notes, 109:2 (2021), 317–319

Congratulations, Andrey!

С Днем Рождения!!!