On Extremal Controls in the Sub-Riemannian Problem on the Group of Rigid Body Motions

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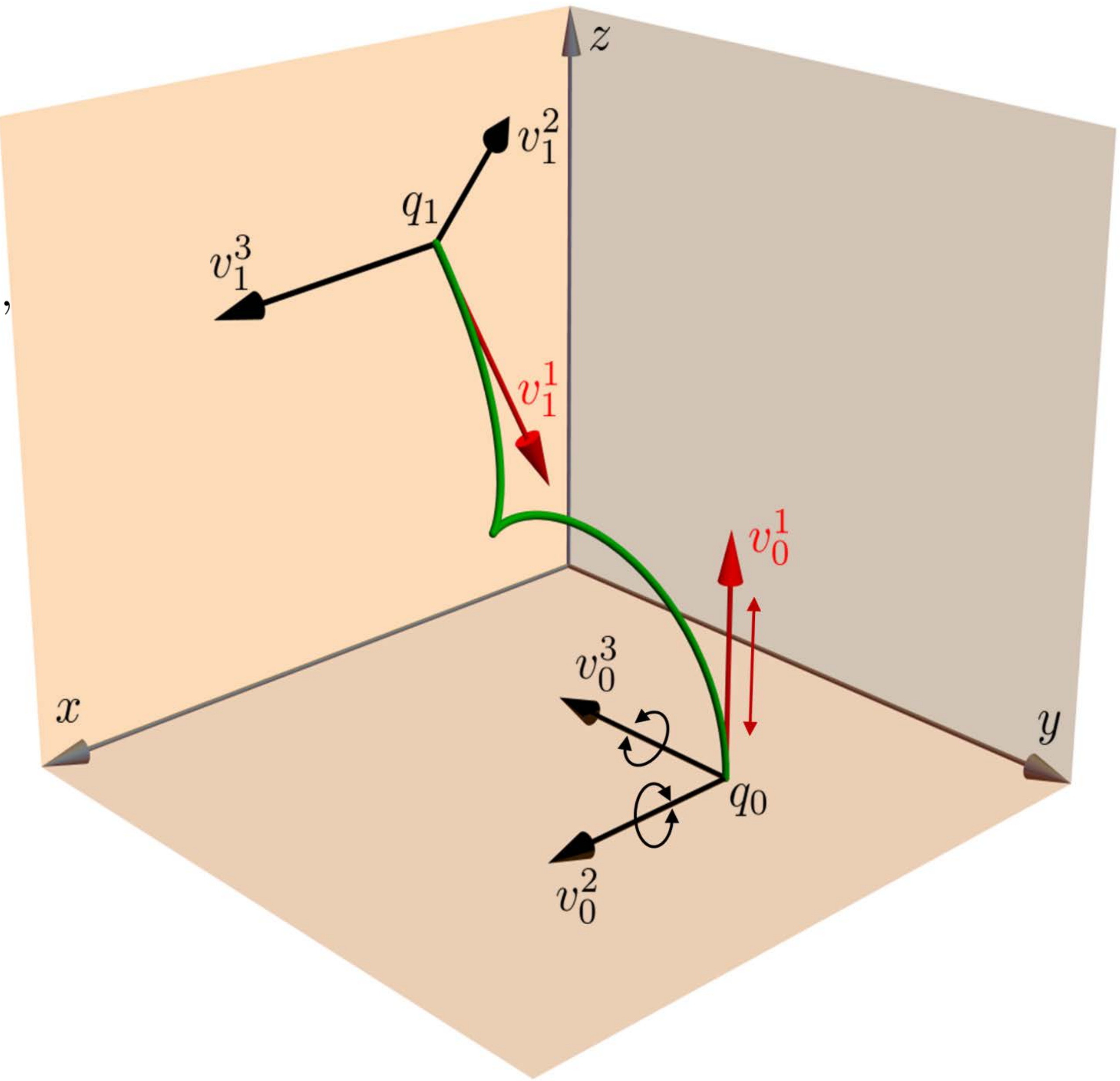
Statement of the Problem

By given two orthonormal frames $N_0 = \{v_0^1, v_0^2, v_0^3\}$ and $N_1 = \{v_1^1, v_1^2, v_1^3\}$ attached respectively at given points $q_0 = (x_0, y_0, z_0)$ and $q_1 = (x_1, y_1, z_1)$ in space $\mathbf{R}^3$, the goal is to find the optimal motion of $\mathbf{R}^3$ that transfers q_0 to q_1 such that N_0 is transferred to N_1. The frame can move forward or backward along one of the vector chosen in the frame and simultaneously rotate around the remaining two (of three) vectors. The required motion should be optimal in the sense of minimal length in $SE(3) \cong \mathbf{R}^3 \times SO(3)$

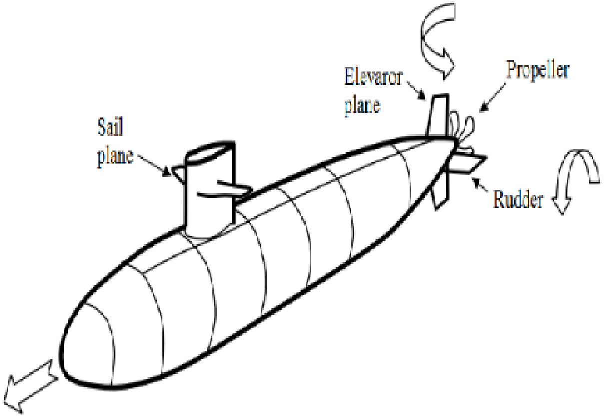


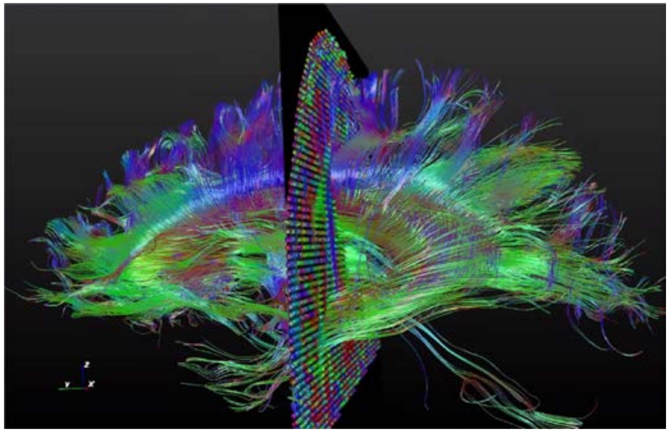
Motivation: Applications in Robotics and Image Processing

- **Control of UAVs**
  J. Jamieson, W. Holderbaum, Trajectory generation using sub-Riemannian curves for quadrotor UAVs, *European Control Conference (ECC)*, 2015.

- **Control of AUVs**
  J. Biggs, W. Holderbaum, Optimal Kinematic Control of an Autonomous Underwater Vehicle, *IEEE Transactions on Automatic Control*, 2009

- **Detection of salient lines in 3D images**
  R. Duits, T. C. J. Dela Haije, E. J. Creusen and A. Ghosh, Morphological and Linear Scale Spaces for Fiber Enhancement in DW-MRI, *Journal of mathematical imaging and vision*, 2013

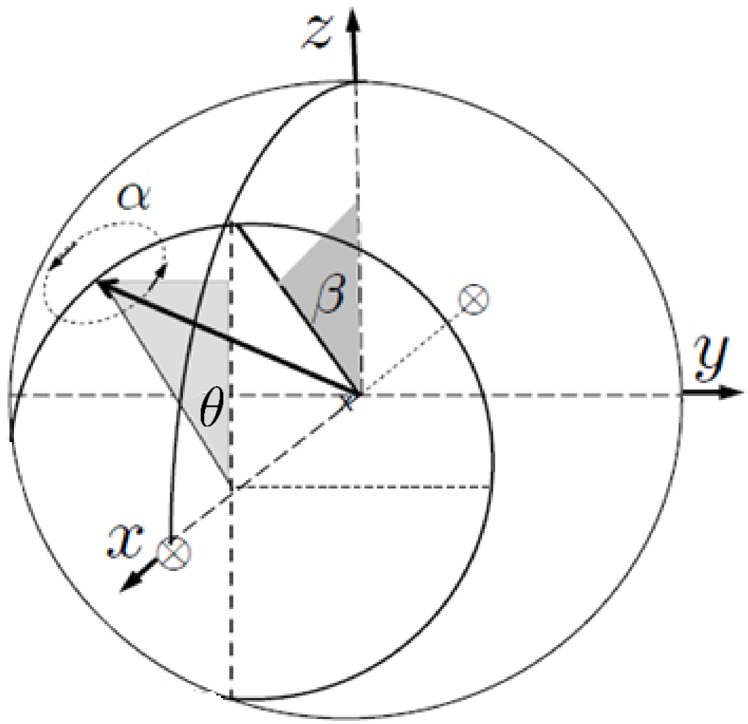




Group of Rigid Body Motions

The group of Euclidean motions of 3-dimensional space

$$g = (\mathbf{x}, R) \in \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$$



Group operations

$$g_1 g_2 = (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) = (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1 R_2),$$

$$g^{-1} = (-R^T\mathbf{x}, R^T).$$

We use the parameterization of SE(3)

$$\mathbf{x} = (x, y, z) \in \mathbb{R}^3,$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in (-\pi, \pi]$, $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in (-\pi, \pi]$

Left-invariant Vector Fields on SE(3)

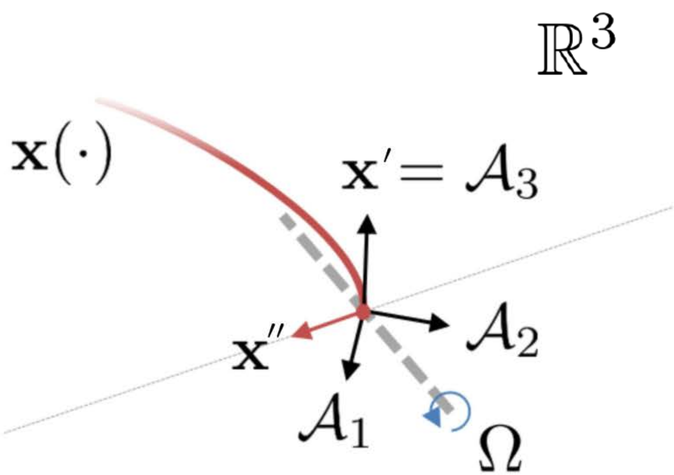
$$(\mathbf{x}, R) \in SE(3) = \mathbb{R}^3 \rtimes SO(3)$$

$$se(3) = T_e\, SE(3) = \text{span}\{A_1, A_2, A_3, A_4, A_5, A_6\}$$

$$\mathcal{A}_i\big|_g = (L_g)_* A_i, \quad i \in \{1, \ldots, 6\}, \quad L_g h = gh$$

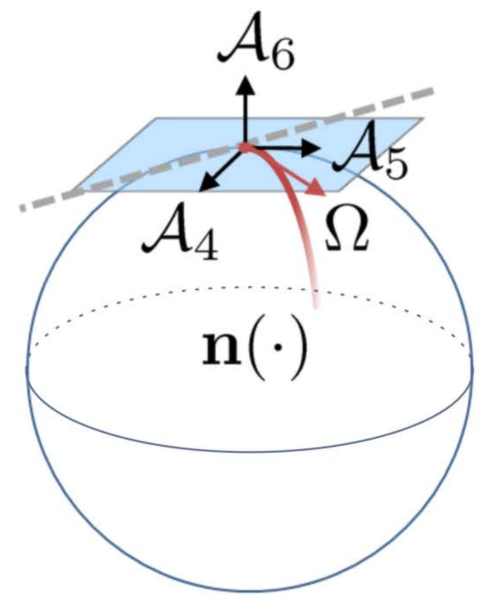
$$\text{Co-frame } \{\omega^1, \ldots, \omega^6\}: \ \langle \omega^i, \mathcal{A}_j \rangle = \delta^i_j, \quad i, j \in \{1, \ldots, 6\}.$$

$$\mathbb{S}^2 = SO(3)/SO(2)$$



$\mathbf{x}'$ spatial velocity:
$\mathbf{x}' = \langle \omega^3|_\gamma, \gamma' \rangle\, \mathcal{A}_3|_\gamma = \mathcal{A}_3|_\gamma$

$\mathbf{x}''$ spatial curvature:
$\mathbf{x}'' = \langle \omega^5|_\gamma, \gamma' \rangle\, \mathcal{A}_1|_\gamma - \langle \omega^4|_\gamma, \gamma' \rangle\, \mathcal{A}_2|_\gamma$



Ω angular velocity:
$\Omega = \langle \omega^4, \gamma' \rangle \mathcal{A}_4 + \langle \omega^5, \gamma' \rangle \mathcal{A}_5$

Sub-Riemannian Problem in SE(3)

SR structure (SR manifold):

$(M, \Delta, \mathcal{G}_\xi)$ $\quad M = \mathrm{SE}(3), \qquad \Delta = \mathrm{span}\{\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\},$

$$\mathcal{G}_\xi = \xi^2 \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5$$

SR distance (Carnot-Carathéodory distance):

$$d(g,h) = \min_{\substack{\gamma \in \mathrm{Lip}([0,T], \mathrm{SE}(3)), T \geq 0, \\ \dot{\gamma} \in \Delta, \ \gamma(0) = g, \gamma(T) = h}} \int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t.$$

$\mathbf{P_{MEC}}$(SE(3)): to find a Lipschitzian curve $\gamma : [0,T] \to \mathrm{SE}(3)$, s.t.

$$\gamma(0) = e := (\mathbf{0}, I), \quad \gamma(T) = (\mathbf{x}_1, R_1) \in \mathrm{SE}(3),$$

$$\dot{\gamma}(t) \in \Delta \text{ for a.e. } t \in [0,T],$$

and $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t \to \min$ (with free T).

Optimal Control Formulation of SR-Problem in SE(3)

Control system

$$\dot{\gamma}(t) = u^3(t)\mathcal{A}_3|_{\gamma(t)} + u^4(t)\mathcal{A}_4|_{\gamma(t)} + u^5(t)\mathcal{A}_5|_{\gamma(t)}$$

Boundary conditions

$$\gamma(0) = e, \quad \gamma(T) = g_1 \in \mathrm{SE}(3)$$

Minimizing functional (here action functional)

$$\int_0^T \frac{1}{2}\left(\xi^2(u^3(t))^2 + (u^4(t))^2 + (u^5(t))^2\right)\, \mathrm{d}t \to \min .$$

In coordinates

$$\begin{cases} \dot{x} = u^3 \sin\beta, \\ \dot{y} = -u^3 \cos\beta \sin\theta, \\ \dot{z} = u^3 \cos\beta \cos\theta, \\ \dot{\theta} = \sec\beta(u^4 \cos\alpha - u^5 \sin\alpha), \\ \dot{\beta} = u^4 \sin\alpha + u^5 \cos\alpha, \\ \dot{\alpha} = -(u^4 \cos\alpha - u^5 \sin\alpha)\tan\beta, \end{cases}$$

$$(x(0), y(0), z(0), \theta(0), \beta(0), \alpha(0)) = \mathbf{0}$$

$$(x(T), y(T), z(T), \theta(T), \beta(T), \alpha(T)) = (x^1, y^1, z^1, \theta^1, \beta^1, \alpha^1)$$

- Complete controllability (Chow-Rashevski)
- Existance of minimizers (Filippov)
- No abnormal extremals: $\dim[\Delta, \Delta] = \dim(\mathrm{SE}(3))$
- The minimizers are analytic
- By scaling homothety ξ can be set $\xi = 1$.

Pontryagin Maximum Principle

- Left Invariant Hamiltonians $\lambda_i = \langle p, \mathcal{A}_i \rangle$, $i = 1, \dots, 6$, where $p = p_1 \mathrm{d}x|_g + p_2 \mathrm{d}y|_g + p_3 \mathrm{d}z|_g + p_4 \mathrm{d}\theta|_g + p_5 \mathrm{d}\beta|_g + p_6 \mathrm{d}\alpha|_g$
- Control dependent Hamiltonian $H_u = u^3\lambda_3 + u^4\lambda_4 + u^5\lambda_5 - \frac{1}{2}\left((u^3)^2 + (u^4)^2 + (u^5)^2\right)$
- Maximality Condition $u^3 = \lambda_3, \quad u^4 = \lambda_4, \quad u^5 = \lambda_5.$
- The (maximized) Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- The Hamiltonian system of PMP (via Poisson brackets $\dot{\lambda}_i = \{H, \lambda_i\}$)

$$\begin{cases} \dot{\lambda}_1 = -\lambda_3\lambda_5, \\ \dot{\lambda}_2 = \lambda_3\lambda_4, \\ \dot{\lambda}_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \\ \dot{\lambda}_4 = \lambda_2\lambda_3 - \lambda_5\lambda_6, \\ \dot{\lambda}_5 = \lambda_4\lambda_6 - \lambda_1\lambda_3, \\ \dot{\lambda}_6 = 0, \end{cases} \qquad \begin{cases} \dot{x} = \lambda_3 \sin\beta, \\ \dot{y} = -\lambda_3 \cos\beta \sin\theta, \\ \dot{z} = \lambda_3 \cos\beta \cos\theta, \\ \dot{\theta} = \sec\beta(\lambda_4 \cos\alpha - \lambda_5 \sin\alpha), \\ \dot{\beta} = \lambda_4 \sin\alpha + \lambda_5 \cos\alpha, \\ \dot{\alpha} = -(\lambda_4 \cos\alpha - \lambda_5 \sin\alpha)\tan\beta, \end{cases}$$

— vertical part, — horizontal part.

Liouville Integrability of the Hamiltonian System

First Integrals:

- the Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- Left-invariant basis Hamiltonian λ_6
- Casimir functions $W = -\lambda_1\lambda_4 - \lambda_2\lambda_5 - \lambda_3\lambda_6, \quad \mathfrak{c}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- Right-invariant Hamiltonians

$$\rho_1 = -\lambda_1 \cos\alpha \cos\beta + \lambda_2 \cos\beta \sin\alpha - \lambda_3 \sin\beta,$$
$$\rho_2 = -\cos\gamma(\lambda_2 \cos\alpha + \lambda_1 \sin\alpha) + (\lambda_3 \cos\beta + (-\lambda_1 \cos\alpha + \lambda_2 \sin\alpha)\sin\beta)\sin\gamma,$$
$$\rho_3 = -\lambda_3 \cos\beta \cos\gamma + \cos\gamma(\lambda_1 \cos\alpha - \lambda_2 \sin\alpha)\sin\beta - (\lambda_2 \cos\alpha + \lambda_1 \sin\alpha)\sin\gamma,$$
$$\rho_4, \quad \rho_5, \quad \rho_6.$$

Complete system of first Integrals: $\mathrm{I} = (H,\ \lambda_6,\ W,\ \rho_1,\ \rho_2,\ \rho_3)$

$$\{I_i, I_j\} = 0 \qquad \frac{\partial(\rho_1,\rho_2,\rho_3,W,H,\lambda_6)}{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)}(q,\lambda) = \quad -\lambda_2\lambda_4 + \lambda_1\lambda_5 \not\equiv 0$$

Theorem *The Hamiltonian system of PMP for sub-Riemanian problem on* SE(3) *is Liouville integrable.*

Vertical Part: System on Extremal Controls

If we consider the mechanical system "body + fluid" and denote by $\gamma = (\lambda_1, \lambda_2, \lambda_3)$ the impulse of this system, $M = (\lambda_4, \lambda_5, \lambda_6)$ the kinetic momentum, and $H = \frac{\lambda_3^2 + \lambda_4^2 + \lambda_5^2}{2}$ the Hamiltonian, which is the kinetic energy of the system, then the vertical part takes a form of Kirchhoff equation describing the motion of a solid body in fluid

$$\begin{cases} \dot{M} = M \times \dfrac{\partial H}{\partial M} + \gamma \times \dfrac{\partial H}{\partial \gamma}, \\ \dot{\gamma} = \gamma \times \dfrac{\partial H}{\partial M} \end{cases} \Leftrightarrow \boxed{\begin{cases} \dot{\lambda}_1 = -\lambda_3 \lambda_5, \\ \dot{\lambda}_2 = \lambda_3 \lambda_4, \\ \dot{\lambda}_3 = \lambda_1 \lambda_5 - \lambda_2 \lambda_4, \\ \dot{\lambda}_4 = \lambda_2 \lambda_3 - \lambda_5 \lambda_6, \\ \dot{\lambda}_5 = \lambda_4 \lambda_6 - \lambda_1 \lambda_3, \\ \dot{\lambda}_6 = 0, \end{cases}}$$

- Kirchhoff, G. R., *Vorlesungen uber mathematische Physik.* Mechanik. Leipzig, 1874.
- Halphen, G.-H., Sur le mouvement d'un solide dans un liquide, *Journal de mathematiques pures et appliques*, 1888.
- A. V. Borisov, I. S. Mamaev, *Rigid body dynamics. Hamiltonian methods, integrability, chaos*, Moscow - Izhevsk: Institute of Computer Science, 2005.

Integration of the Vertical Part in Special Case

Theorem *Suppose $\lambda_6(0) = 0$; then the vertical part is given by*

$$\dot\lambda_1 = -\lambda_3\lambda_5, \quad \dot\lambda_2 = \lambda_3\lambda_4, \quad \dot\lambda_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \quad \dot\lambda_4 = \lambda_2\lambda_3, \quad \dot\lambda_5 = -\lambda_1\lambda_3.$$

The momenta λ_4, λ_5 are expressed via $U(t) = \int_0^t \lambda_3(\tau)\,\mathrm{d}\,\tau$ and the initial values

$$\lambda_4(t) = \frac{\lambda_2(0) + \lambda_4(0)}{2}\exp\left(U\left(t\right)\right) - \frac{\lambda_2(0) - \lambda_4(0)}{2}\exp\left(-U\left(t\right)\right),$$
$$\lambda_5(t) = \frac{\lambda_1(0) + \lambda_5(0)}{2}\exp\left(-U\left(t\right)\right) - \frac{\lambda_1(0) - \lambda_5(0)}{2}\exp\left(U\left(t\right)\right).$$

The momentum λ_3 is expressed via the initial values depending on several cases. For the cases $\lambda_1(0) = \pm\lambda_5(0)$, $\lambda_2(0) = \mp\lambda_4(0)$, we have

$$\lambda_3(t) = \frac{(b + \lambda_3(0))\,\mathrm{e}^{\pm bt} - (b - \lambda_3(0))\,\mathrm{e}^{\mp bt}}{\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}}, \qquad U(t) = -\ln\left(\frac{1}{2}\left[\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}\right]\right),$$

where $b = \sqrt{\lambda_3^2(0) + \lambda_4^2(0) + \lambda_5^2(0)}$. Otherwise, we have

$$\lambda_3(t) = -\frac{P}{2}\,\mathrm{sn}\left(\psi_t, k\right), \qquad U(t) = \frac{1}{2}\ln\left(\frac{A}{B} + \frac{P^2}{2B}\left(\mathrm{cn}^2\left(\psi_t, k\right) + \frac{1}{k}\,\mathrm{cn}\left(\psi_t, k\right)\mathrm{dn}\left(\psi_t, k\right)\right)\right),$$

where $A = (\lambda_1(0) + \lambda_5(0))^2 + (\lambda_2(0) - \lambda_4(0))^2$, $B = (\lambda_1(0) - \lambda_5(0))^2 + (\lambda_2(0) + \lambda_4(0))^2$,
$P = \sqrt{4\lambda_3^2\left(0\right) + \left(\sqrt{A} - \sqrt{B}\right)^2}$, $Q = \sqrt{4\lambda_3^2\left(0\right) + \left(\sqrt{A} + \sqrt{B}\right)^2}$,

$$\psi_t = F\left(p_0, k\right) + \frac{Q}{2}t, \quad k = \frac{P}{Q}, \qquad p_0 = \begin{cases} -\arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \textit{if } B \geq A, \\ \pi + \arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \textit{if } B < A. \end{cases}$$

Integration of the Vertical Part in General Case: Spatial Component of Extremal Control

Theorem *The extremal control* λ_3 *in the sub-Riemannian problem on the group of rigid body motions is given by a solution to the Cauchy problem*

$$\ddot{\lambda}_3(t) = 2\lambda_3^3(t) - c_1\lambda_3(t) + c_2,$$
$$\lambda_3(0) = \lambda_3^0, \quad \dot{\lambda}_3(0) = \lambda_1^0\lambda_5^0 - \lambda_2^0\lambda_4^0,$$

where the constants c_1 *and* c_2 *are given by*

$$c_1 = H + C + \lambda_6^2, \qquad c_2 = \lambda_6 W.$$

Integration of the Vertical Part in General Case: Angular Components of Extremal Control

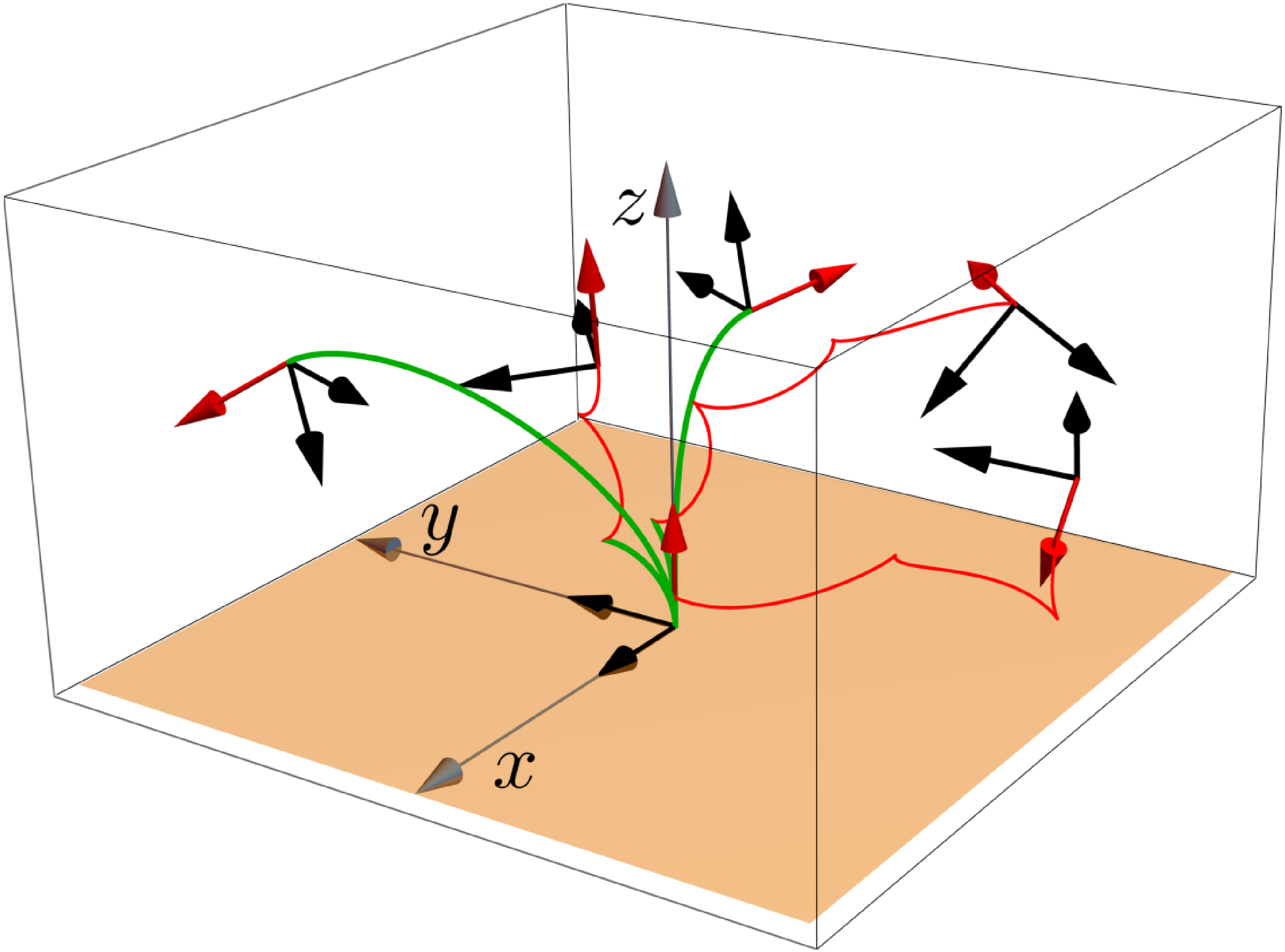
Theorem *The extremal controls λ_1, λ_2, λ_4, λ_5 in the sub-Riemannian problem on the group of rigid body motions are given by*

$$\begin{aligned}
\lambda_1(t) &= F_1 - \lambda_6(I+\lambda_6^2L^2)^{-1}\big[M[G] - \lambda_6 ML[F]\big],\\
\lambda_2(t) &= G_1 - \lambda_6(I+\lambda_6^2L^2)^{-1}\big[M[F] + \lambda_6 ML[G]\big],\\
\lambda_4(t) &= (I+\lambda_6^2L^2)^{-1}\big[G - \lambda_6 L[F]\big],\\
\lambda_5(t) &= (I+\lambda_6^2L^2)^{-1}\big[F + \lambda_6 L[G]\big],
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= F_1(t) = \tfrac{\lambda_1^0+\lambda_5^0}{2}\,\mathrm{e}^{-U(t)} - \tfrac{\lambda_5^0-\lambda_1^0}{2}\,\mathrm{e}^{U(t)},\\
G_1 &= G_1(t) = \tfrac{\lambda_2^0+\lambda_4^0}{2}\,\mathrm{e}^{U(t)} - \tfrac{\lambda_4^0-\lambda_2^0}{2}\,\mathrm{e}^{-U(t)},\\
F &= F(t) = \tfrac{\lambda_1^0+\lambda_5^0}{2}\,\mathrm{e}^{-U(t)} + \tfrac{\lambda_5^0-\lambda_1^0}{2}\,\mathrm{e}^{U(t)},\\
G &= G(t) = \tfrac{\lambda_2^0+\lambda_4^0}{2}\,\mathrm{e}^{U(t)} + \tfrac{\lambda_4^0-\lambda_2^0}{2}\,\mathrm{e}^{-U(t)},\\
L[f] &= L[f](t) = \textstyle\int_0^t f(\tau)\cosh(U(t)-U(\tau))d\tau,\\
M[f] &= M[f](t) = \textstyle\int_0^t f(\tau)\sinh(U(t)-U(\tau))d\tau,\\
U(t) &= \int\limits_0^t \lambda_3(\tau)d\tau.
\end{aligned}$$

Spatial projection of SR geodesics in SE(3) can have singularities (the cusp points)

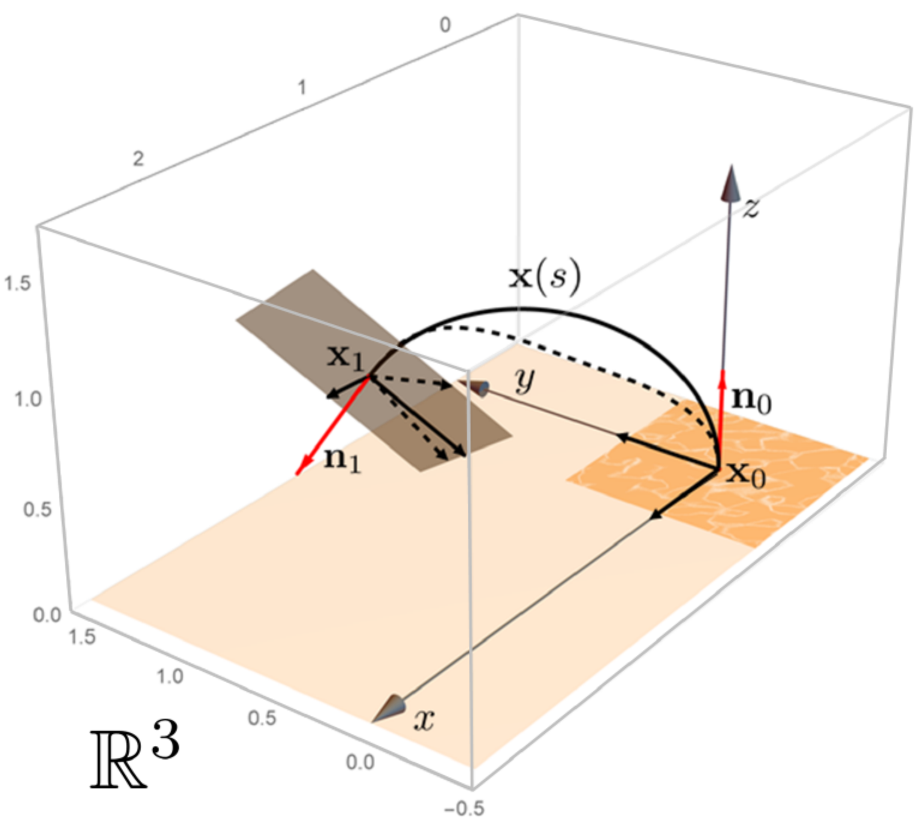


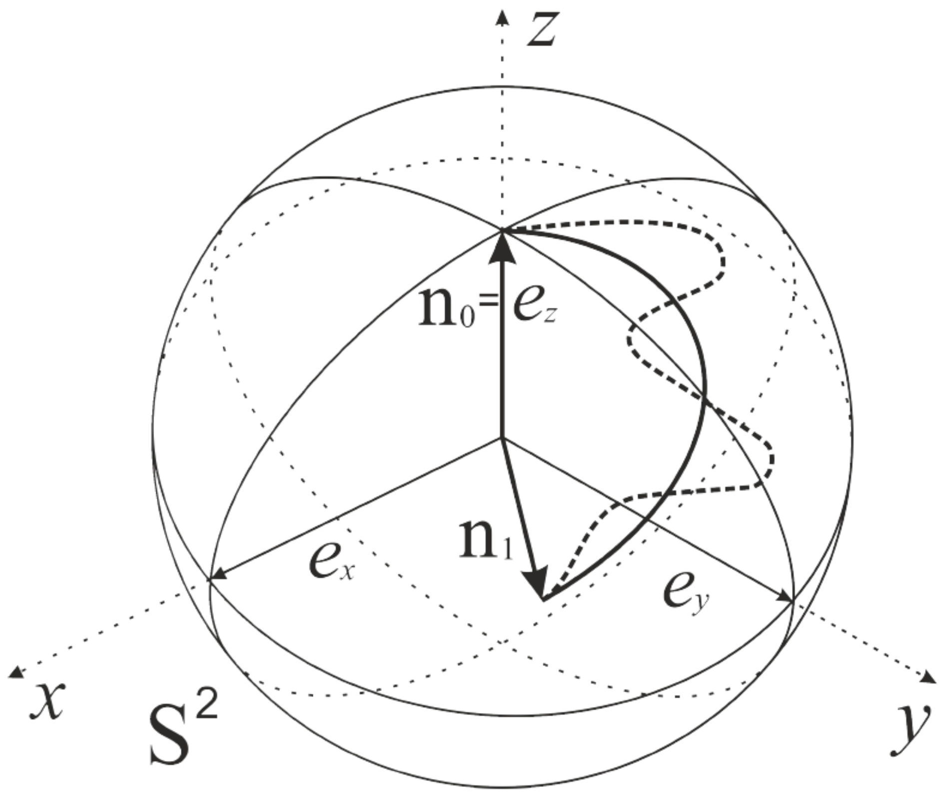
Back to applications. **Pcurve($\mathbb{R}^3$)**: Shortest Path on $R^3 \times S^2$

Given $\xi > 0, \quad \mathbf{x}_i \in \mathbb{R}^3, \; \mathbf{n}_i \in S^2, \; i \in \{0,1\}.$

Find a smooth curve $\mathbf{x} \in C^\infty([0,L], \mathbb{R}^3)$ s.t. $\mathbf{x}(0) = \mathbf{x}_0, \; \mathbf{x}(L) = \mathbf{x}_1 \in \mathbb{R}^3,$ $\mathbf{x}'(0) = \mathbf{n}_0, \; \mathbf{x}'(L) = \mathbf{n}_1 \in S^2,$

and $E(\mathbf{x}) := \int_0^L \sqrt{\xi^2 + \kappa^2(s)} \, \mathrm{d}s \to \min,$ where $\kappa(s) = \|\mathbf{x}''(s)\|.$





SR problem Pmec($\mathbf{R}^3 \times \mathbf{S}^2$) in Quotient SE(3)/({0}xSO(2))

Well-defined distance on the quotient $\mathbb{R}^3 \rtimes S^2$

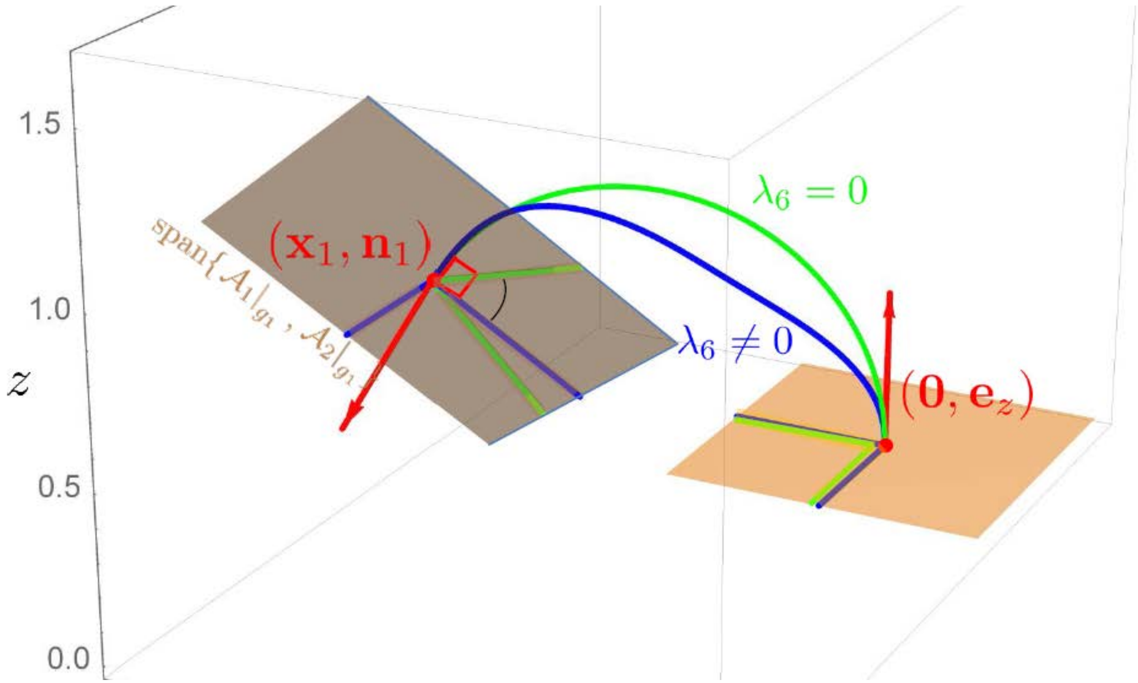
$$
\begin{aligned}
d_{\mathbb{R}^3 \rtimes S^2}((\mathbf{0}, \mathbf{e}_z), (\mathbf{y}_1, \mathbf{n}_1)) &= \min_{\alpha^1, \alpha^2 \in [0, 2\pi)} d(e h_{\alpha^1}, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2}) \\
&= \min_{\alpha^1, \alpha^2 \in [0, 2\pi)} d(e, h_{\alpha^1}^{-1} (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2 - \alpha^1} h_{\alpha^1}) \\
&= \min_{\alpha \in [0, 2\pi)} d(e, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha})
\end{aligned}
$$

$\mathbf{P}_{\mathbf{mec}}(\mathbb{R}^3 \times S^2)$: Let $(\mathbf{y}_1, \mathbf{n}_1) \in \mathbb{R}^3 \rtimes S^2$. Find

$$
[0, T] \ni t \mapsto (\mathbf{x}(t), \mathbf{n}(t)) = \gamma(t) \odot (\mathbf{0}, \mathbf{e}_z) \in \mathbb{R}^3 \rtimes S^2,
$$

with γ a Lipschitzian curve in SE(3) with velocity $\dot{\gamma} \in \Delta$, such that sub-Riemannian length $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} dt$ is minimal under boundary conditions $\gamma(0) = (\mathbf{0}, I)$ and $\gamma(T) = (\mathbf{y}_1, R_{\mathbf{n}_1} R_{\mathbf{e}_z, \alpha})$, where both $T \geq 0$ and $\alpha \in [0, 2\pi)$ are free variables in the optimization process.

Relation of Pcurve($\mathbb{R}^3$), Pmec($\mathbb{R}^3 \times S^2$) and PMEC(SE(3))



Theorem *If* $g_1 = (\mathbf{x}_1, R_1) \in \text{SE}(3)$ *is chosen s.t. a corresponding minimizer* γ^* *of* $\mathbf{P}_{\mathbf{MEC}}$ *satisfies* $u^3(t) := \langle \omega^3|_{\gamma^*(t)}, \dot{\gamma}^*(t) \rangle > 0,\ t \in (0,T)$, *then* γ^* *can be parameterized by spatial arclength* s, *and its spatial projection does not exhibit a cusp. If moreover* g_1 *is chosen s.t.* γ^* *has* $\lambda_6(0) = 0$ *then this yields the required minimum choice of* α, *and* $\gamma^*(t)$ *provides the minimizer* $(\mathbf{x}^*(t), \mathbf{n}^*(t)) = \gamma^*(t) \odot (\mathbf{0}, \mathbf{e}_z)$ *of* $\mathbf{P}_{\mathbf{mec}}$.

Under these two requirements the spatial projection $\mathbf{x}^*(\cdot)$ *of* $\gamma^*(\cdot) = (\mathbf{x}^*(\cdot), R^*(\cdot))$ *coincides with a minimizer of problem* $\mathbf{P}_{\mathbf{curve}}$.

For details see

- Duits, R., Ghosh, A., Dela Haije, T. and Mashtakov, A., On sub-Riemannian geodesics in $SE(3)$ whose spatial projections do not have cusps, *Journal of dynamical and control systems*, 2016.
- Mashtakov, A. P. and Popov, A. Yu., Extremal Controls in the Sub-Riemannian Problem on the Group of Motions of Euclidean Space, *Regular and Chaotic Dynamics*, 2017.
- Mashtakov, A. P. and Popov, A. Yu., Asymptotics of Extremal Controls in the Sub-Riemannian Problem on the Group of Motions of Euclidean Space, *Russian Journal of Nonlinear Dynamics*, 2020.
- A. Mashtakov, On Extremal Controls in the Sub-Riemannian Problem on the Group of Rigid Body Motions, *Proceedings of STAB2020.*

Conclusion

- Sub-Riemannian Problem in the Group of Rigid Body motions appears in robotics (kinematic models of UAVs and AUVs) and image processing (tracking of salient lines in 3D images).
- The vertical part of the Hamiltonian system of PMP (on extremal controls) is a special case of Kirchhoff equations of motion of a solid body in fluid.
- The Hamiltonian system of PMP is Liouville integrable.
- Exact expressions for extremal controls in the special case.
- Exact expression for spatial control and expressions in operator form for angular controls in the general case.
- Relation with problem Pcurve: shortest curve in R3 x S2 is given by cuspless SR geodesics in SE(3).

Thank you for your attention!