Modelling of the Poggendorff Illusion via Sub-Riemannian Geodesics in the Roto-Translation Group

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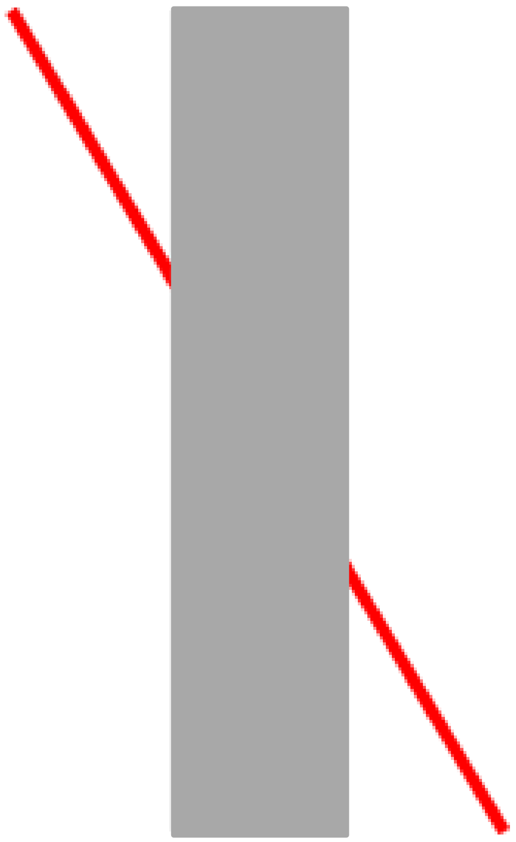
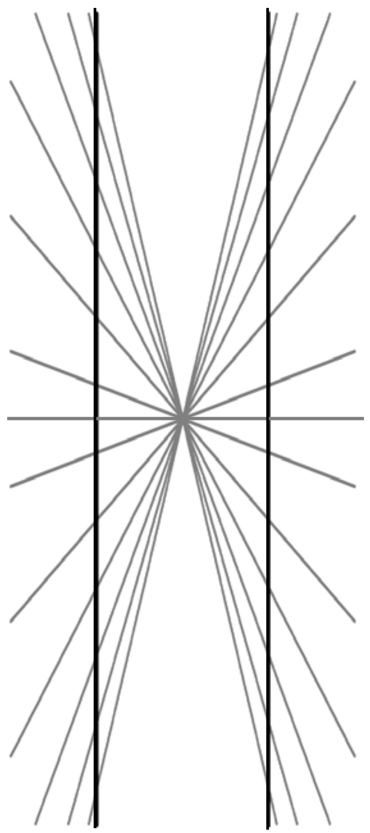
Geometrical Optical Illusions

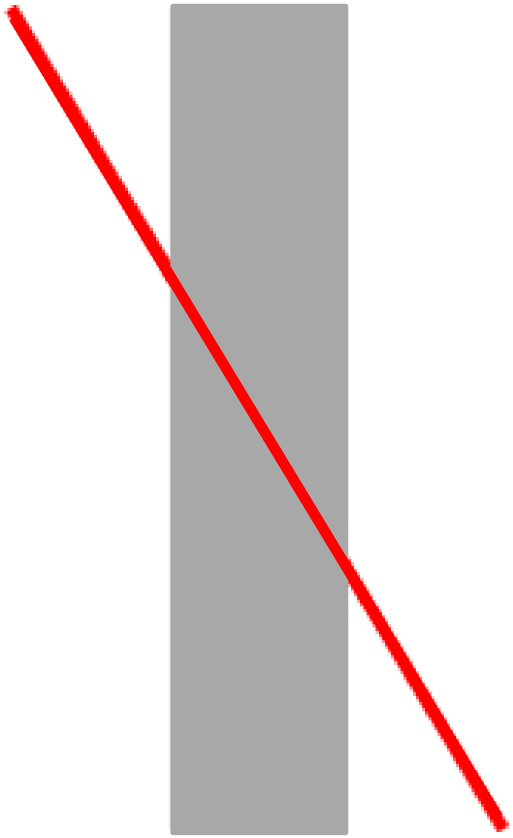


Poggendorff illusion

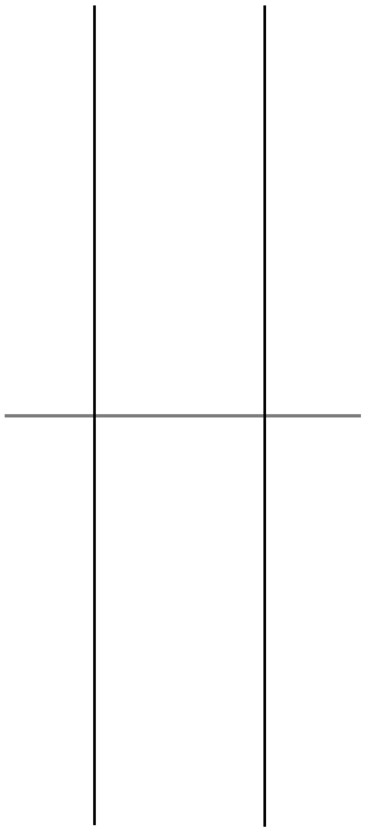


Hering illusion

Geometrical Optical Illusions

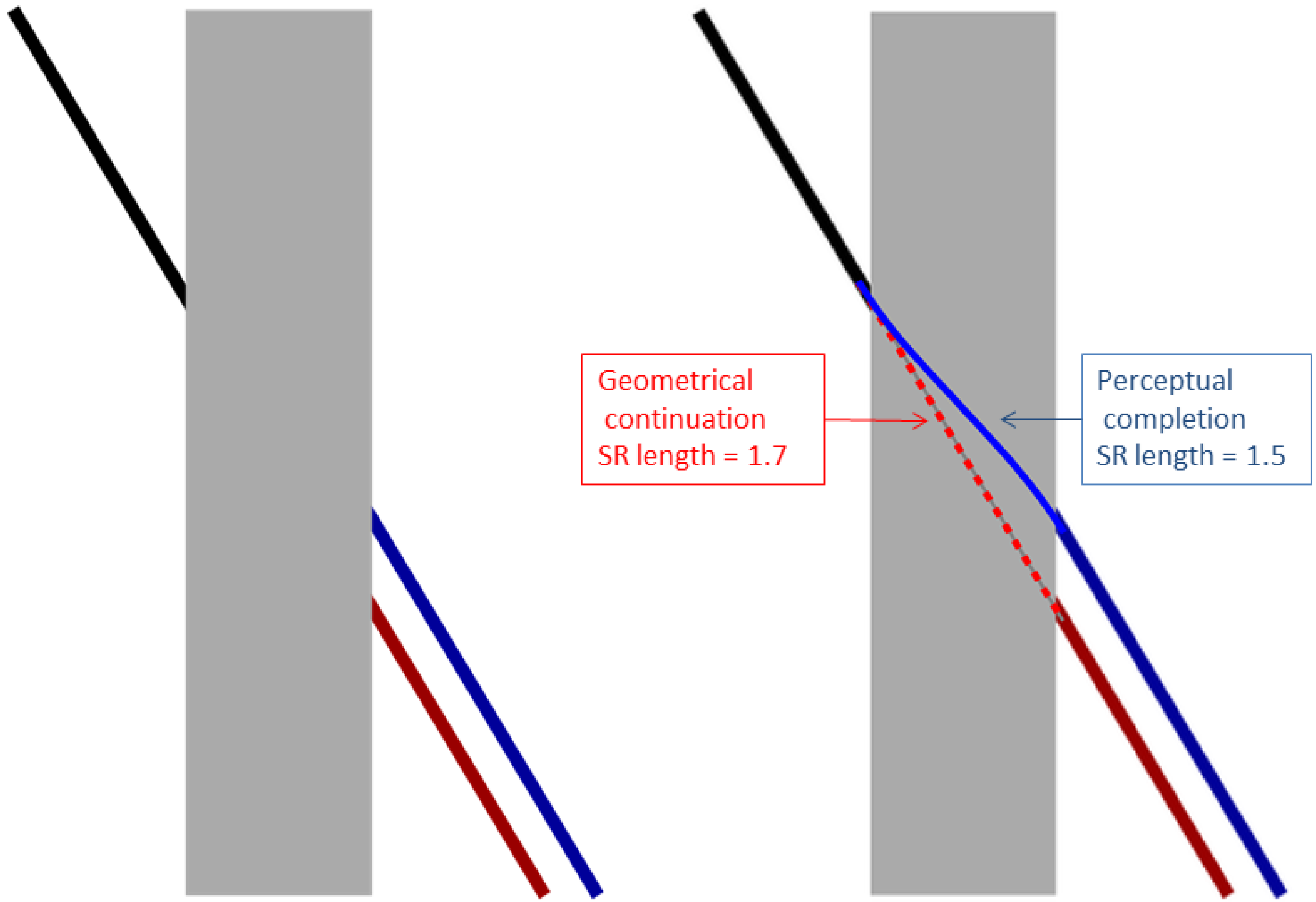


Poggendorff illusion



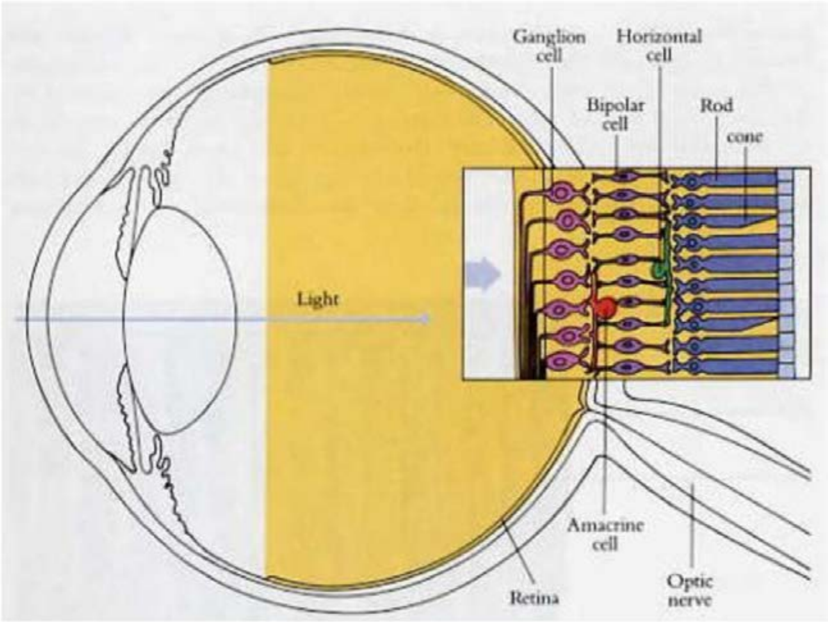
Hering illusion

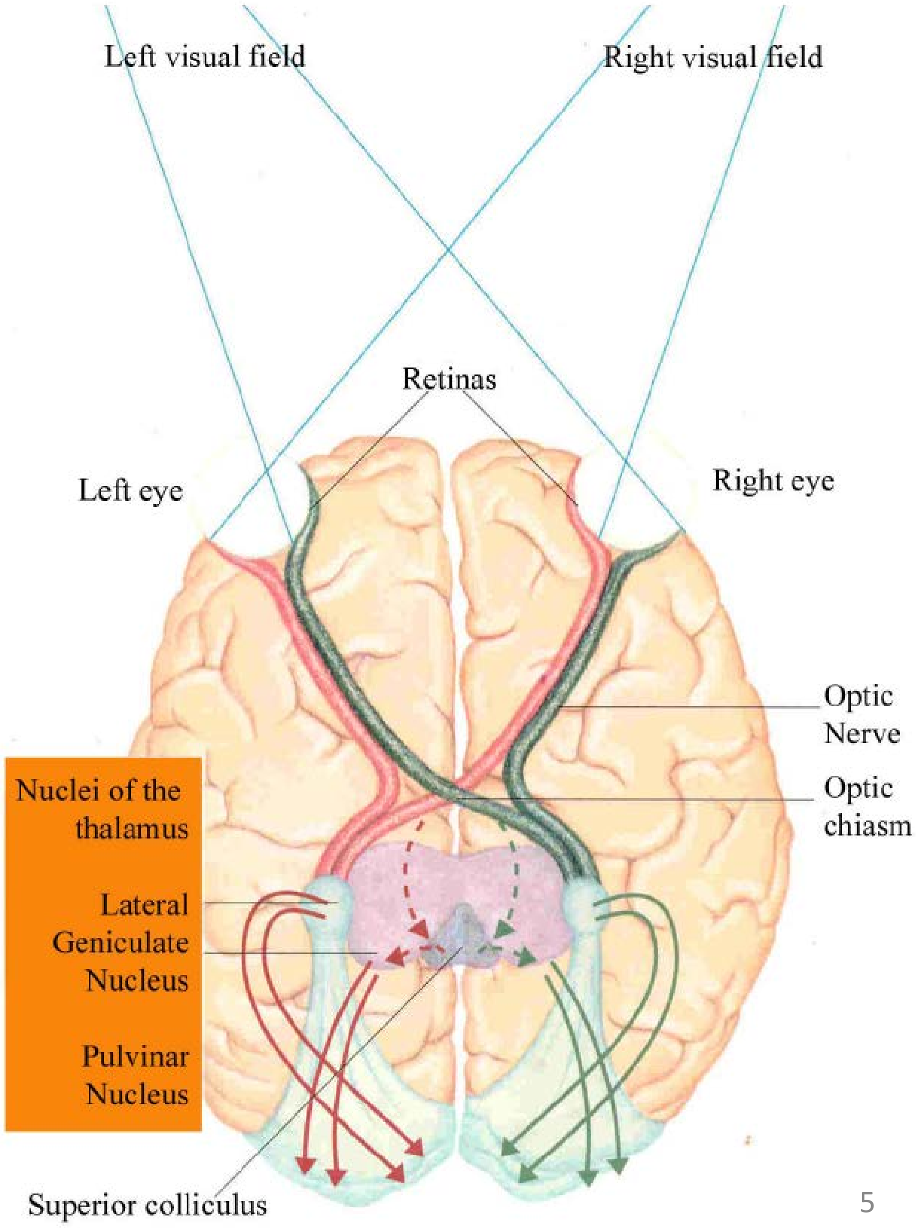
Modelling of Illusory Contour



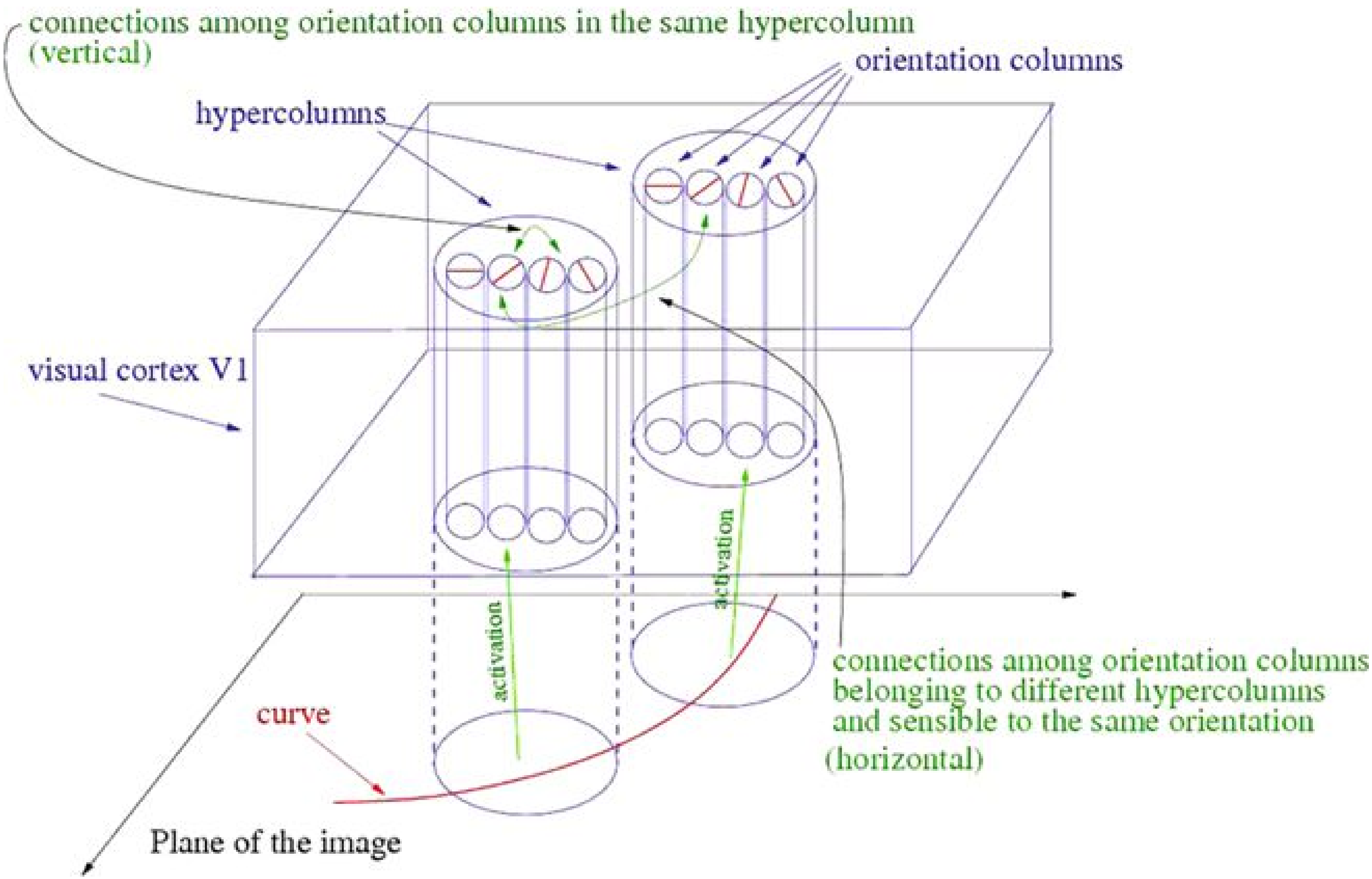
Idea: The illusory contour appears as a geodesic in a metric induced by visual stimulus.

Perception of Visual Information in Human Brain





Architecture of Primary Visual Cortex V1

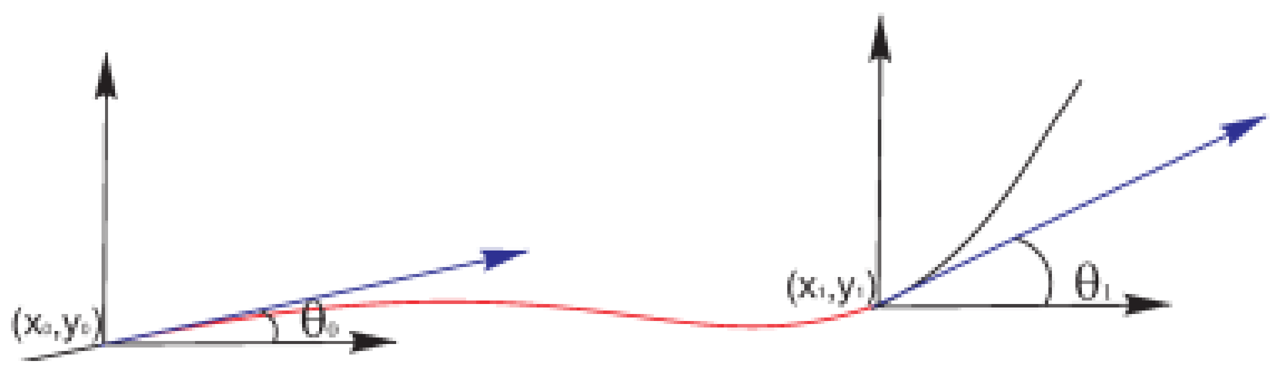


Replicated from R. Duits, U. Boscain, F. Rossi, Y. Sachkov,
Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2), JMIV, 2013.

Cortical Based Model of Perceptual Completion

- D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.
- Sub-Riemanian structures in neurogeometry of the vision:
  - J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003. (Heisenberg group.)
  - G. Citti and A. Sarti, A Cortical Based Model of Perceptual Completion in the Roto-Translation Space, 2006. ($SE(2)$ group.)
- Variational principle: recovered arc has minimal length in the space (x, y, θ):

$$\int \sqrt{\xi^2\left(\dot{x}^2+\dot{y}^2\right)+\dot{\theta}^2}\,dt \to \min, \ \text{under constraint } \dot{\theta} = \arg\left(\dot{x} + \mathrm{i}\ \dot{y}\right)$$

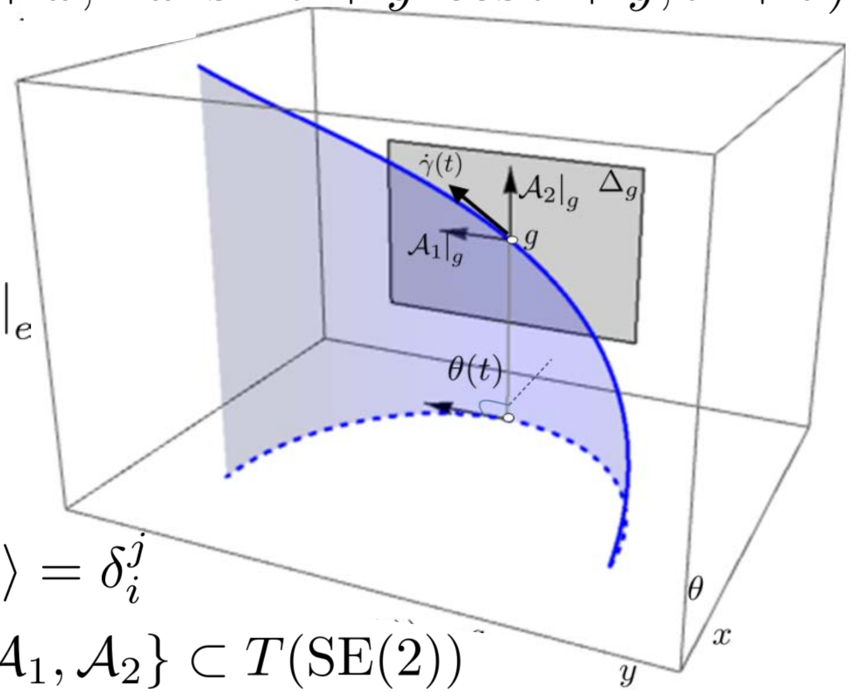


Sub-Riemannian Structure in SE(2)

Lie group $\mathrm{SE}(2) \ni g \sim (x, y, \theta) \in \mathbb{R}^2 \times S^1$,

$$L_g g' = g g' = (x' \cos\theta + y' \sin\theta + x, -x' \sin\theta + y' \cos\theta + y, \theta' + \theta).$$

Basis left-invariant vector fields



$$\mathcal{A}_1|_g = \cos\theta\, \partial_x|_g + \sin\theta\, \partial_y|_g = (L_g)_* \,\partial_x|_e$$

$$\mathcal{A}_3|_g = -\sin\theta\, \partial_x|_g + \cos\theta\, \partial_y|_g = (L_g)_* \,\partial_y|_e$$

$$\mathcal{A}_2|_g = \partial_\theta|_g = (L_g)_* \,\partial_\theta|_e$$

Basis left-invariant one forms $\langle \omega^i, \mathcal{A}_j \rangle = \delta^j_i$

Left-invariant distribution $\Delta = \mathrm{span}\{\mathcal{A}_1, \mathcal{A}_2\} \subset T(\mathrm{SE}(2))$

Metric tensor $\mathcal{G}|_g = \mathcal{C}^2(g) \left(\xi^2 \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2\right)\big|_g$ on Δ,

with external cost $\mathcal{C} : \mathrm{SE}(2) \to [\delta, +\infty)$, $\delta > 0$, and $\xi > 0$.

SR-distance: Inf among Lipschitzian curves $\gamma : [0, T] \to \mathrm{SE}(2)$

$$d(e, g) = \inf\{\int_0^T \sqrt{\mathcal{G}|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\mathrm{d}\,t \mid \begin{array}{l}\gamma(0) = e, \\ \gamma(T) = g,\end{array} \;\; \dot{\gamma}(t) \in \Delta|_{\gamma(t)}\}.$$

Construction of External Cost

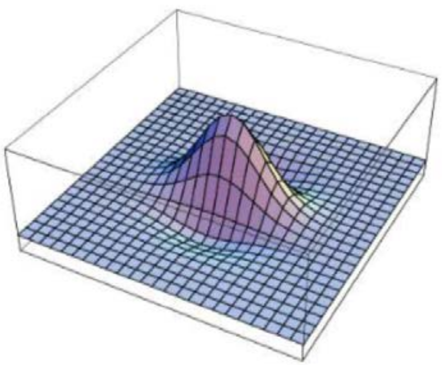
Retinal Plane $\mathbb{R}^2 \ni (x, y)$. Stimulus of Intensity $I(x, y) : M \subset \mathbb{R}^2 \to \mathbb{R}^+$.

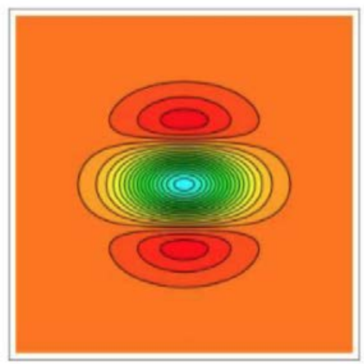
Local coordinates $\chi = (\chi_1, \chi_2) \in M$ centered at (x, y).

Gabor Filters:

$$\psi_0(\chi) = \psi_0(\chi_1, \chi_2) = \frac{\alpha}{2\pi\sigma^2} e^{\frac{-(\chi_1^2+\alpha^2\chi_2^2)}{2\sigma^2}} e^{\frac{2i\chi_2}{\lambda}},$$

$\lambda > 0$ spatial wavelength, $\alpha > 0$ spatial aspect ratio, $\sigma > 0$ deviation.

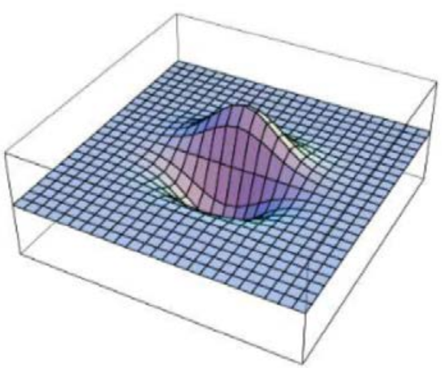


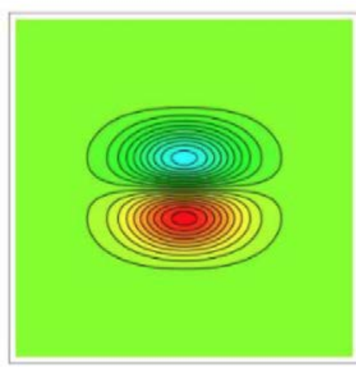


Even part

$$\mathrm{Re}(\psi_0(\chi)) = \frac{\alpha}{2\pi\sigma^2} e^{\frac{-(\chi_1^2+\alpha^2\chi_2^2)}{2\sigma^2}} \cos \frac{2\chi_2}{\lambda}$$

detection of contours.





Odd part

$$\mathrm{Im}(\psi_0(\chi)) = \frac{\alpha}{2\pi\sigma^2} e^{\frac{-(\chi_1^2+\alpha^2\chi_2^2)}{2\sigma^2}} \sin \frac{2\chi_2}{\lambda}$$

detection of boundaries.

Construction of External Cost

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$\lambda > 0$ spatial wavelength, $\alpha > 0$ spatial aspect ratio, $\sigma > 0$ deviation.

Translations and rotations: $A_{(x,y,\theta)}(\chi) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.$

General Receptive Profile:

$$\psi_{(x,y,\theta)}(\chi_1, \chi_2) = \psi_0(A^{-1}_{(x,y,\theta)}(\chi_1, \chi_2)),$$

Output of Receptive Profiles:

$$O(x, y, \theta) = \int_M I(\chi_1, \chi_2)\, \psi_{(x,y,\theta)}\, (\chi_1, \chi_2)\, d\chi_1 d\chi_2.$$

External Cost in SR metric:

$$\mathcal{C}(x, y, \theta) = (1 + \mathrm{Im}(O(x, y, \theta)))^{-\frac{1}{2}}$$

Sub-Riemannian Length Minimizers

Theorem[BDMS15]. Let $\mathcal{W}(g)$ be a viscosity solution of eikonal system

$$\begin{cases} \frac{1}{\xi^2}\left(\mathcal{A}_1|_g(\mathcal{W})\right)^2 + \left(\mathcal{A}_2|_g(\mathcal{W})\right)^2 = \mathcal{C}^2(g), \text{ for } g \neq e, \\ \mathcal{W}(e) = 0. \end{cases}$$

Then $\mathcal{S}_t = \{g \in \mathrm{SE}(2) \mid \mathcal{W}(g) = t\}$ are SR-spheres of radius t.
SR-minimizer $\gamma(t)$ starting from e and ending at g is given by $\gamma(t) = \gamma_b(\mathcal{W}(g) - t)$, which is found by integration for $t \in [0, \mathcal{W}(g)]$

$$\dot{\gamma}_b(t) = -u_1(t)\, \mathcal{A}_1|_{\gamma_b(t)} - u_2(t)\, \mathcal{A}_2|_{\gamma_b(t)}, \qquad \gamma_b(0) = g,$$

where $u_1(t) = \dfrac{\mathcal{A}_1|_{\gamma_b(t)}(\mathcal{W})}{(\xi\, \mathcal{C}(\gamma_b(t)))^2}$ and $u_2(t) = \dfrac{\mathcal{A}_2|_{\gamma_b(t)}(\mathcal{W})}{(\mathcal{C}(\gamma_b(t)))^2}$.

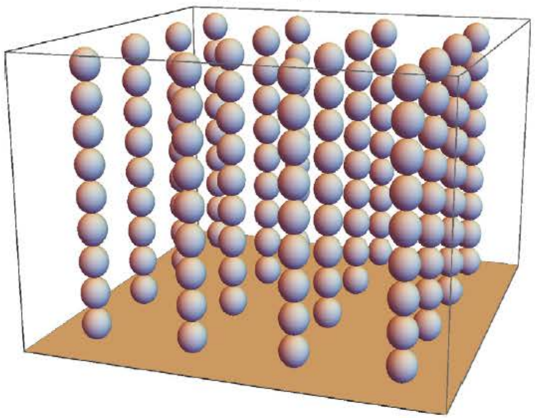
[BDMS15] E.J. Bekkers, R. Duits, A. Mashtakov and G.R. Sanguinetti, *A PDE Approach to Data-driven SR Geodesics in* SE(2), SIIMS, 2015.

Riemannian Approximation and Fast Marching

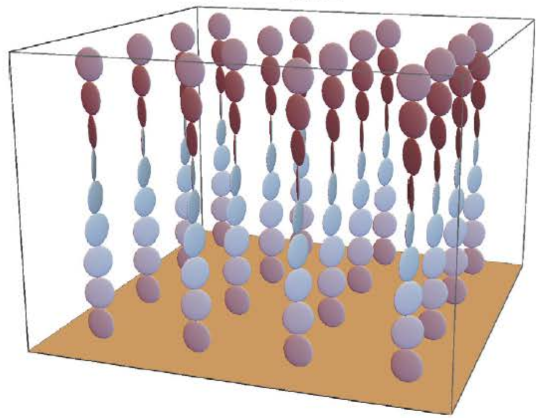
Riemannian metric tensor

$$\mathcal{G}_\varepsilon = \mathcal{C}^2(\cdot)\left(\xi^2\omega^1\otimes\omega^1+\omega^2\otimes\omega^2+\xi^2\varepsilon^{-2}\omega^3\otimes\omega^3\right)$$

$\varepsilon = 1$



$\varepsilon = 0.1$



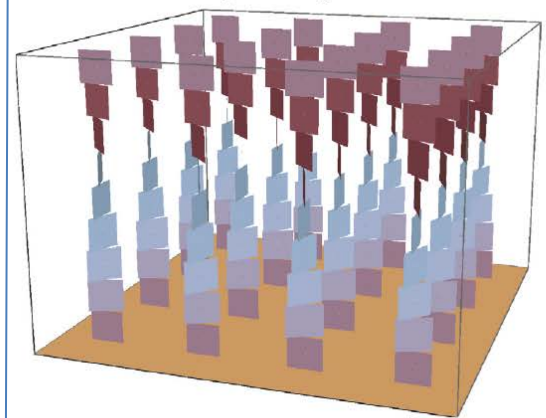
SR Fast Marching for highly anisotropic Riemannian eikonal equation

$$\begin{cases} \text{For } g \neq e: \\ \frac{\mathcal{A}_1|_g(\mathcal{W}_\varepsilon)^2}{\xi^2} + \mathcal{A}_2|_g(\mathcal{W}_\varepsilon)^2 + \varepsilon^2\frac{\mathcal{A}_3|_g(\mathcal{W}_\varepsilon)^2}{\xi^2} = \mathcal{C}^2(g), \\ \text{For } g = e: \quad \mathcal{W}_\varepsilon(e) = 0. \end{cases}$$

Sub-Riemannian metric tensor

$$\mathcal{G} = \mathcal{C}^2(\cdot)\left(\xi^2\omega^1\otimes\omega^1+\omega^2\otimes\omega^2\right)$$

$\varepsilon \to 0$



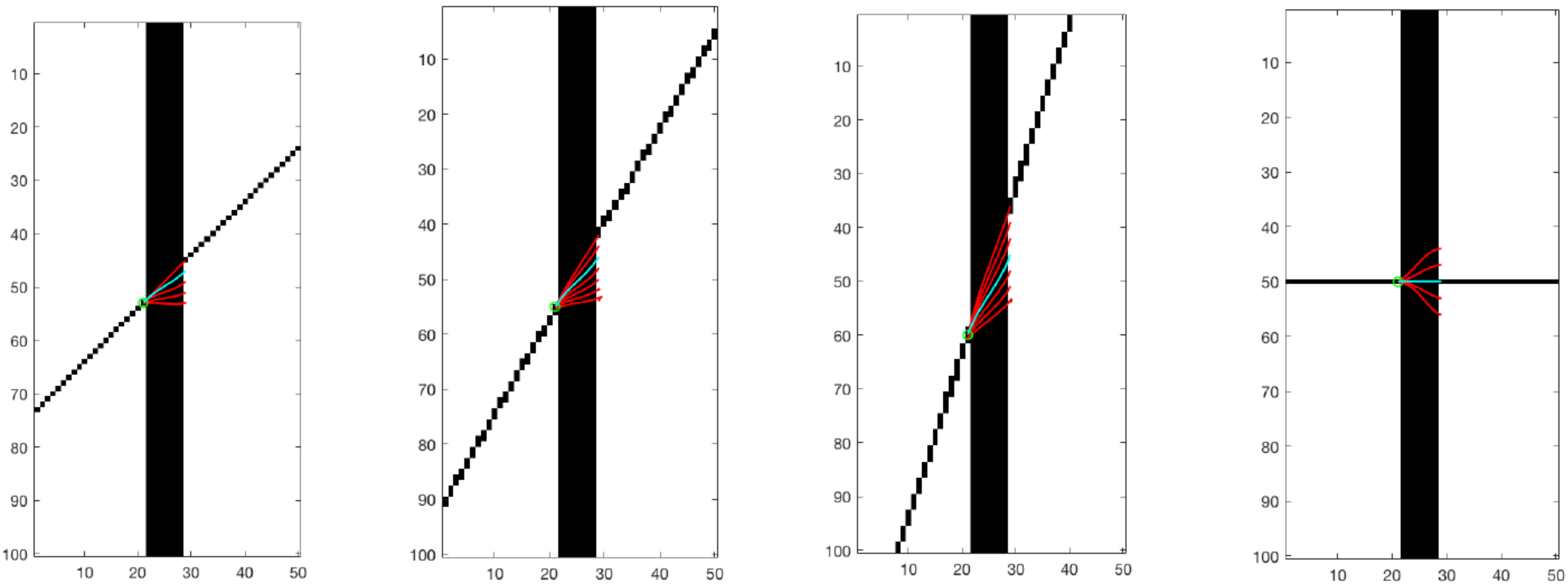
HJB system can be written as SR eikonal equation

$$\begin{cases} \text{For } g \neq e: \\ \frac{\mathcal{A}_1|_g(\mathcal{W})^2}{\xi^2} + \mathcal{A}_2|_g(\mathcal{W})^2 = \mathcal{C}^2(g), \\ \text{For } g = e: \quad \mathcal{W}(e) = 0. \end{cases}$$

[**SDBJMM15**] G. Sanguinetti, R. Duits, E. Bekkers, M. Janssen, A. Mashtakov, J-M. Mirebeau, *Sub-Riemannian Fast Marching in* SE(2), Proc. CIARP, 2015.

Simulation of Illusory Contour

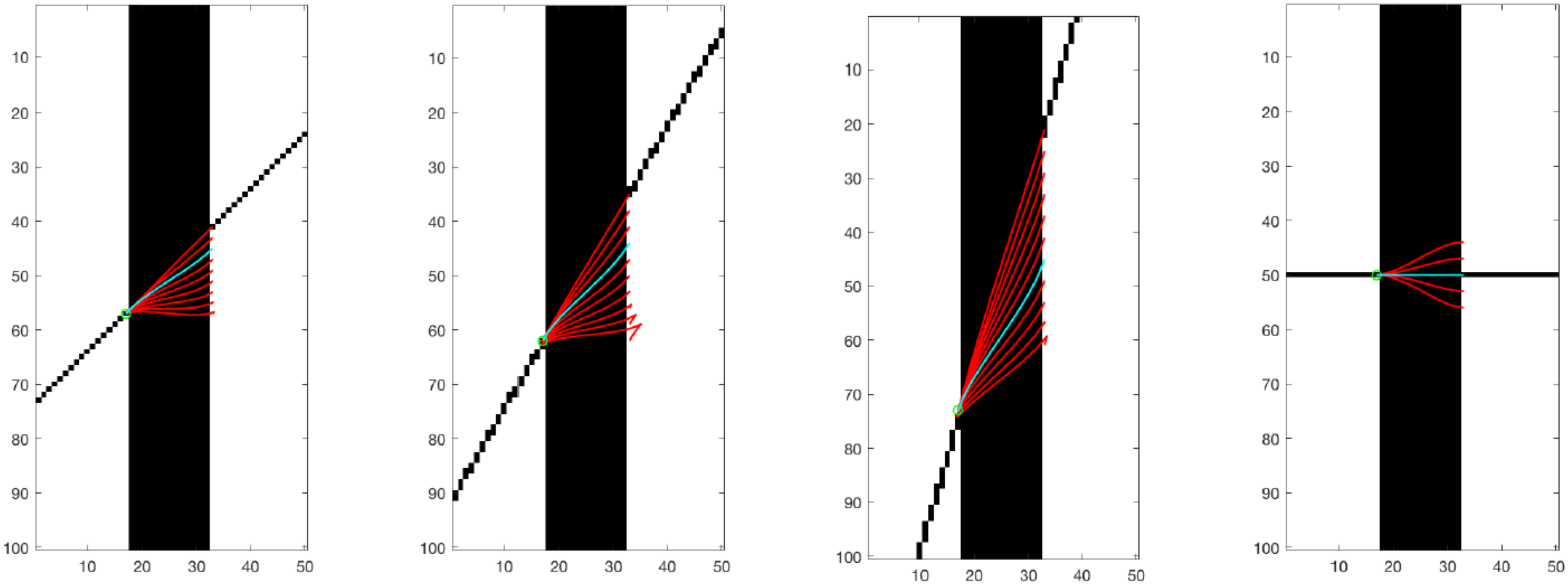
Width of the stripe 9 px.



Illusory contour (in cyan) is given by SR-minimizer with fixed initial condition and terminal set on the right edge of the stripe with the same angle.

Simulation of Illusory Contour

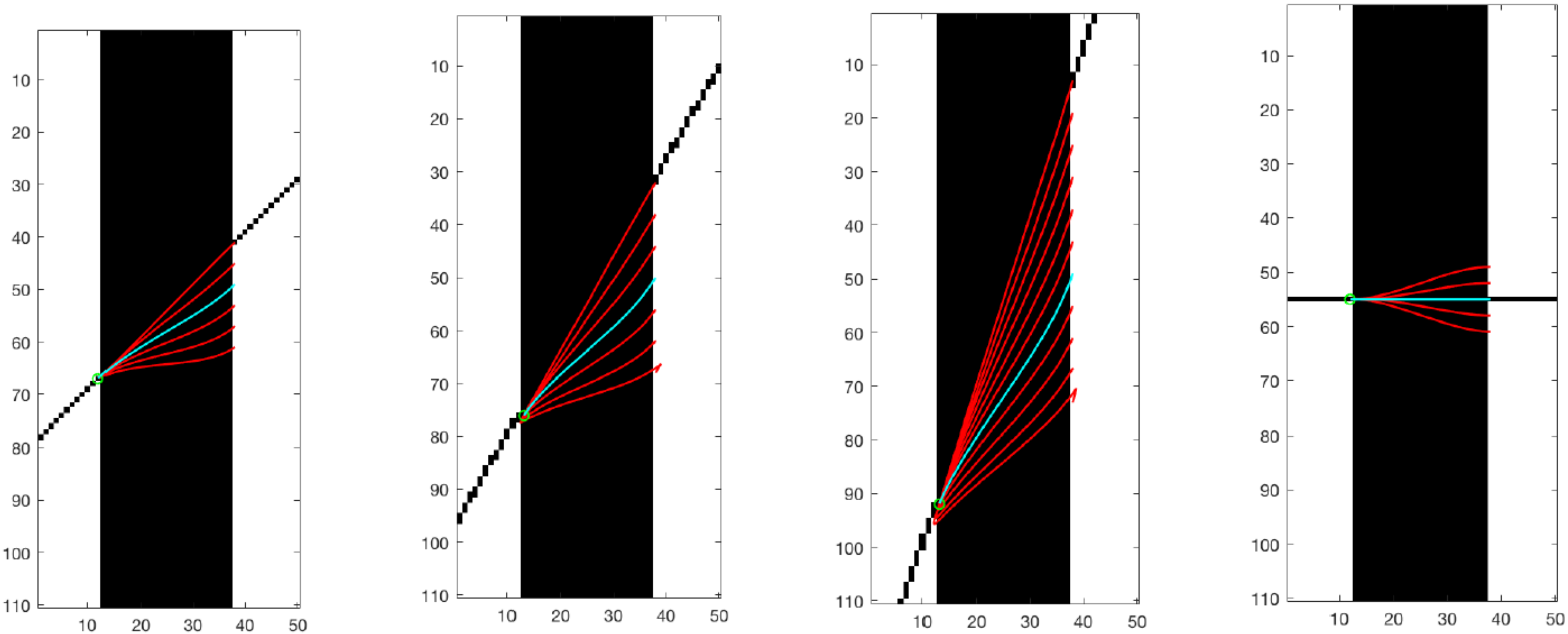
Width of the stripe 15 px.



Illusory contour (in cyan) is given by SR-minimizer with fixed initial condition and terminal set on the right edge of the stripe with the same angle.

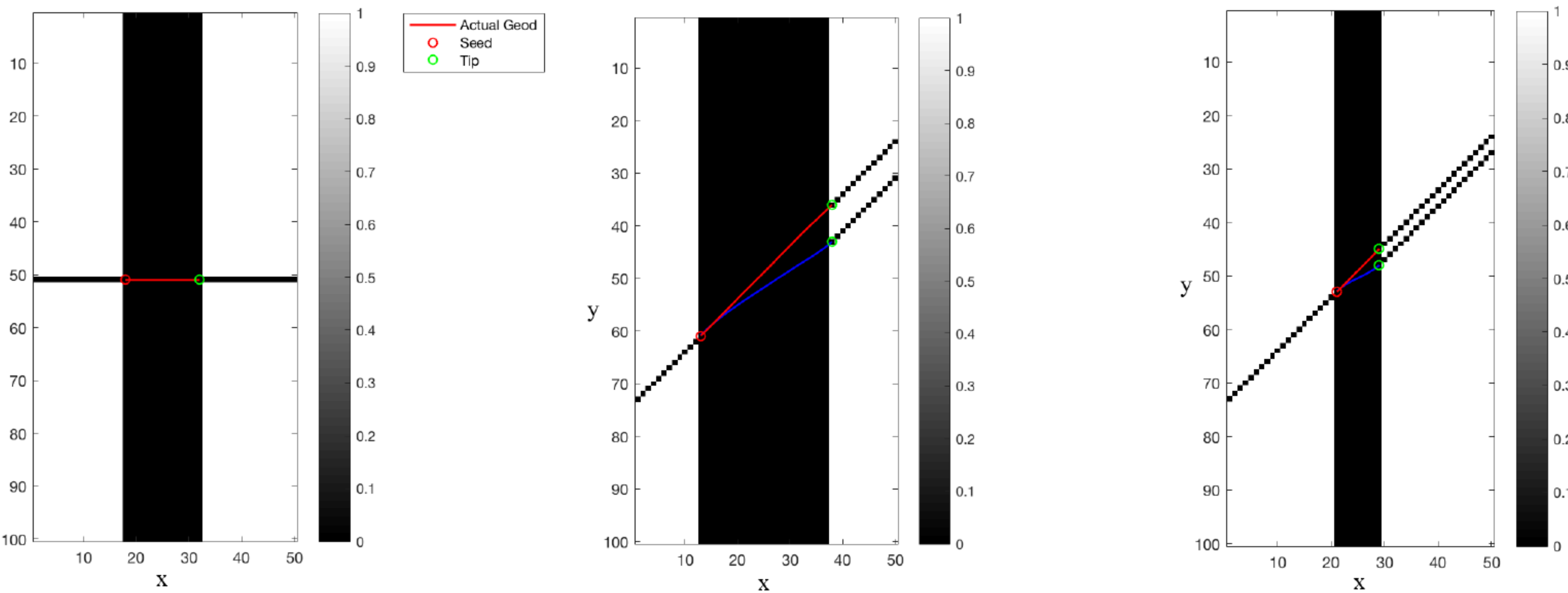
Simulation of Illusory Contour

Width of the stripe 25 px.



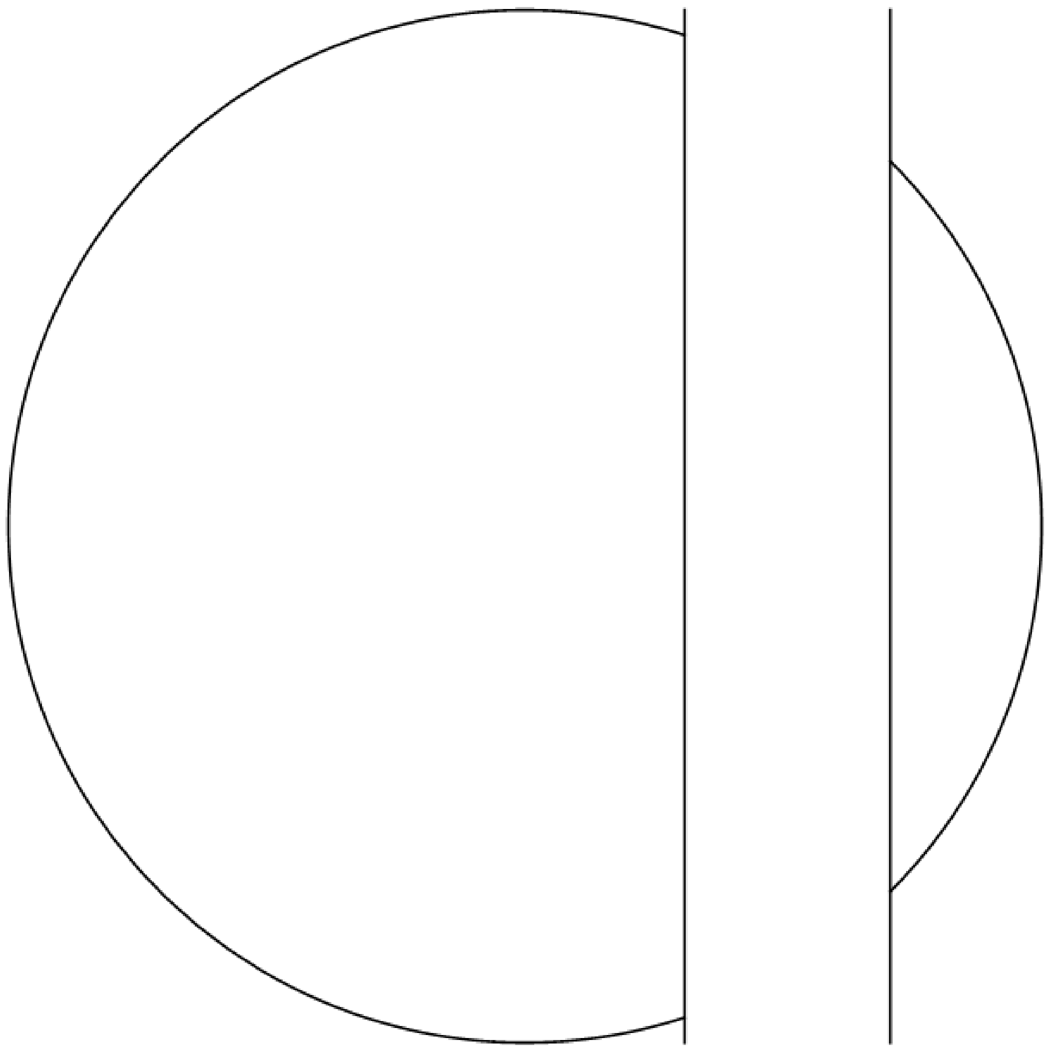
Illusory contour (in cyan) is given by SR-minimizer with fixed initial condition and terminal set on the right edge of the stripe with the same angle.

Simulation of Illusory Contour

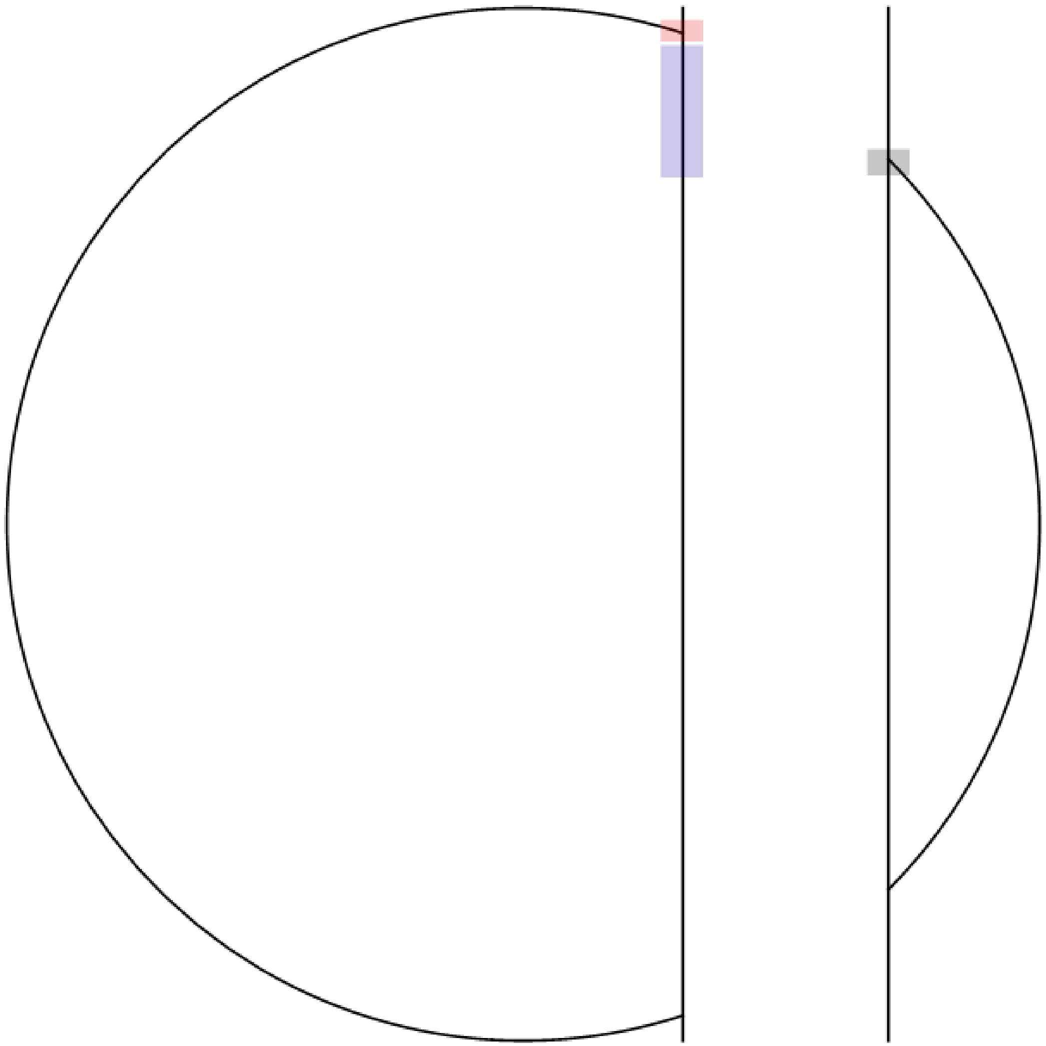


Type of curve	Width = 9 pixels	Width = 15 pixels	Width = 25 pixels
Percep. curve $\theta = \pi/4$	1.0366	1.8094	3.1113
Actual curve $\theta = \pi/4$	1.1369	2.0480	3.5354
Percep. curve $\theta = \pi/10$	2.1033	3.4719	4.9411
Actual curve $\theta = \pi/10$	2.8925	4.4927	7.3924
Percep. curve $\theta = \pi/2$	1.0320	1.4412	2.5196

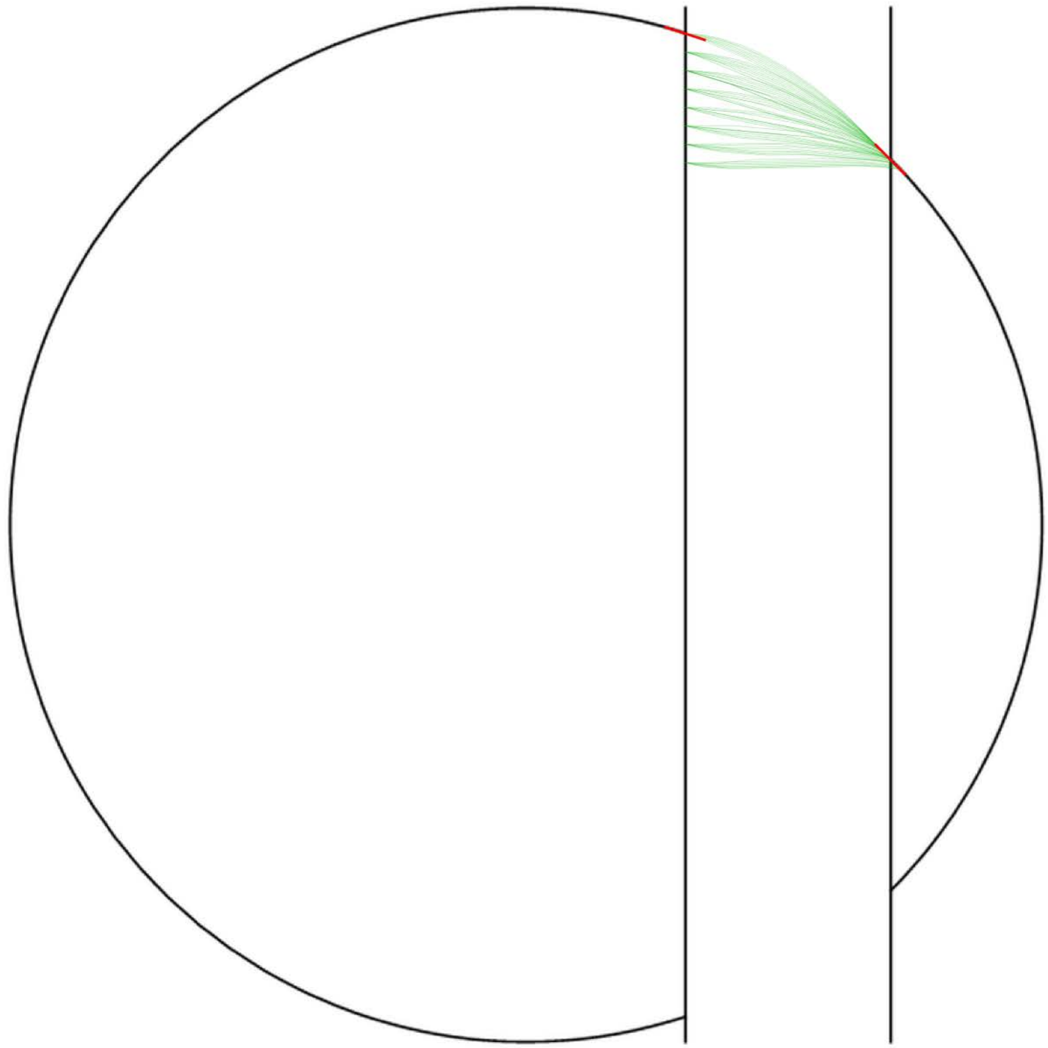
Round Poggendorff Illusion



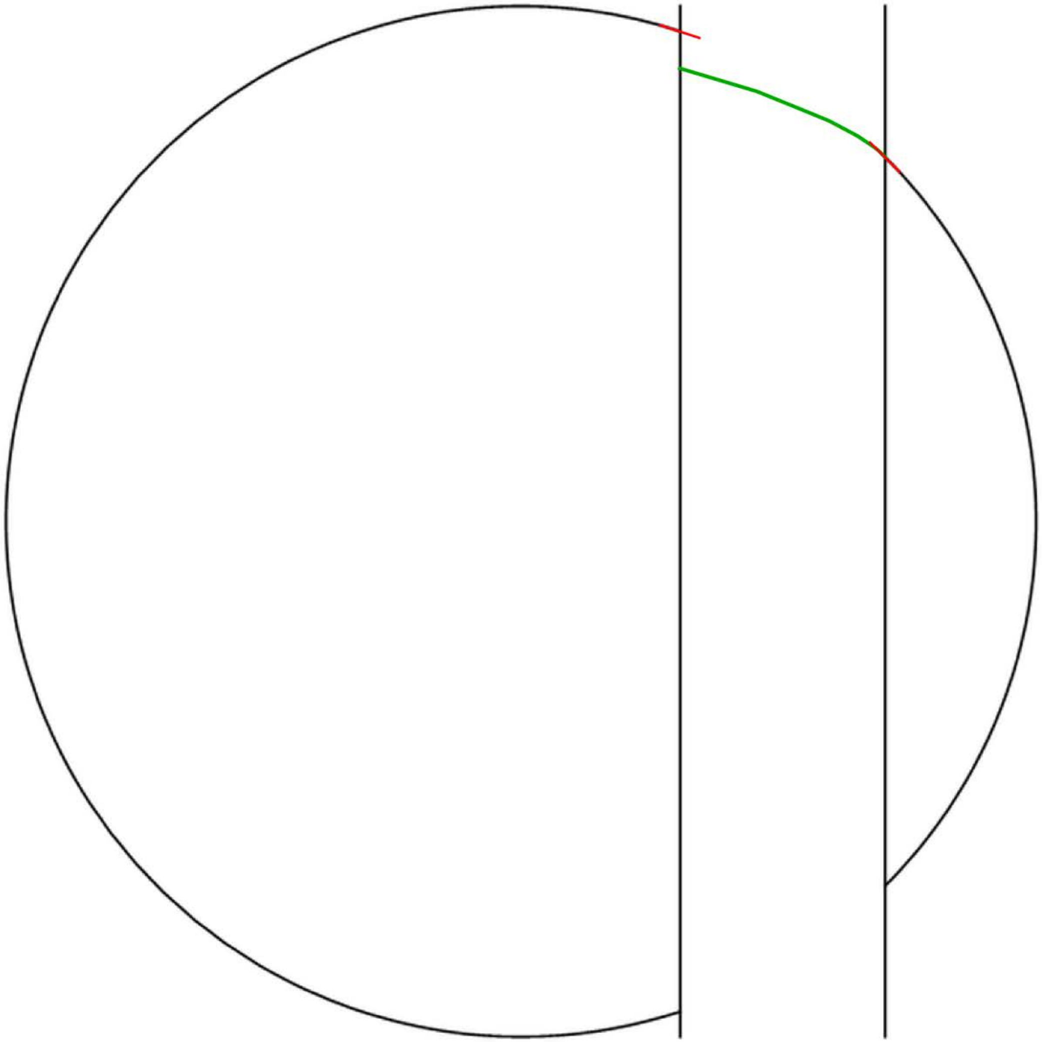
Round Poggendorff Illusion



Round Poggendorff Illusion



Round Poggendorff Illusion



Conclusion

- neuro-mathematical model for the perceptual Poggendorff phenomenon
- realistic external cost based on output of receptive profiles
- computation of SR-minimizers via SR-Fast Marching

Continuation

- extension to Hering, Zollner and Wundt illusions
- phenomenological values of the parameters

Thank you for your attention!