SUB-RIEMANNIAN GEODESICS ON THE GROUP OF RIGID BODY MOTIONS

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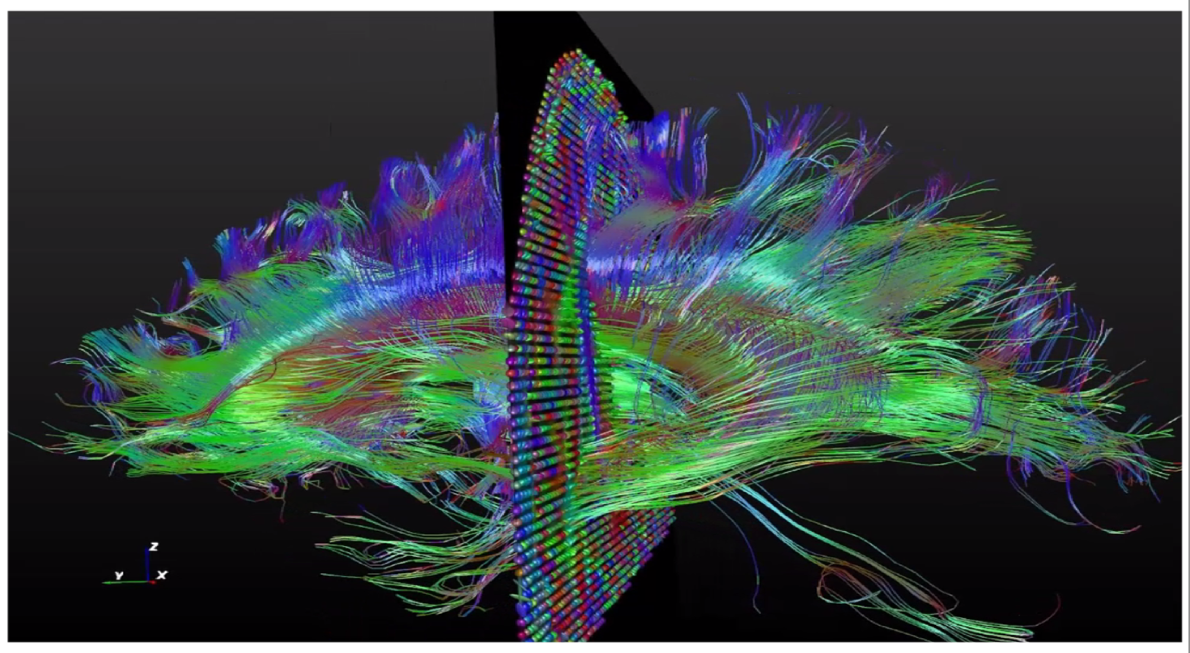
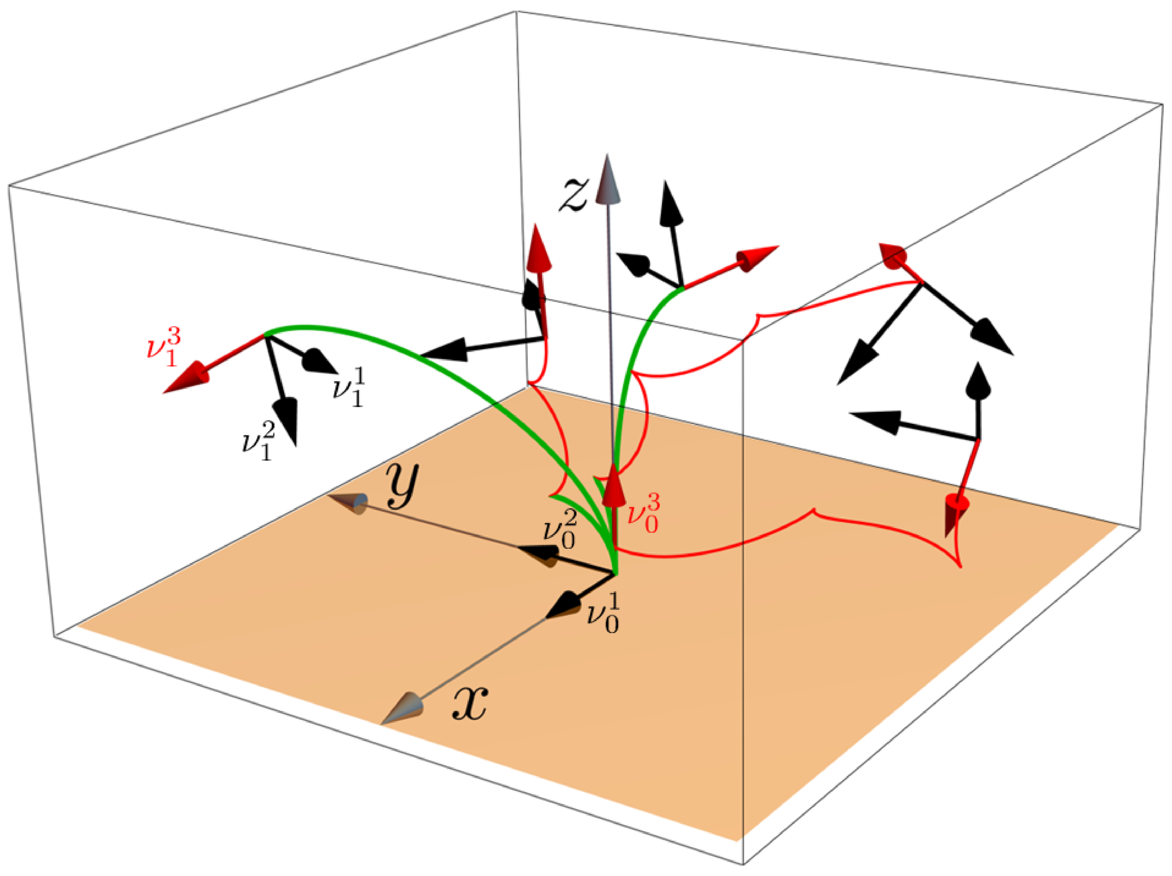
INTRODUCTION

We study a sub-Riemannian (SR) problem on the Lie group SE_3 of rigid body motions in $\mathbb{R}^3$. Solution curves have applications in image processing (tracking of neural fibers and blood vessels in DW-MRI images of human brain); and in robotics (motion planing problem for an aircraft, moving forward/backward).



It can be seen as a problem of optimal motion of a rigid body with nonintegrable constraints. By given two orthonormal frames $N_0 = \{\nu_0^1, \nu_0^2, \nu_0^3\}$, $N_1 = \{\nu_1^1, \nu_1^2, \nu_1^3\}$ attached respectively at two given points $q_0 = (x_0, y_0, z_0)$, $q_1 = (x_1, y_1, z_1)$ in $\mathbb{R}^3$, we aim to find an optimal motion that transfers q_0 to q_1 such that the frame N_0 is transferred to the frame N_1. The frame can move forward or backward along one of the vector chosen in the frame and rotate simultaneously via the remaining two (of three) prescribed axes. The required motion should be optimal in the sense of minimal length in the space $\mathrm{SE}_3 \simeq \mathbb{R}^3 \rtimes \mathrm{SO}_3$.



GROUP OF RIGID BODY MOTIONS IN 3D

Group element:

$g = (\mathbf{x}, R) \in \mathrm{SE}_3 = \mathbb{R}^3 \rtimes \mathrm{SO}_3.$

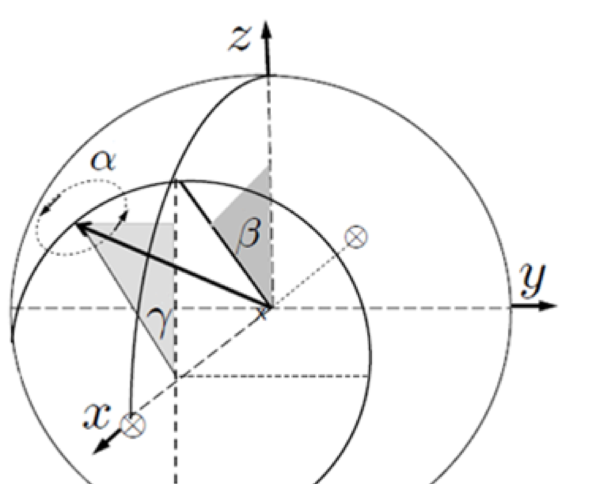
Group operations:

$g_1 g_2 = (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)$

$\quad = (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2),$

$g^{-1} = (-R^T\mathbf{x}, R^T),$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$,



$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the angles $\alpha \in (-\pi, \pi]$, $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in (-\pi, \pi]$.

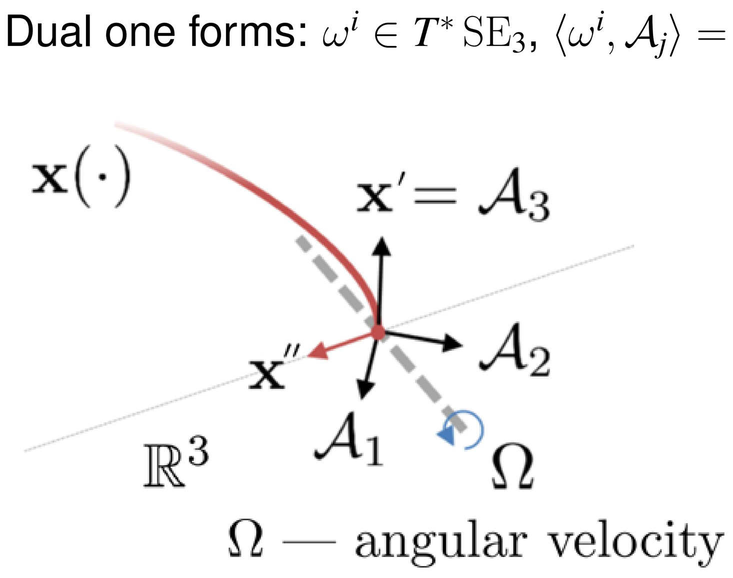
LEFT-INVARIANT VECTOR FIELDS

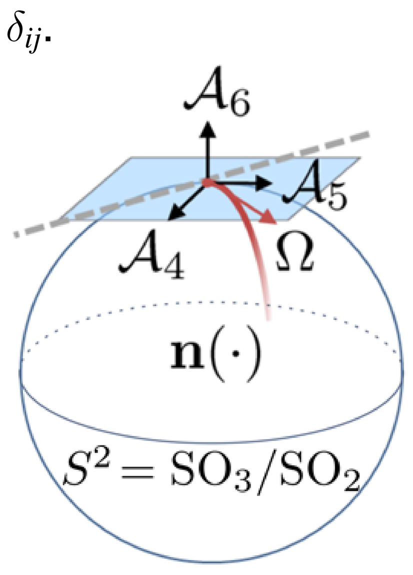
Lie algebra: $\mathfrak{se}_3 = T_e\mathrm{SE}_3 = \mathrm{span}(A_1, \dots, A_6)$

Left shift on the group: $L_g h = gh$,

Left-invariant vector fields: $\mathcal{A}_i|_g = (L_g)_* A_i$,

Dual one forms: $\omega^i \in T^*\mathrm{SE}_3$, $\langle \omega^i, \mathcal{A}_j \rangle = \delta_{ij}$.





SUB-RIEMANNIAN PROBLEM IN SE_3

Left-invariant distribution: $\Delta = \mathrm{span}(\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) \subset T\,\mathrm{SE}_3$

Metric tensor: $\mathcal{G}_\xi = \xi^2\omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5$ on Δ

A constant $\xi > 0$ balancing spatial and angular displacement

SR-distance: Inf among Lipschitzian curves $\gamma : [0, T] \to \mathrm{SE}_3$

$$d(e, g) = \inf\left\{ \int_0^T \sqrt{\mathcal{G}_\xi(\dot\gamma(t), \dot\gamma(t))}\,\mathrm{d}\,t \;\middle|\; \dot\gamma(t) \in \Delta|_{\gamma(t)},\ \begin{matrix}\gamma(0) = e, \\ \gamma(T) = g\end{matrix} \right\}.$$

SR-minimizers are solutions to the optimal control problem

$\mathbf{P_{mec}}$: Boundary conditions: $\gamma(0) = e, \quad \gamma(T) = g,$

Control system: $\dot\gamma(t) = u^3(t)\mathcal{A}_3|_{\gamma(t)} + u^4(t)\mathcal{A}_4|_{\gamma(t)} + u^5(t)\mathcal{A}_5|_{\gamma(t)}$

Cost functional: $\frac{1}{2}\int_0^T \xi^2 u_3(t)^2 + u_4(t)^2 + u_5(t)^2\,\mathrm{d}\,t \to \min.$

- Complete controllability (Chow-Rashevski)
- Existence of minimizers (Filippov theorem)
- No abnormal extremals: $\dim[\Delta, \Delta] = \dim(\mathrm{SE}_3)$
- Minimizers are analytic
- Scaling homothety: $\xi = 1$

Control system in coordinates:

$$\begin{cases} \dot x = u^3 \sin\beta, \\ \dot y = -u^3\cos\beta\sin\theta, \\ \dot z = u^3\cos\beta\cos\theta, \\ \dot\theta = \sec\beta(u^4\cos\alpha - u^5\sin\alpha), \\ \dot\beta = u^4\sin\alpha + u^5\cos\alpha, \\ \dot\alpha = -(u^4\cos\alpha - u^5\sin\alpha)\tan\beta. \end{cases}$$

PONTRYAGIN MAXIMUM PRINCIPLE

Left Invariant Hamiltonians: $\lambda_i = \langle p, \mathcal{A}_i \rangle$

$p = p_1\,\mathrm{d}x + p_2\,\mathrm{d}y + p_3\,\mathrm{d}z + p_4\,\mathrm{d}\theta + p_5\,\mathrm{d}\beta + p_6\,\mathrm{d}\alpha \in T^*\mathrm{SE}_3$

Control dependent Hamiltonian:

$H_u = u^3\lambda_3 + u^4\lambda_4 + u^5\lambda_5 - \frac{1}{2}\left((u^3)^2 + (u^4)^2 + (u^5)^2\right)$

Maximality Condition: $u^3 = \lambda_3, \quad u^4 = \lambda_4, \quad u^5 = \lambda_5$

The (maximized) Hamiltonian: $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$

The Hamiltonian system (via Poisson brackets $\dot\lambda_i = \{H, \lambda_i\}$):

$$\begin{cases} \dot\lambda_1 = -\lambda_3\lambda_5, \\ \dot\lambda_2 = \lambda_3\lambda_4, \\ \dot\lambda_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \\ \dot\lambda_4 = \lambda_2\lambda_3 - \lambda_5\lambda_6, \\ \dot\lambda_5 = \lambda_4\lambda_6 - \lambda_1\lambda_3, \\ \dot\lambda_6 = 0, \end{cases} \qquad \begin{cases} \dot x = \lambda_3\sin\beta, \\ \dot y = -\lambda_3\cos\beta\sin\theta, \\ \dot z = \lambda_3\cos\beta\cos\theta, \\ \dot\theta = \sec\beta(\lambda_4\cos\alpha - \lambda_5\sin\alpha), \\ \dot\beta = \lambda_4\sin\alpha + \lambda_5\cos\alpha, \\ \dot\alpha = -(\lambda_4\cos\alpha - \lambda_5\sin\alpha)\tan\beta, \end{cases}$$

LIOUVILLE INTEGRABILITY

First integrals the Hamiltonian system:

- The Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- Left-invariant basis Hamiltonian λ_6
- Casimirs $W = -\lambda_1\lambda_4 - \lambda_2\lambda_5 - \lambda_3\lambda_6$, $\mathcal{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- Right-invariant Hamiltonians $\rho_i = \langle \lambda, Y_i \rangle$, Y_i right invariant v.f.

Complete system of first Integrals: $(H,\ \lambda_6,\ W,\ \rho_1,\ \rho_2,\ \rho_3)$

Theorem. The Hamiltonian system is Liouville integrable.

EXTREMAL CONTROLS FOR $\lambda_6 = 0$

Theorem. Suppose $\lambda_6(0) = 0$; then λ_4, λ_5 are expressed via $U(t) = \int_0^t \lambda_3(\tau)\,\mathrm{d}\,\tau$ and initial values

$$\lambda_4(t) = \frac{\lambda_2(0) + \lambda_4(0)}{2}\exp(U(t)) - \frac{\lambda_2(0) - \lambda_4(0)}{2}\exp(-U(t)),$$

$$\lambda_5(t) = \frac{\lambda_1(0) + \lambda_5(0)}{2}\exp(-U(t)) - \frac{\lambda_1(0) - \lambda_5(0)}{2}\exp(U(t)).$$

λ_3 is expressed via initial values depending on several cases. For $\lambda_1(0) = \pm\lambda_5(0)$, $\lambda_2(0) = \mp\lambda_4(0)$, we have

$$\lambda_3(t) = \frac{(b + \lambda_3(0))\,\mathrm{e}^{\pm bt} - (b - \lambda_3(0))\,\mathrm{e}^{\mp bt}}{\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}},\ b = \sqrt{\sum_{i=3}^{5}\lambda_i^2(0)}$$

$$U(t) = -\ln\left(\frac{1}{2}\left[\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}\right]\right),$$

Otherwise, we have $\lambda_3(t) = -\frac{P}{2}\,\mathrm{sn}(\psi_t, k)$,

$$U(t) = \frac{1}{2}\ln\left(\frac{A}{B} + \frac{P^2}{2B}\left(\mathrm{cn}^2(\psi_t, k) + \frac{1}{k}\mathrm{cn}(\psi_t, k)\,\mathrm{dn}(\psi_t, k)\right)\right),$$

$$A = (\lambda_1(0) + \lambda_5(0))^2 + (\lambda_2(0) - \lambda_4(0))^2,\ B = (\lambda_1(0) - \lambda_5(0))^2 + (\lambda_2(0) + \lambda_4(0))^2$$

$$P = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} - \sqrt{B}\right)^2},\ Q = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} + \sqrt{B}\right)^2}$$

$$\psi_t = F(p_0, k) + \frac{Q}{2}t,\ k = \frac{P}{Q},\ p_0 = \begin{cases} -\arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B \ge A, \\ \pi + \arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B < A. \end{cases}$$

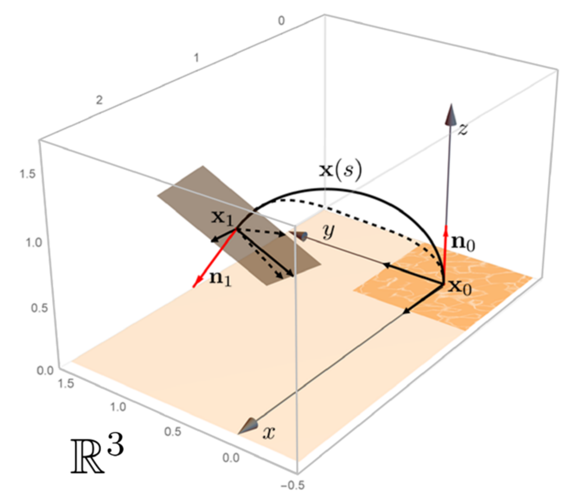
SHORTEST PATH ON $\mathbb{R}^3 \times S^2$

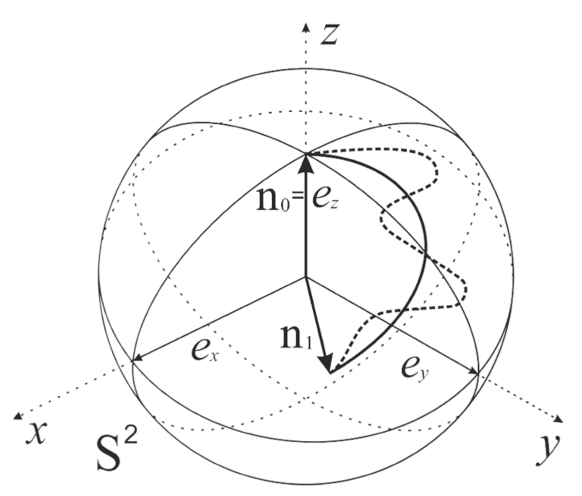
Original problem that comes from processing of 3D images:

$\mathbf{P_{curve}}$: Given: $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$, $\mathbf{n}_1, \mathbf{n}_2 \in S^2$

To find: $\mathbf{x} \in C^\infty([0, L], \mathbb{R}^3)$, s.t. $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(L) = \mathbf{x}_1 \in \mathbb{R}^3$, $\mathbf{x}'(0) = \mathbf{n}_0$, $\mathbf{x}'(L) = \mathbf{n}_1 \in S^2$

and $E(\mathbf{x}) := \int_0^L \sqrt{1 + \kappa^2(s)}\,\mathrm{d}\,s \to \min$, with $\kappa(s) = \|\mathbf{x}''(s)\|$.





RELATION BETWEEN $\mathbf{P_{curve}}$ AND $\mathbf{P_{mec}}$

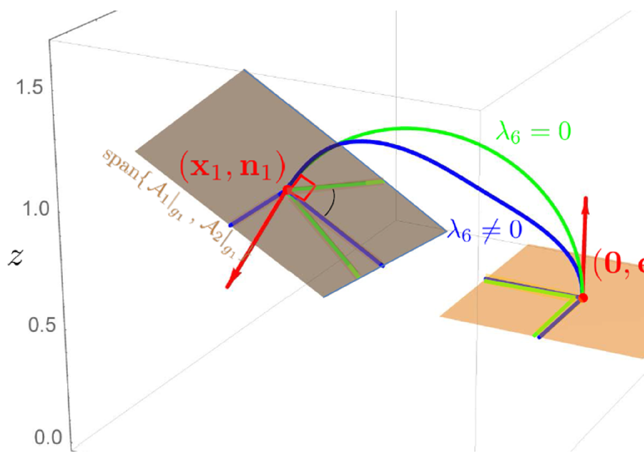
We define the quotient $\mathrm{SE}_3/\{0\} \times \mathrm{SO}_2$ and the distance on it:

$d_{\mathbb{R}^3 \times S^2}((\mathbf{0}, \mathbf{e}_z), (\mathbf{y}_1, \mathbf{n}_1)) = \min\limits_{\alpha \in [0, 2\pi)} d(e, (\mathbf{y}_1, R_{\mathbf{n}_1})h_\alpha)$

$\mathbf{P_{quot}}$ Let $(\mathbf{y}_1, \mathbf{n}_1) \in \mathbb{R}^3 \times S^2$. Find $(\mathbf{x}(t), \mathbf{n}(t)) = \gamma(t) \odot (\mathbf{0}, \mathbf{e}_z)$, with γ is minimizer of $\mathbf{P_{mec}}$ under boundary conditions $\gamma(0) = (\mathbf{0}, I)$ and $\gamma(T) = (\mathbf{y}_1, R_{\mathbf{n}_1}R_{\mathbf{e}_z,\alpha})$, where both $T \ge 0$ and $\alpha \in [0, 2\pi)$ are free variables in the optimization process.

Theorem. If $g_1 = (\mathbf{x}_1, R_1) \in \mathrm{SE}_3$ is chosen s.t. corresponding minimizer γ of $\mathbf{P_{mec}}$ satisfies $u^3(t) > 0$, $t \in (0, T)$, then γ can be parameterized by spatial arclength s and its spatial projection does not exhibit a cusp. If moreover g_1 is chosen s.t. γ has $\lambda_6 = 0$ then this yields required minimum choice of α, and $\gamma(t)$ provides the minimizer $(\mathbf{x}(t), \mathbf{n}(t))$ of $\mathbf{P_{quot}}$.

Under these two requirements the spatial projection $\mathbf{x}(\cdot)$ of $\gamma(\cdot) = (\mathbf{x}(\cdot), R(\cdot))$ coincides with a minimizer of $\mathbf{P_{curve}}$.



GEOMETRIC PROPERTIES OF GEODESICS

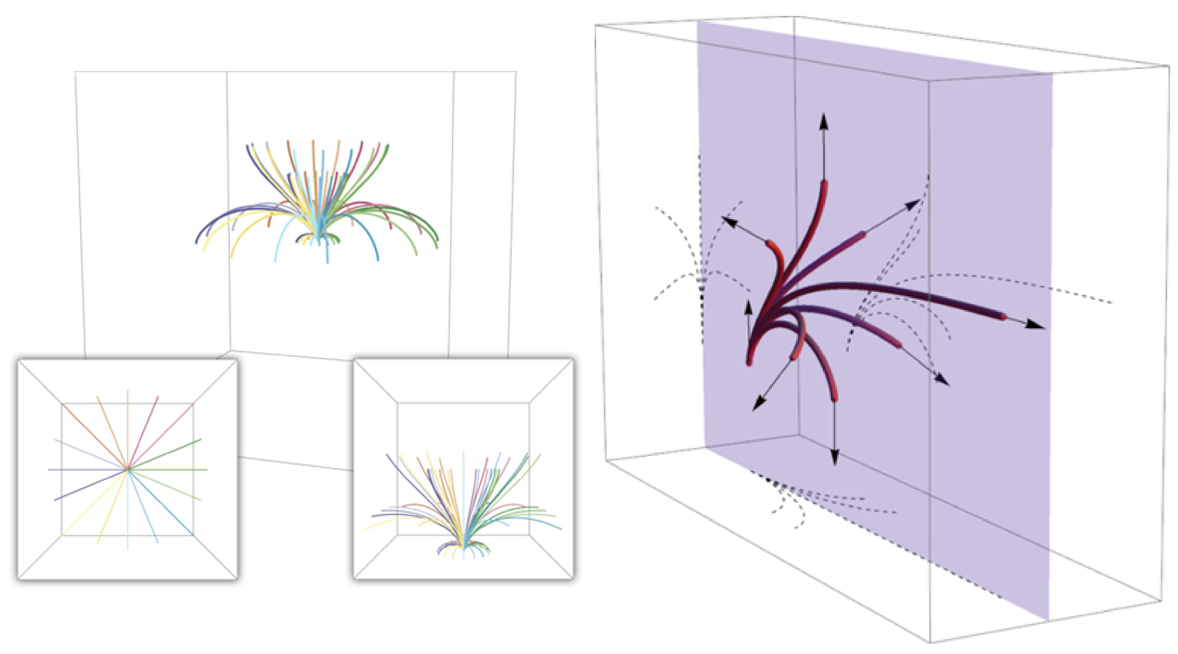
Corollary. Curvature and torsion in $\mathbf{P_{curve}}$: $\kappa = \frac{\sqrt{1 - \lambda_3^2}}{\lambda_3}$, $\tau = \frac{W}{\lambda_4^2 + \lambda_5^2}$.

$\forall s \in [0, s_{max}]$ torsion is bounded $|W| \le |\tau(s)| \le \frac{2|W|}{\sqrt{(1 - \mathcal{C}^2)^2 + 4W^2} + 1 - \mathcal{C}^2}$

Corollary. The cuspless spatial projections of SR geodesics of $\mathbf{P_{mec}}$ (i.e. geodesics of $\mathbf{P_{curve}}$) are planar iff $W = 0$.

Corollary. Given admissible coplanar end conditions for $\mathbf{P_{curve}}$, the unique cuspless geodesic connecting them is planar.

Corollary. All cuspless SR geodesics in SE_3 with $\lambda_6 = 0$ and $\sum_{i=1}^3 \lambda_i^2(0) \neq 0$ stay in the upper half space $z \ge 0$.



RESULTS

For the SR problem in SE_3 ($\mathbf{P_{mec}}$) we derive the Hamiltonian system of PMP, prove Liouville integrability and find explicit expression for extremal controls in the case $\lambda_6 = 0$.

We establish a relation between problems $\mathbf{P_{mec}}$ and $\mathbf{P_{curve}}$, appeared in imaging applications. We provide explicit expressions for solution curves of $\mathbf{P_{curve}}$, evaluate the first cusp time and study admissible boundary conditions reachable by cuspless geodesics. We also study geometrical properties of the solution curves: bounds on torsion, planarity conditions, symmetries. Numerical investigation shows absence of conjugate points on cuspless geodesics.

BIBLIOGRAPHY

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