ASYMPTOTICS OF EXTREMAL CONTROLS IN THE SUB-RIEMANNIAN PROBLEM ON THE GROUP OF RIGID BODY MOTIONS

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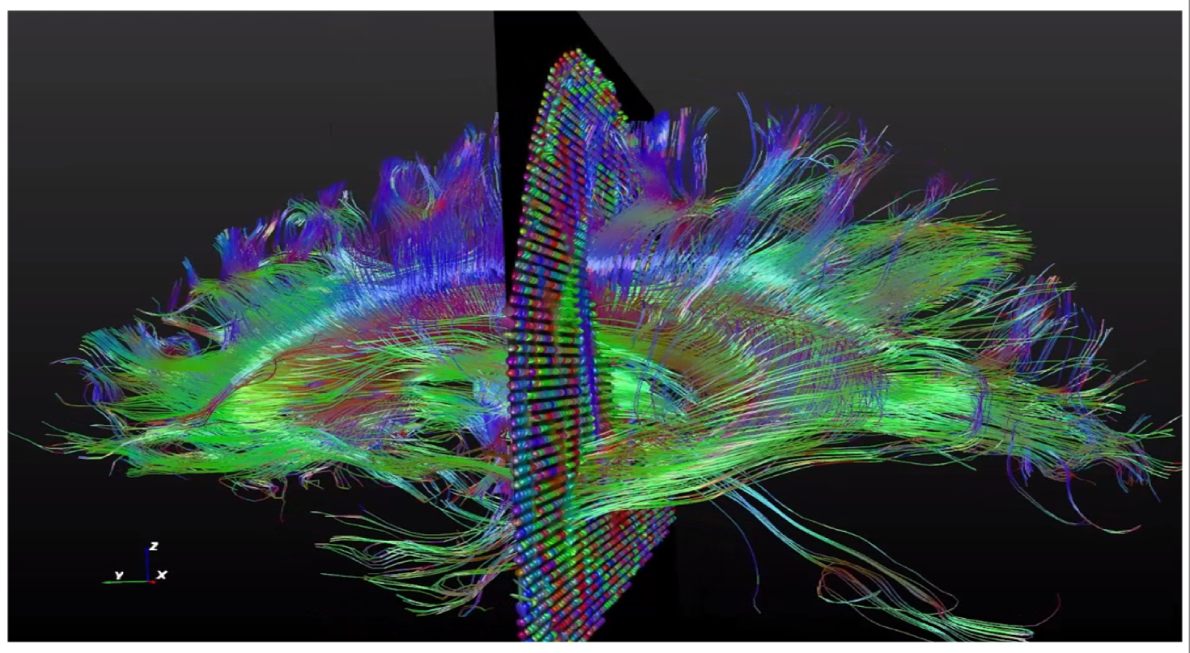
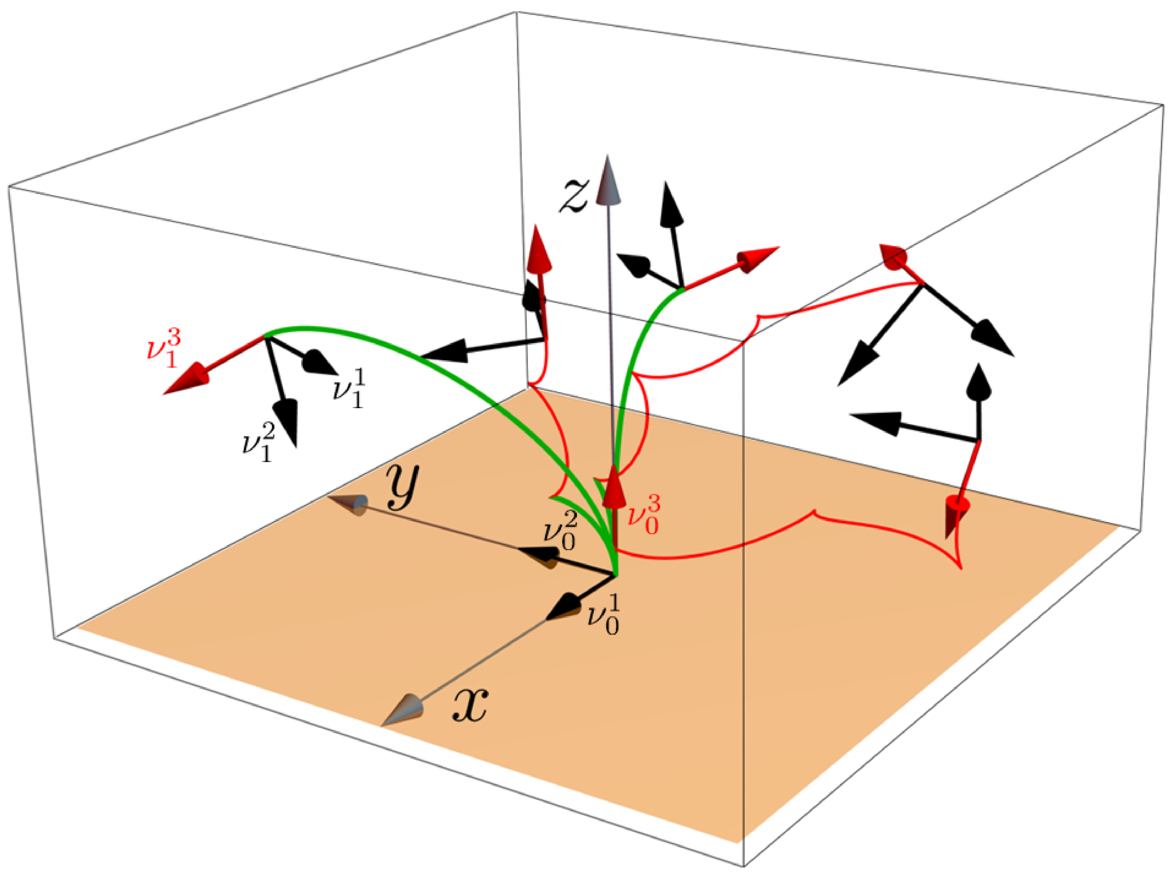
INTRODUCTION

We study a sub-Riemannian(SR) problem on the Lie group SE_3 of rigid body motions in $\mathbb{R}^3$. Solution curves have applications in image processing (tracking of neural fibers and blood vessels in DW-MRI images of human brain); and in robotics (motion planing problem for an aircraft, moving forward/backward).



It can be seen as a problem of optimal motion of a rigid body with nonintegrable constraints. By given two orthonormal frames $N_0 = \{\nu_0^1, \nu_0^2, \nu_0^3\}$, $N_1 = \{\nu_1^1, \nu_1^2, \nu_1^3\}$ attached respectively at two given points $q_0 = (x_0, y_0, z_0)$, $q_1 = (x_1, y_1, z_1)$ in $\mathbb{R}^3$, we aim to find an optimal motion that transfers q_0 to q_1 such that the frame N_0 is transferred to the frame N_1. The frame can move forward or backward along one of the vector chosen in the frame and rotate simultaneously via the remaining two (of three) prescribed axes. The required motion should be optimal in the sense of minimal length in the space $\mathrm{SE}_3 \simeq \mathbb{R}^3 \rtimes \mathrm{SO}_3$.



GROUP OF RIGID BODY MOTIONS IN 3D

Group element:

$g = (\mathbf{x}, R) \in \mathrm{SE}_3 = \mathbb{R}^3 \rtimes \mathrm{SO}_3$.

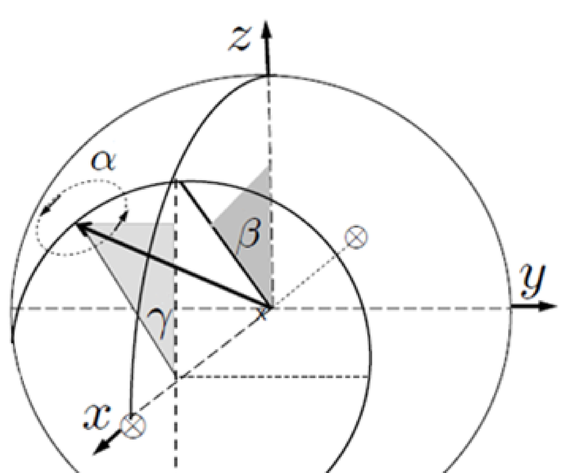
Group operations:

$g_1 g_2 = (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)$

$\quad = (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2)$,

$g^{-1} = (-R^T\mathbf{x}, R^T)$,

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$,



$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the angles $\alpha \in (-\pi, \pi]$, $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in (-\pi, \pi]$.

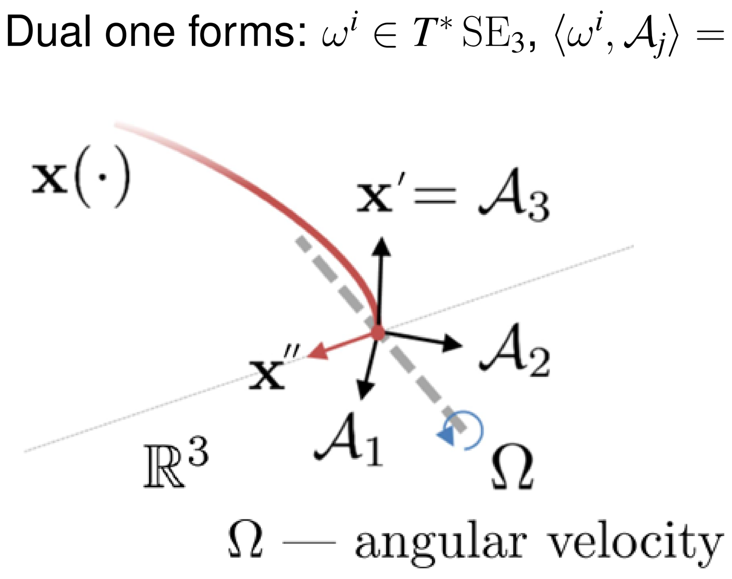
LEFT-INVARIANT VECTOR FIELDS

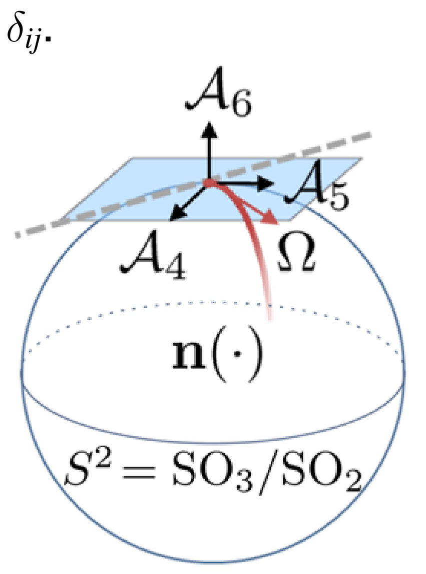
Lie algebra: $\mathfrak{se}_3 = T_e\mathrm{SE}_3 = \mathrm{span}(A_1, \dots, A_6)$

Left shift on the group: $L_g h = gh$,

Left-invariant vector fields: $\mathcal{A}_i|_g = (L_g)_* A_i$,

Dual one forms: $\omega^i \in T^*\mathrm{SE}_3$, $\langle \omega^i, \mathcal{A}_j \rangle = \delta_{ij}$.





SUB-RIEMANNIAN PROBLEM IN SE_3

Left-invariant distribution: $\Delta = \mathrm{span}(\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) \subset T\,\mathrm{SE}_3$

Metric tensor: $\mathcal{G}_\xi = \xi^2\omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5$ on Δ

A constant $\xi > 0$ balancing spatial and angular displacement

SR-distance: Inf among Lipschitzian curves $\gamma : [0, T] \to \mathrm{SE}_3$

$$d(e, g) = \inf\left\{ \int_0^T \sqrt{\mathcal{G}_\xi(\dot\gamma(t), \dot\gamma(t))}\,\mathrm{d}\,t \;\middle|\; \dot\gamma(t) \in \Delta|_{\gamma(t)},\ \begin{matrix}\gamma(0) = e, \\ \gamma(T) = g\end{matrix} \right\}.$$

SR-minimizers are solutions to the optimal control problem

$\mathbf{P}_{\mathbf{mec}}$: Boundary conditions: $\gamma(0) = e, \quad \gamma(T) = g$,

Control system: $\dot\gamma(t) = u^3(t)\mathcal{A}_3|_{\gamma(t)} + u^4(t)\mathcal{A}_4|_{\gamma(t)} + u^5(t)\mathcal{A}_5|_{\gamma(t)}$

Cost functional: $\frac{1}{2}\int_0^T \xi^2 u_3(t)^2 + u_4(t)^2 + u_5(t)^2\,\mathrm{d}\,t \to \min$.

- Complete controllability (Chow-Rashevski)
- Existence of minimizers (Filippov theorem)
- No abnormal extremals: $\dim[\Delta, \Delta] = \dim(\mathrm{SE}_3)$
- Minimizers are analytic
- Scaling homothety: $\xi = 1$

Control system in coordinates:

$$\begin{cases} \dot x = u^3 \sin\beta, \\ \dot y = -u^3\cos\beta\sin\theta, \\ \dot z = u^3\cos\beta\cos\theta, \\ \dot\theta = \sec\beta(u^4\cos\alpha - u^5\sin\alpha), \\ \dot\beta = u^4\sin\alpha + u^5\cos\alpha, \\ \dot\alpha = -(u^4\cos\alpha - u^5\sin\alpha)\tan\beta. \end{cases}$$

PONTRYAGIN MAXIMUM PRINCIPLE

Left Invariant Hamiltonians: $\lambda_i = \langle p, \mathcal{A}_i \rangle$

$p = p_1\,\mathrm{d}x + p_2\,\mathrm{d}y + p_3\,\mathrm{d}z + p_4\,\mathrm{d}\theta + p_5\,\mathrm{d}\beta + p_6\,\mathrm{d}\alpha \in T^*\mathrm{SE}_3$

Control dependent Hamiltonian:

$H_u = u^3\lambda_3 + u^4\lambda_4 + u^5\lambda_5 - \frac{1}{2}\left((u^3)^2 + (u^4)^2 + (u^5)^2\right)$

Maximality Condition: $u^3 = \lambda_3$, $\quad u^4 = \lambda_4$, $\quad u^5 = \lambda_5$

The (maximized) Hamiltonian: $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$

The Hamiltonian system (via Poisson brackets $\dot\lambda_i = \{H, \lambda_i\}$):

$$\begin{cases} \dot\lambda_1 = -\lambda_3\lambda_5, \\ \dot\lambda_2 = \lambda_3\lambda_4, \\ \dot\lambda_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \\ \dot\lambda_4 = \lambda_2\lambda_3 - \lambda_5\lambda_6, \\ \dot\lambda_5 = \lambda_4\lambda_6 - \lambda_1\lambda_3, \\ \dot\lambda_6 = 0, \end{cases} \qquad \begin{cases} \dot x = \lambda_3\sin\beta, \\ \dot y = -\lambda_3\cos\beta\sin\theta, \\ \dot z = \lambda_3\cos\beta\cos\theta, \\ \dot\theta = \sec\beta(\lambda_4\cos\alpha - \lambda_5\sin\alpha), \\ \dot\beta = \lambda_4\sin\alpha + \lambda_5\cos\alpha, \\ \dot\alpha = -(\lambda_4\cos\alpha - \lambda_5\sin\alpha)\tan\beta, \end{cases}$$

LIOUVILLE INTEGRABILITY

First integrals the Hamiltonian system:

- The Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- Left-invariant basis Hamiltonian λ_6
- Casimirs $W = -\lambda_1\lambda_4 - \lambda_2\lambda_5 - \lambda_3\lambda_6$, $\mathcal{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- Right-invariant Hamiltonians $\rho_i = \langle \lambda, Y_i \rangle$, Y_i right invariant v.f.

Complete system of first Integrals: $(H, \lambda_6, W, \rho_1, \rho_2, \rho_3)$

Theorem. The Hamiltonian system is Liouville integrable.

EXTREMAL CONTROLS FOR $\lambda_6 = 0$

Theorem. Suppose $\lambda_6(0) = 0$; then
λ_4, λ_5 are expressed via $U(t) = \int_0^t \lambda_3(\tau)\,\mathrm{d}\,\tau$ and initial values

$$\lambda_4(t) = \frac{\lambda_2(0) + \lambda_4(0)}{2}\exp(U(t)) - \frac{\lambda_2(0) - \lambda_4(0)}{2}\exp(-U(t)),$$

$$\lambda_5(t) = \frac{\lambda_1(0) + \lambda_5(0)}{2}\exp(-U(t)) - \frac{\lambda_1(0) - \lambda_5(0)}{2}\exp(U(t)).$$

λ_3 is expressed via initial values depending on several cases.

For $\lambda_1(0) = \pm\lambda_5(0)$, $\lambda_2(0) = \mp\lambda_4(0)$, we have

$$\lambda_3(t) = \frac{(b + \lambda_3(0))\,\mathrm{e}^{\pm bt} - (b - \lambda_3(0))\,\mathrm{e}^{\mp bt}}{\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}},\ b = \sqrt{\sum_{i=3}^{5}\lambda_i^2(0)}$$

$$U(t) = -\ln\left(\frac{1}{2}\left[\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}\right]\right),$$

Otherwise, we have $\lambda_3(t) = -\frac{P}{2}\,\mathrm{sn}(\psi_t, k)$,

$$U(t) = \frac{1}{2}\ln\left(\frac{A}{B} + \frac{P^2}{2B}\left(\mathrm{cn}^2(\psi_t, k) + \frac{1}{k}\mathrm{cn}(\psi_t, k)\,\mathrm{dn}(\psi_t, k)\right)\right),$$

$A = (\lambda_1(0) + \lambda_5(0))^2 + (\lambda_2(0) - \lambda_4(0))^2$, $B = (\lambda_1(0) - \lambda_5(0))^2 + (\lambda_2(0) + \lambda_4(0))^2$

$P = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} - \sqrt{B}\right)^2}$, $Q = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} + \sqrt{B}\right)^2}$

$$\psi_t = F(p_0, k) + \frac{Q}{2}t,\ k = \frac{P}{Q},\ p_0 = \begin{cases} -\arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B \geq A, \\ \pi + \arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B < A. \end{cases}$$

SOLUTION IN OPERATOR FORM

Theorem. Extremal controls in the sub-Riemannian problem on the group of motions of Euclidean 3D space have the form

$$\begin{aligned} \lambda_1(t) &= F_1 - \lambda_6(I + \lambda_6^2L^2)^{-1}\left[M[G] - \lambda_6 ML[F]\right], \\ \lambda_2(t) &= G_1 - \lambda_6(I + \lambda_6^2L^2)^{-1}\left[M[F] + \lambda_6 ML[G]\right], \\ \lambda_3(t) &= \dot U(t), \\ \lambda_4(t) &= (I + \lambda_6^2L^2)^{-1}\left[G - \lambda_6 L[F]\right], \\ \lambda_5(t) &= (I + \lambda_6^2L^2)^{-1}\left[F + \lambda_6 L[G]\right], \end{aligned}$$

where

$$\begin{aligned} F_1 = F_1(t) &= \tfrac{\lambda_1(0)+\lambda_5(0)}{2}\,\mathrm{e}^{-U(t)} - \tfrac{\lambda_5(0)-\lambda_1(0)}{2}\,\mathrm{e}^{U(t)}, \\ G_1 = G_1(t) &= \tfrac{\lambda_2(0)+\lambda_4(0)}{2}\,\mathrm{e}^{U(t)} - \tfrac{\lambda_4(0)-\lambda_2(0)}{2}\,\mathrm{e}^{-U(t)}, \\ F = F(t) &= \tfrac{\lambda_1(0)+\lambda_5(0)}{2}\,\mathrm{e}^{-U(t)} + \tfrac{\lambda_5(0)-\lambda_1(0)}{2}\,\mathrm{e}^{U(t)}, \\ G = G(t) &= \tfrac{\lambda_2(0)+\lambda_4(0)}{2}\,\mathrm{e}^{U(t)} + \tfrac{\lambda_4(0)-\lambda_2(0)}{2}\,\mathrm{e}^{-U(t)}, \\ L[f] = L[f](t) &= \textstyle\int_0^t f(\tau)\mathrm{ch}(U(t) - U(\tau))d\tau, \\ M[f] = M[f](t) &= \textstyle\int_0^t f(\tau)\mathrm{sh}(U(t) - U(\tau))d\tau, \end{aligned}$$

and the function $U(t)$ satisfies the Cauchy problem

$$\ddot U(t) = \lambda_1(t)\lambda_5(t) - \lambda_2(t)\lambda_4(t), \quad U(0) = 0, \quad \dot U(0) = u_3(0).$$

ASYMPTOTICS NEAR $\lambda_6 = 0$

Theorem. Let λ_6 be close to zero. Extremal controls in sub-Riemannian problem on the group of motions of Euclidean 3D space are approximately expressed up to $O(\lambda_6^2)$ as $\lambda_3(t) = \dot U(t)$,

$$\lambda_1(t) = F_1(t) - \lambda_6 M[G](t), \quad \lambda_2(t) = G_1(t) - \lambda_6 M[F](t),$$
$$\lambda_4(t) = G(t) - \lambda_6 L[F](t), \quad \lambda_5(t) = F(t) + \lambda_6 L[G](t),$$

where

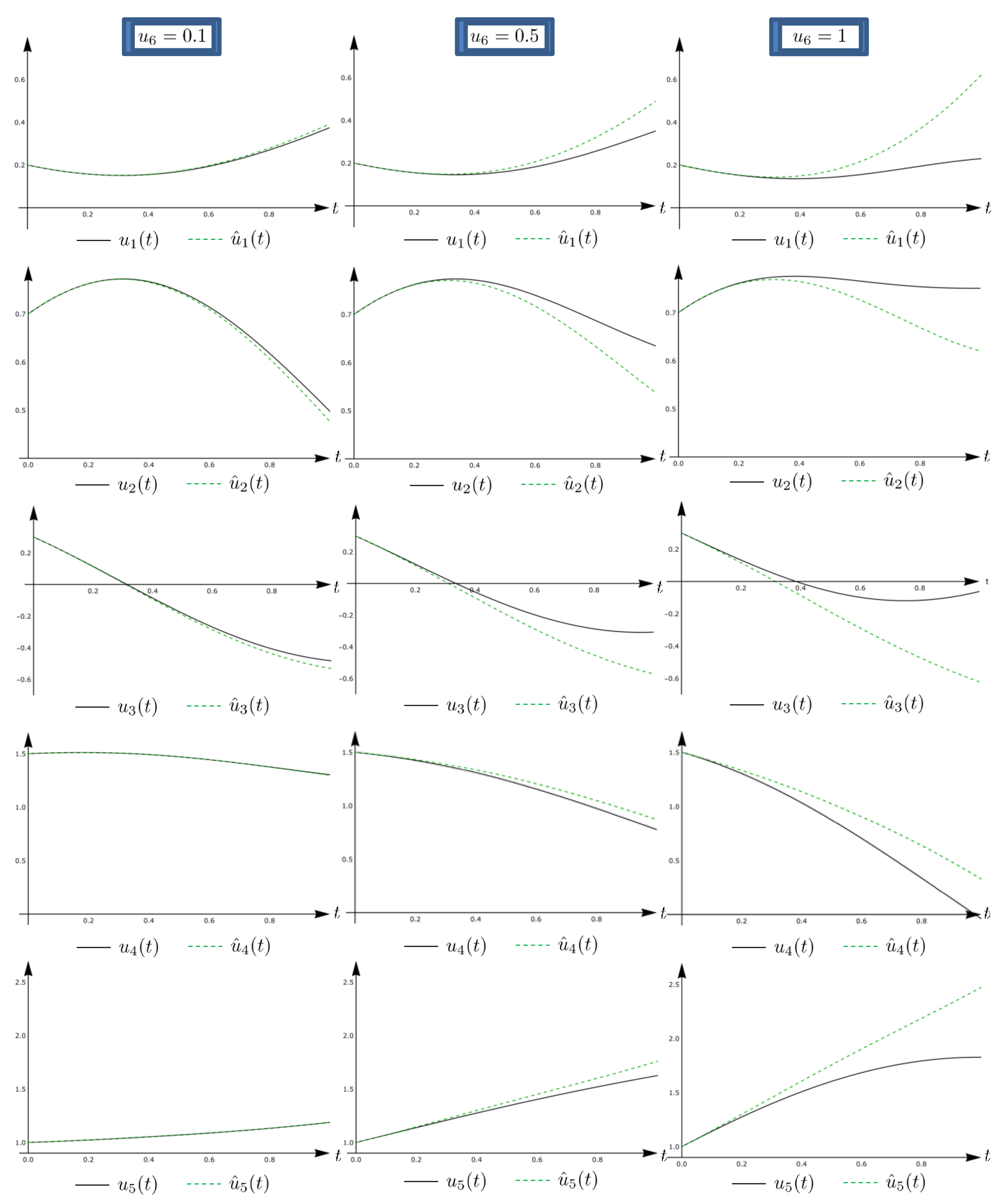
$$F_1(t) = \alpha\,\mathrm{e}^{-U(t)} - \beta\,\mathrm{e}^{U(t)}, \quad G_1(t) = \alpha\,\mathrm{e}^{U(t)} - \beta\,\mathrm{e}^{-U(t)},$$
$$F(t) = \alpha\,\mathrm{e}^{-U(t)} + \beta\,\mathrm{e}^{U(t)}, \quad G(t) = \alpha\,\mathrm{e}^{U(t)} + \beta\,\mathrm{e}^{-U(t)},$$
$$L[f](t) = \int_0^t f(\tau)\mathrm{ch}(U(t) - U(\tau))d\tau,$$
$$M[f](t) = \int_0^t f(\tau)\mathrm{sh}(U(t) - U(\tau))d\tau,$$
$$\alpha = \frac{\lambda_2(0) + \lambda_4(0)}{2},\ \beta = \frac{\lambda_4(0) - \lambda_2(0)}{2},$$
$$\alpha = \frac{\lambda_1(0) + \lambda_5(0)}{2},\ \beta = \frac{\lambda_5(0) - \lambda_1(0)}{2}.$$

and the function $U(t)$ satisfies the Cauchy problem

$$\begin{cases} \ddot U(t) = f_0(U(t)) + u_6\left(tf_1(U(t)) + k\int_0^t \mathrm{ch}(2U(t) - 2U(\tau))d\tau\right), \\ U(0) = 0, \quad \dot U(0) = u_3(0), \end{cases}$$

where $k = \alpha\alpha - \beta\beta$, $f_0(U) = (\alpha^2 + \beta^2)\,\mathrm{e}^{-2U} - (\beta^2 + \alpha^2)\,\mathrm{e}^{2U}$, $f_1(U) = k - \alpha\beta\,\mathrm{e}^{2U} - \alpha\beta\,\mathrm{e}^{-2U}$.

NUMERICAL SIMULATIONS



RESULTS

For the SR problem in SE_3 ($\mathbf{P}_{\mathbf{mec}}$) we derive the Hamiltonian system of PMP, prove Liouville integrability and find explicit expression for extremal controls in the case $\lambda_6 = 0$. For general $\lambda_6 \in \mathbb{R}$ we present the solution in operator form. We derive the asymptotic formulas for the extremal controls in the case, when $|\lambda_6|$ is small. The practical applicability of obtained formulas has been justified by the results of numerical simulations

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