On Extremal Controls in the Sub-Riemannian Problem on the Group of Rigid Body Motions

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***Abstract*—We consider the sub-Riemannian problem on the group of rigid body motions in three–dimensional space. Such a problem is encountered in the analysis of 3D images as well as in describing the motion of a solid body in a fluid. Mathematically, this problem reduces to solving a Hamiltonian system, the vertical part of which is a system of six differential equations with unknown functions — extremal controls. We derive an ordinary differential equation for one of the components of the extremal control vector. The obtained equation admits a solution in elliptic functions. Then we find the expression in the operator form for the remaining components of the extremal control vector.**

***Index Terms*—rigid body motions, extremal controls, Hamiltonian system, Pontryagin maximum principle, sub-Riemannian**

I. Introduction

In mathematical formalization of control problems for aircraft and floating robots, a sub-Riemannian (SR) problem arises on the Lie group SE_3 of rigid body motions in three–dimensional space [1], [2]. SR problems also occur in studies related to image processing. E.g., the SR problem in SE_3 appeared in [3], where the authors aim to detect the nerve fibers and blood vessels in three-dimensional MRI (magnetic resonance imaging) images of the human brain.

The SR problem on the Lie group SE_3 of rigid body motions in space $\mathbb{R}^3$ can be seen as follows. By given two orthonormal frames $N_0 = \{v_0^1, v_0^2, v_0^3\}$ and $N_1 = \{v_1^1, v_1^2, v_1^3\}$ attached respectively at two given points $q_0 = (x_0, y_0, z_0)$ and $q_1 = (x_1, y_1, z_1)$ in space $\mathbb{R}^3$, the goal is to find the optimal motion of $\mathbb{R}^3$ that transfers q_0 to q_1 such that the frame N_0 is transferred to the frame N_1. The frame can move forward or backward along one of the vector chosen in the frame and simultaneously rotate around the remaining two (of three) vectors. The required motion should be optimal in the sense of minimal length in the space $\mathrm{SE}_3 \cong \mathbb{R}^3 \times \mathrm{SO}_3$.

Equivalently, the problem is to find a Lipschitzian curve $\gamma : [0, t_1] \to \mathrm{SE}_3$, such that

$$\dot{\gamma} = u_3 \mathcal{A}_3 + u_4 \mathcal{A}_4 + u_5 \mathcal{A}_5, \qquad (1)$$
$$\gamma(0) = e, \quad \gamma(t_1) = g,$$
$$l(\gamma) = \int_0^{t_1} \sqrt{\xi^2 u_3^2 + u_4^2 + u_5^2}\, \mathrm{d}t \to \min,$$

where $\mathcal{A}_i$ are left-invariant vector fields on the group SE_3, the controls u_3, u_4, u_5 are real-valued $\mathbb{L}^\infty(0, t_1)$ functions, the terminal time $t_1 > 0$ is free, e is the identity transformation of $\mathbb{R}^3$, g is a given element of SE_3 and $\xi > 0$ is a parameter that balances the influence of translation and rotations in $\mathbb{R}^3$ on length of the corresponding trajectory.

Investigation of this problem was initiated in [4], where in particular it was shown that due to the scaling homothety, see [4, Remark 5], the general case $\xi > 0$ reduces to the case $\xi = 1$ by a linear change of variables. Thus, we assume $\xi = 1$ without loss of generality.

The necessary condition of optimality is given by the Pontryagin maximum principle (PMP). Application of PMP to our problem gives an expression of extremal controls in terms of momentums (conjugate variables). Namely, the extremal controls u_3, u_4, u_5 coincide with three certain momentums, see [4, Sec. 3.1]. The remaining three momentums we denote by u_1, u_2 and u_6. Further, to simplify the notation, we call the extremal control the entire vector $\bar{u} = (u_1, u_2, u_3, u_4, u_5, u_6)$ of the momentums. Application of PMP leads to a Hamiltonian system, the horizontal part of which is given by system (1), and the vertical part (on extremal controls) has the following form [4, eq. (3.5)]:

$$\begin{cases} \dot{u}_1 = -u_3 u_5, \\ \dot{u}_2 = u_3 u_4, \\ \dot{u}_3 = u_1 u_5 - u_2 u_4, \\ \dot{u}_4 = u_2 u_3 - u_5 u_6, \\ \dot{u}_5 = u_4 u_6 - u_1 u_3, \\ \dot{u}_6 = 0. \end{cases} \qquad (2)$$

System of equations (2) describes the motion of a solid body in a fluid [5]. In the general case, equations of motion were obtained and studied by G. Kirchhoff [6]. If we consider the mechanical system “body + fluid” [7] and denote by

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$\gamma = (u_1, u_2, u_3)$ the impulse of this system, $M = (u_4, u_5, u_6)$ the kinetic momentum, and $\frac{H}{2} = \frac{u_3^2+u_4^2+u_5^2}{2}$ the Hamiltonian, which is the kinetic energy of the system, then the Kirchhoff equation takes form (2).

Such a system of equations has been rigorously studied by G.-H. Halphen in [8], where, based on the knowledge of the first integrals, the author parameterized the two components of the impulse of the system "body + fluid" (in our notation u_1 and u_2) by Weierstrass elliptic functions and derived a differential equation on the component u_3. This equation has a complicated form, and it describes the squared derivative $\dot{u}_3^2$. Thus, even if solutions to the equation are found, one faces the problem of choosing the sign of the derivative when passing through critical points.

In [4], the problem was studied for $u_6 = 0$ under the assumption that there are no cusp points on the spacial projections of the geodesics, which is equivalent to $u_3 \neq 0$. In [9], the explicit expressions are obtained for the extremal control (solution to system (2)) without any restrictions on the cusp points, but under the assumption $u_6 = 0$. Such trajectories (with $u_6 = 0$) are the most demanded for the applications in image processing, see [4, Thm. 1]. However, in the theoretical aspect, the case $u_6 \neq 0$ is of great interest as a general case of the model problem. In [10], the extremal controls for arbitrary u_6 were found in operator form, without providing the exact expressions, and asymptotics of the extremal controls were examined for $u_6 \to 0$.

In this paper, we continue to study the general case when the parameter u_6 is arbitrary. We enhance the results of [10] by providing a simple ordinary differential equation (ODE) to the "main" extremal control u_3.

II. Extremal controls

The extremal controls $u_1, \ldots, u_6$ are given by solution to system (2) with given initial values $u_i(0) = u_i^0$. From (2), it immideately follows that the extremal control u_6 is constant. To find the remaining extremal controls we propose the following procedure. First, based of knowledge of the first integrals, we find the extremal control u_3. When u_3 is found, we can obtain the remaining extremal controls u_1, u_2, u_4, u_5 as functions of u_3 and the initial values u_i^0. By this reason, we call u_3 the "main" extremal control.

It was shown in [4, Thm. 2], that Hamiltonian system (1)–(2) is Liouville integrable. Thus, it has a complete set of functionally independent first integrals in involution. In this article, we are interested in the first integrals of vertical part (2).

Theorem 1: System (2) has the following first integrals: u_6, the Hamiltonian H and Casimir functions W, C, given by

$$H = u_3^2 + u_4^2 + u_5^2, \tag{3}$$

$$W = u_1 u_4 + u_2 u_5 + u_3 u_6, \tag{4}$$

$$C = u_1^2 + u_2^2 + u_3^2. \tag{5}$$

Proof: By virtue of (2), we have $\dot{u}_6 = \dot{H} = \dot{W} = \dot{C} = 0$. ■

Now, based on knowledge of the first integrals of system (2), we can derive an ODE on the "main" extremal control u_3.

Theorem 2: The extremal control u_3 in the sub-Riemannian problem on the group of rigid body motions in three-dimensional space is given by a solution to the following Cauchy problem with the second-order ordinary differential equation:

$$\ddot{u}_3(t) = 2u_3^3(t) - c_1 u_3(t) + c_2, \tag{6}$$

$$u_3(0) = u_3^0, \;\; \dot{u}_3(0) = u_1^0 u_5^0 - u_2^0 u_4^0, \tag{7}$$

where the constants c_1 and c_2 are given by

$$c_1 = H + C + u_6^2, \quad c_2 = u_6 W. \tag{8}$$

Proof: Computation of the second derivative of u_3 by virtue of system (2) and substitution (3)–(5) leads to ODE (6). The initial value $u_3(0) = u_3^0$ is given and the expression for $\dot{u}_3(0)$ is obtained from the third equation in system (2). ■

Finally, the the remaining extremal controls u_1, u_2, u_4, u_5 can be found as functions of u_3 and the initial values u_i^0.

Theorem 3: The extremal controls u_1, u_2, u_4, u_5 in the sub-Riemannian problem on the group of rigid body motions in three-dimensional space have the form

$$\begin{aligned} u_1(t) &= F_1 - u_6(I + u_6^2 L^2)^{-1}\Big[M[G] - u_6 ML[F]\Big], \\ u_2(t) &= G_1 - u_6(I + u_6^2 L^2)^{-1}\Big[M[F] + u_6 ML[G]\Big], \\ u_4(t) &= (I + u_6^2 L^2)^{-1}\Big[G - u_6 L[F]\Big], \\ u_5(t) &= (I + u_6^2 L^2)^{-1}\Big[F + u_6 L[G]\Big], \end{aligned}$$

where

$$\begin{aligned} &F_1 = F_1(t) = \tfrac{u_1^0+u_5^0}{2}\mathrm{e}^{-U(t)} - \tfrac{u_5^0-u_1^0}{2}\mathrm{e}^{U(t)}, \\ &G_1 = G_1(t) = \tfrac{u_2^0+u_4^0}{2}\mathrm{e}^{U(t)} - \tfrac{u_4^0-u_2^0}{2}\mathrm{e}^{-U(t)}, \\ &F = F(t) = \tfrac{u_1^0+u_5^0}{2}\mathrm{e}^{-U(t)} + \tfrac{u_5^0-u_1^0}{2}\mathrm{e}^{U(t)}, \\ &G = G(t) = \tfrac{u_2^0+u_4^0}{2}\mathrm{e}^{U(t)} + \tfrac{u_4^0-u_2^0}{2}\mathrm{e}^{-U(t)}, \\ &L[f] = L[f](t) = \textstyle\int_0^t f(\tau)\mathrm{ch}(U(t) - U(\tau))d\tau, \\ &M[f] = M[f](t) = \textstyle\int_0^t f(\tau)\mathrm{sh}(U(t) - U(\tau))d\tau, \\ &U(t) = \int\limits_0^t u_3(\tau)d\tau. \end{aligned}$$

Proof: Follows from [10, Thm. 1]. ■

III. Conclusion

In this paper, we study the problem of finding the extremal controls in the sub-Riemannian problem on the group SE_3 of rigid body motions in three-dimensional space. Here, we improve the results of [10]. The main contribution is given by Theorem 2, where we present a simple ODE for the "main" extremal control u_3. The exact solution can be expressed in elliptic functions by standard methods [11].

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