Sub-Riemannian Geodesics on the Group of Motions of Euclidean Space

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Based on joint works with

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Зимняя геометрическая школа

Переславль-Залесский, 25.01.2018

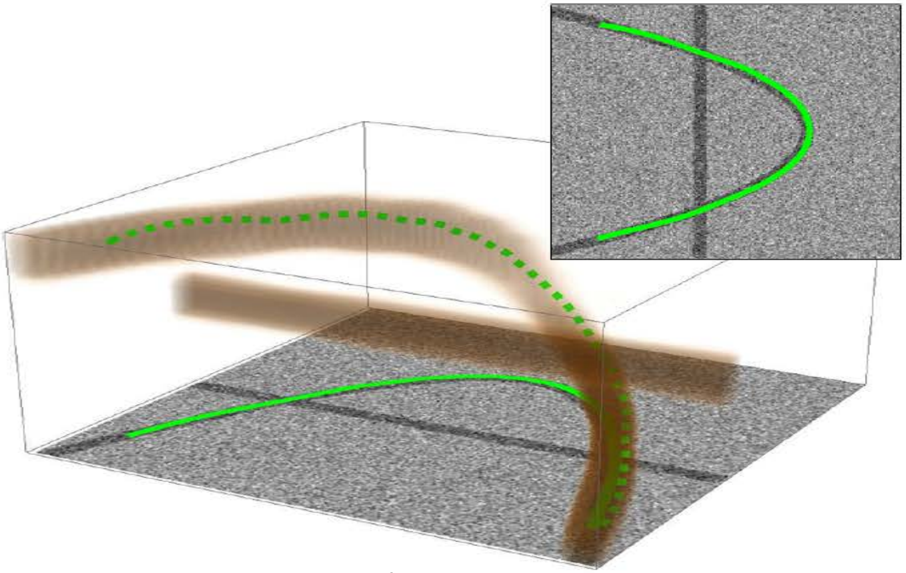
Structure of the talk

- Application of Sub-Riemannian geodesics in image processing
- **Sub-Riemannian problem on SE(3)**
- Brief tour in Sub-Riemannian geometry

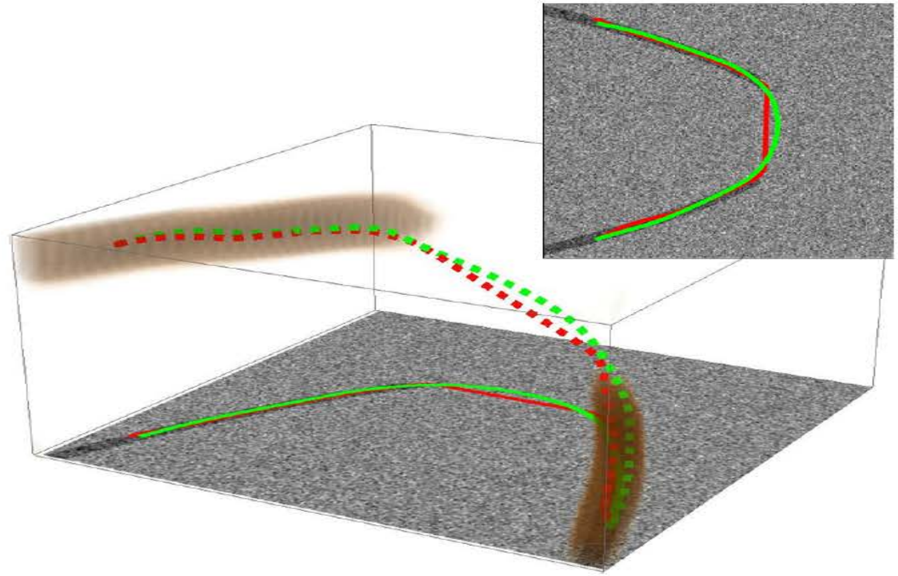
Application of Sub-Riemannian geodesics:

Detection of salient curves in medical images

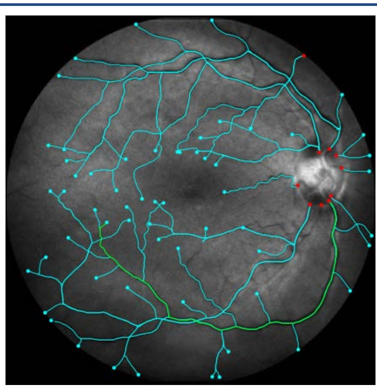
SR geodesics on Lie Groups in Image Analysis



Crossing structures are disentangled

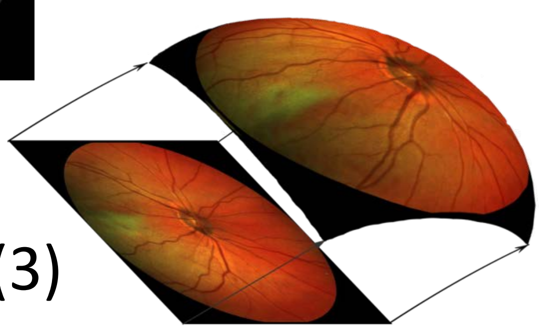


Restoration of corrupted contours based on model of human vision

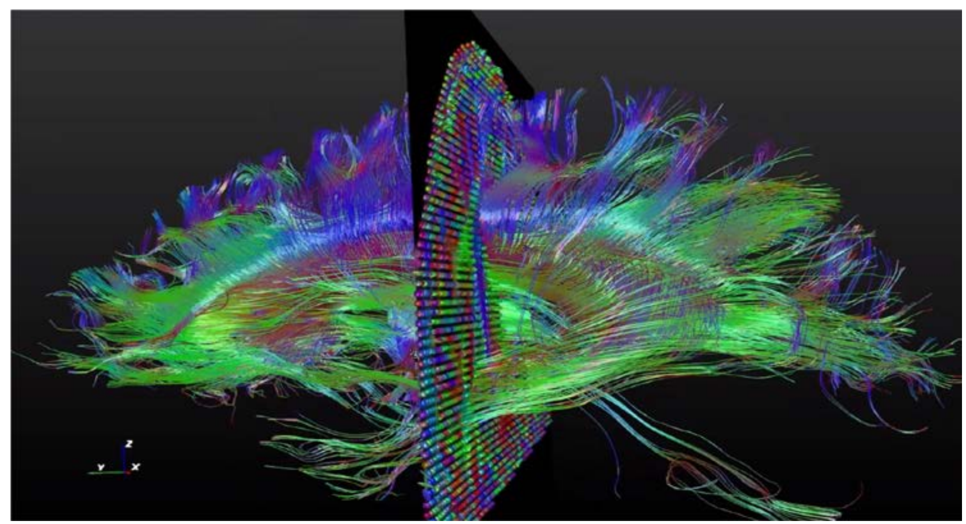


SE(2)

Applications in medical image analysis



SO(3)



SE(3)

Analysis of Images of the Retina

Diabetic retinopathy --- one of the main causes of blindness.
Epidemic forms: 10% people in China suffer from DR.
Patients are found early --> treatment is well possible.
Early warning --- leakage and malformation of blood vessels.
The retina --- excellent view on the microvasculature of the brain.

Healthy retina

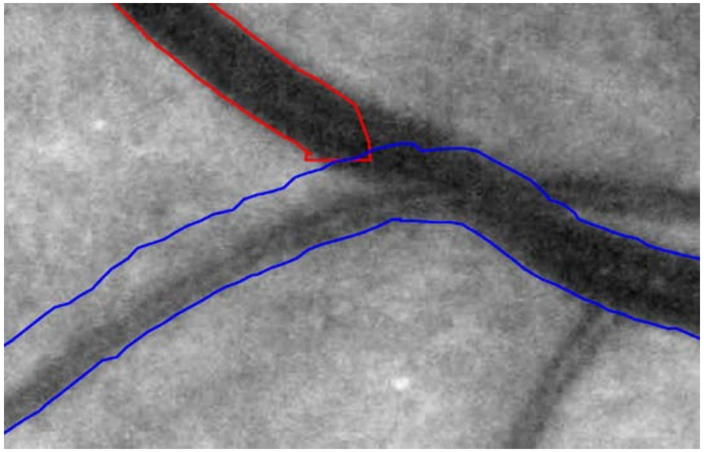
Diabetes Retinopathy with tortuous vessels

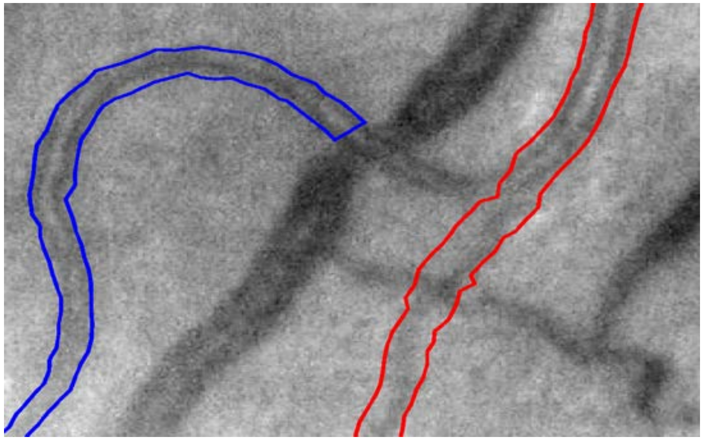
Detection of Vascular Tree in Images of the Retina

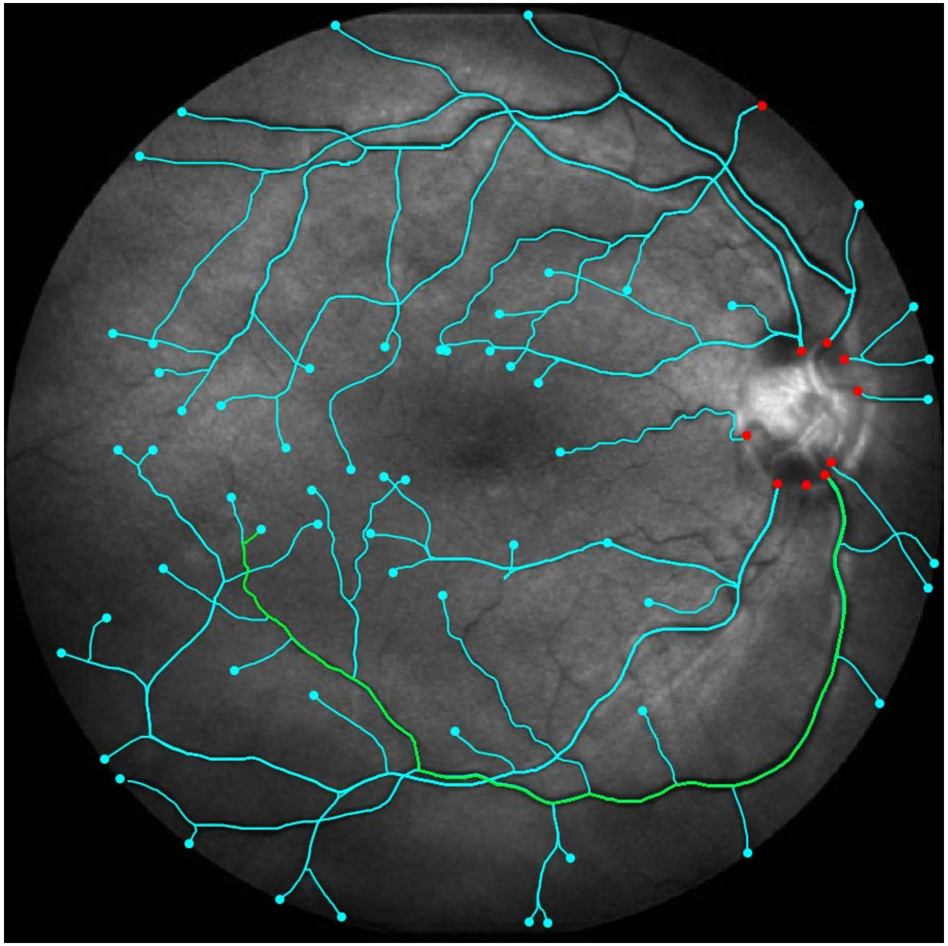
Application: Early diagnosis of diabetes

Problem: Low contrast & crossings & bifurcations

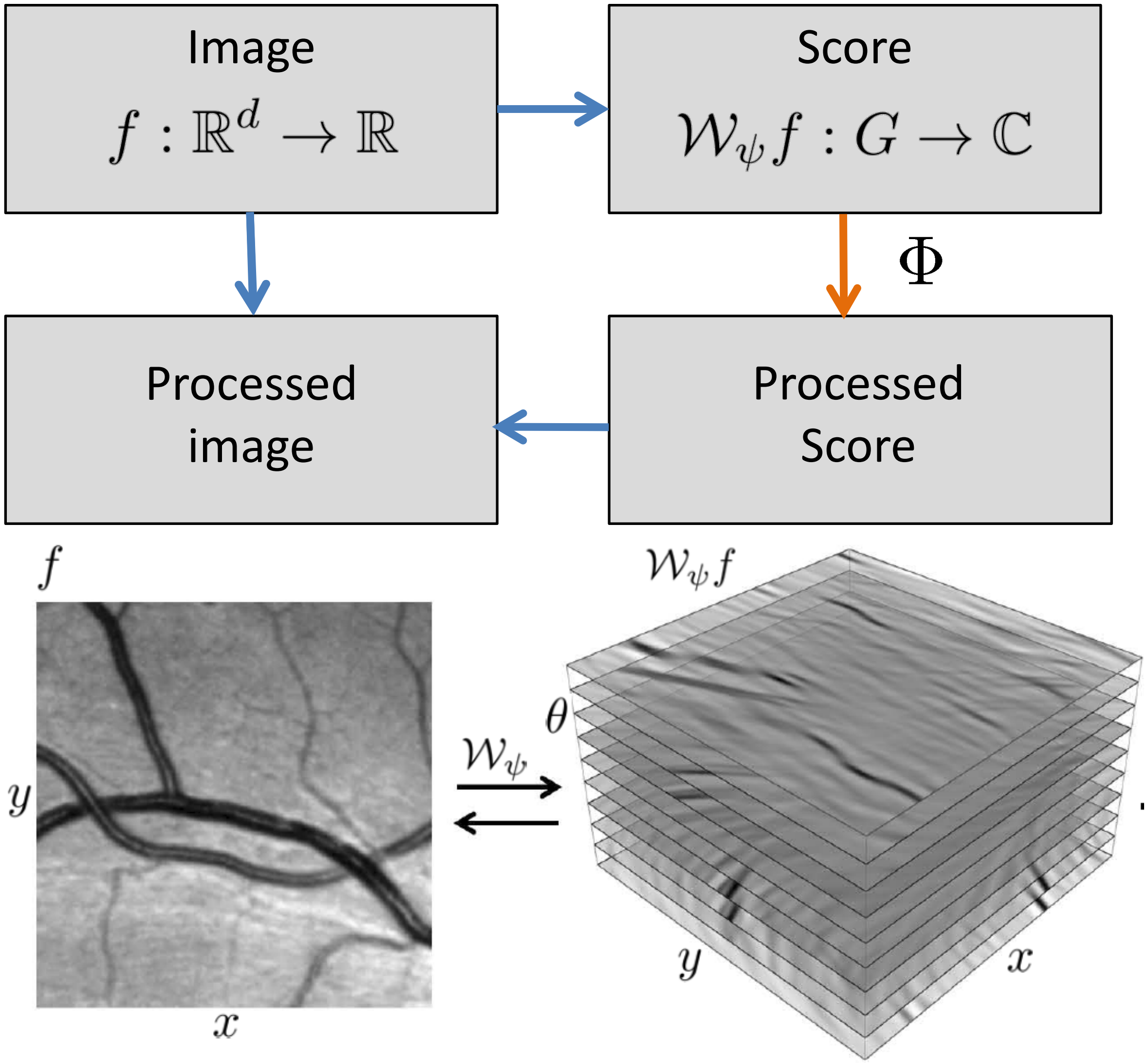
Aim: Reliable tracking of *all* blood vessels in retina





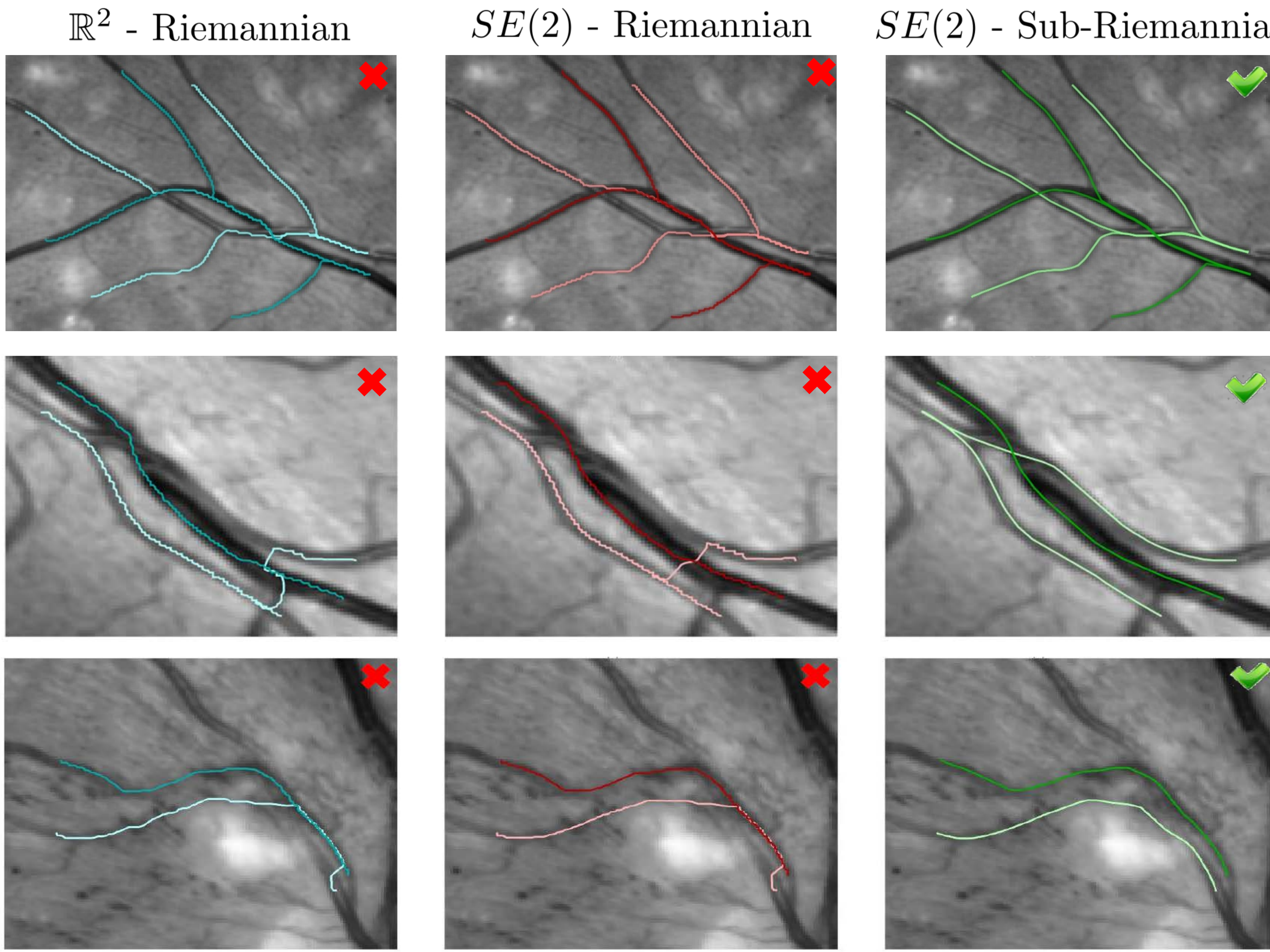


Lie Group Analysis via Invertible Orientation Scores



R. Duits: generic mathematical model for contextual image analysis via scores on Lie groups with many applications.

Comparison with Classical Methods

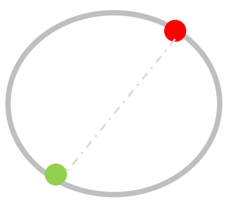


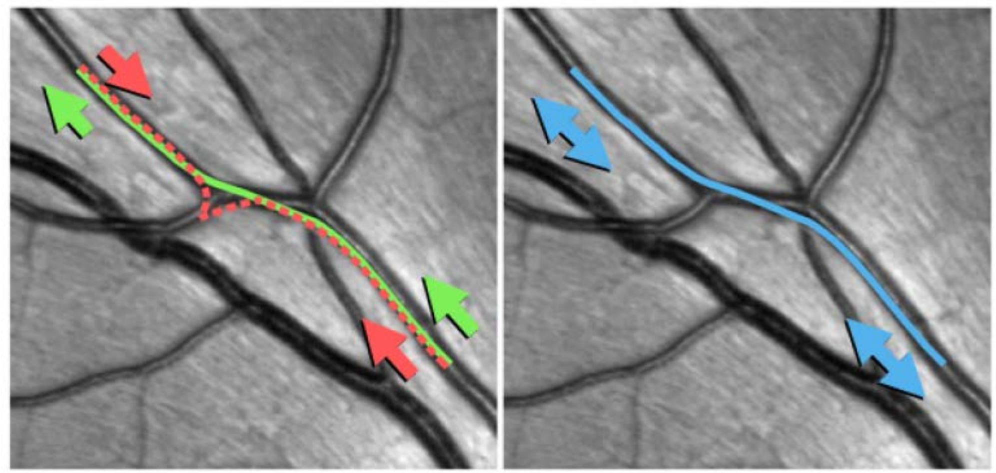
SR-geodesics in Projective Line Bundle

Importance stressed by **Petitot 1999 & Boscain 2010** for cortical modeling
We use it for vessel tracking & analysis:

$$PT(\mathbb{R}^2) \equiv \mathbb{R}^2 \times P^1$$

$$P^1 = S^1/_{\sim}$$

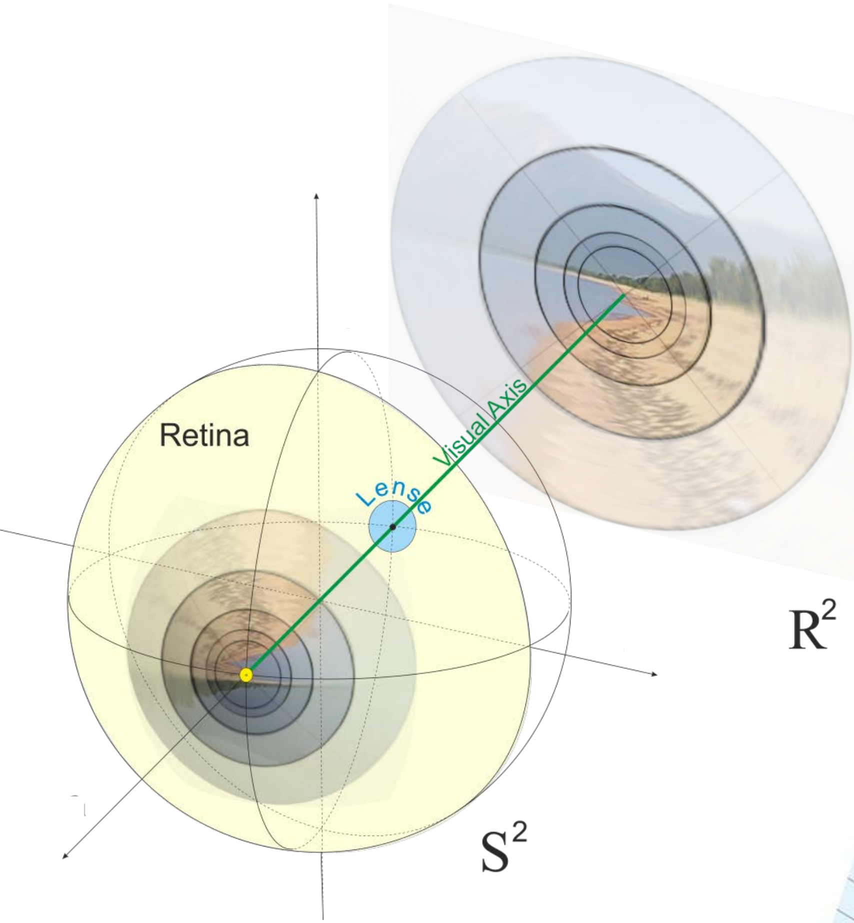




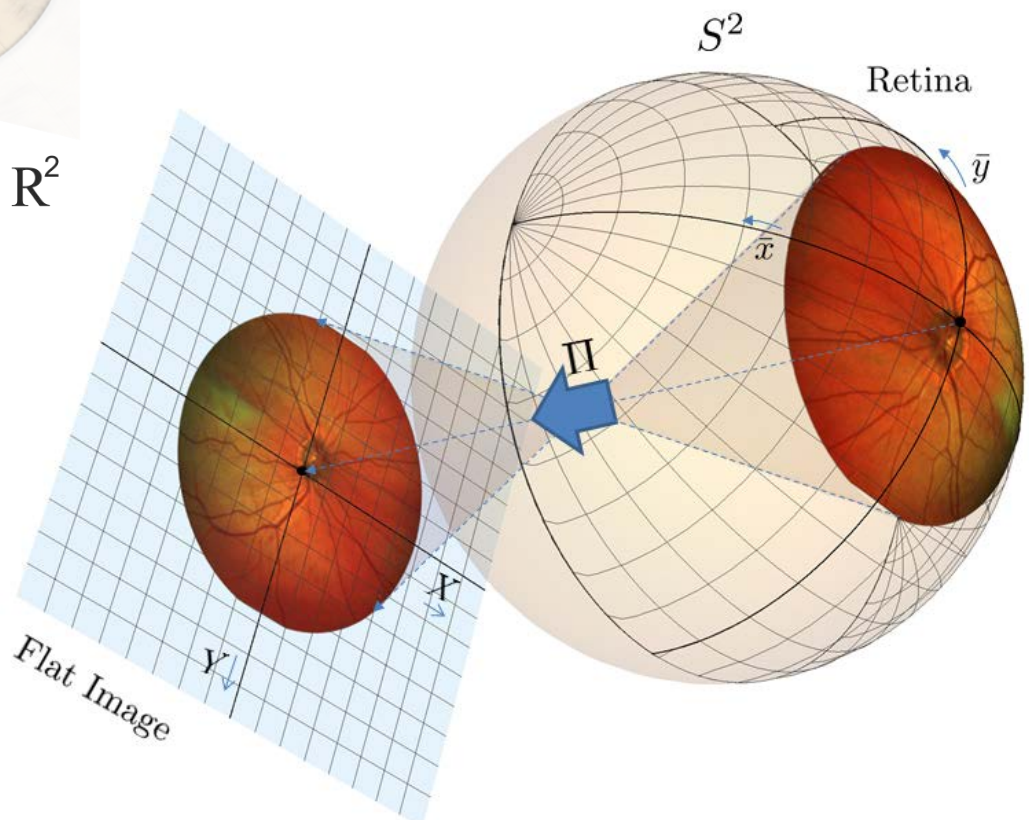
$$\overline{d}(\mathbf{q}_0, \mathbf{q}_1) = \min\{d(\mathbf{q}_0, \mathbf{q}_1), d(\mathbf{q}_0 \odot (0, 0, \pi), \mathbf{q}_1)\}$$

1) One can account for the $\mathrm{PT}(\mathbb{R}^2)$ structure in the building of the distance function before tracking takes place
2) It affects cut-locus, the first Maxwell set and reduces cusps!

Sub-Riemannian Geodesics on SO(3)

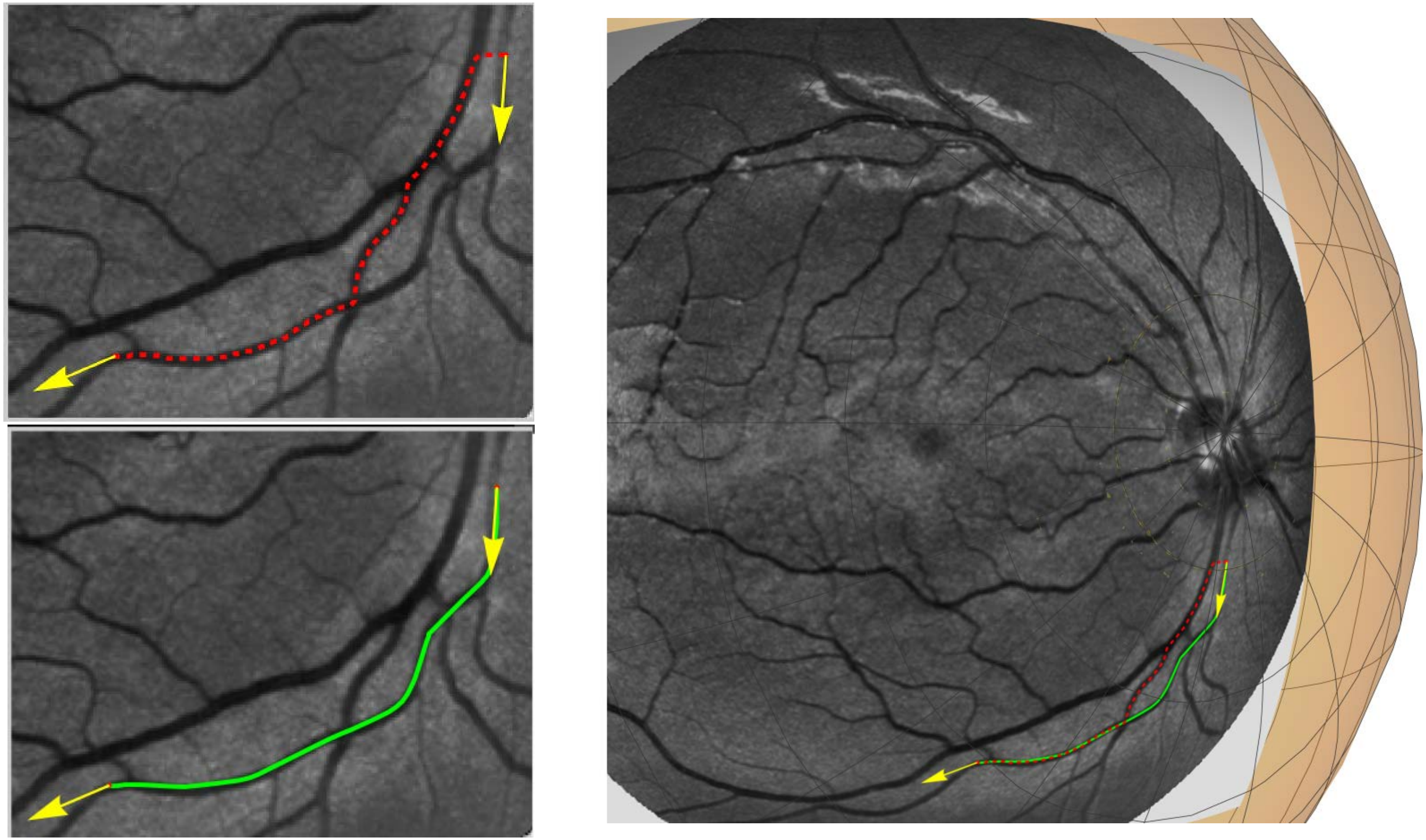


Aim: data-driven SR geodesics on **SO(3)** for detection and analysis of vessel tree in spherical images of retina, to reduce distortion.



Spherical extension of cortical based model of perceptual completion on retinal sphere

Vessel Analysis via Riemannian and SR geometry on SO(3)



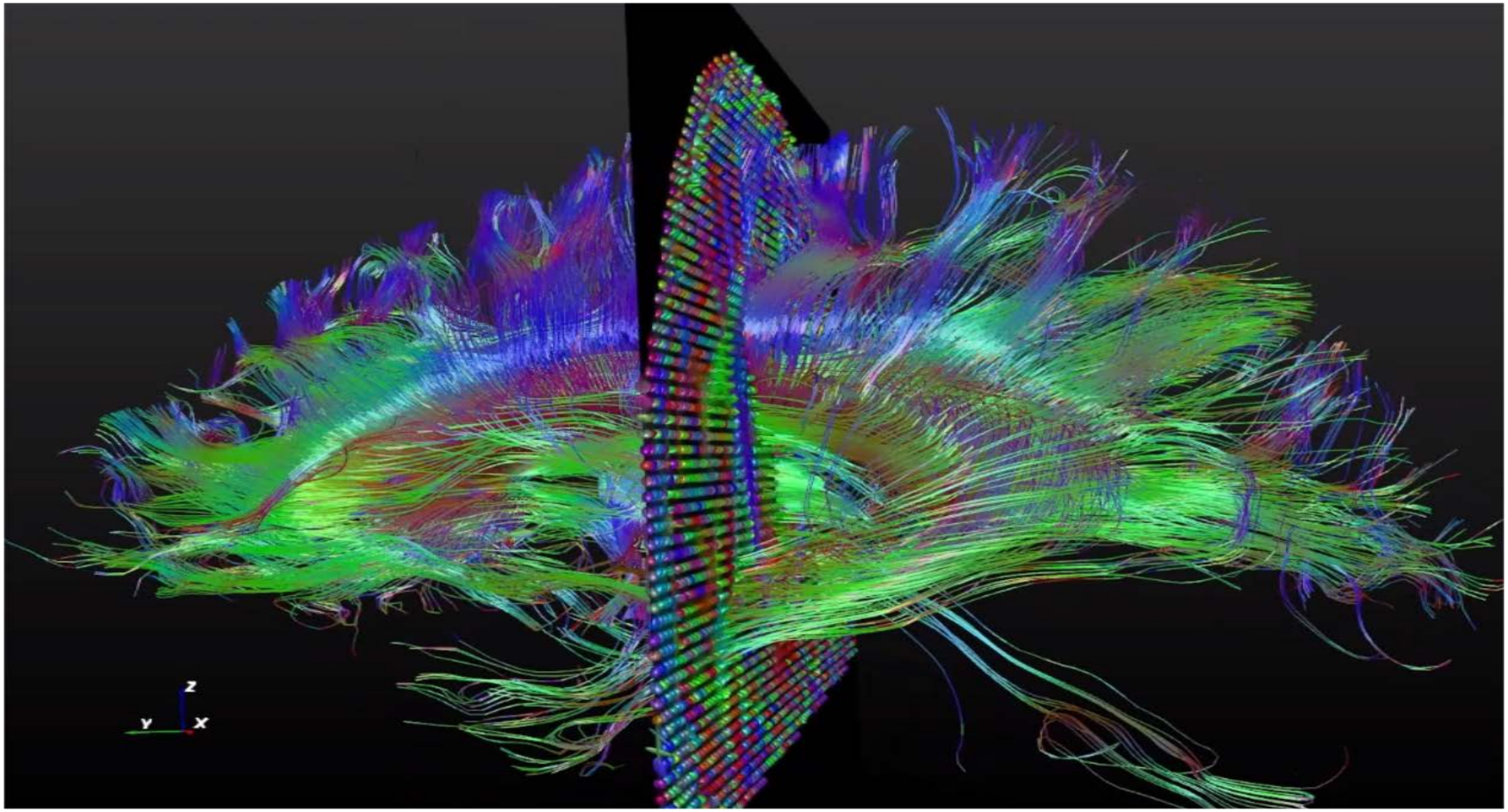
- - - Riemannian geodesic in SO(3)

—— Sub-Riemannian geodesic in SO(3)

Sub-Riemannian problem on SE(3)

Sub-Riemannian Geodesics in **SE(3)**

Data-driven sub-Riemannian geodesics on **SE(3)** are used for detection and analysis of neuron fibers in magnetic resonance images of a human brain.

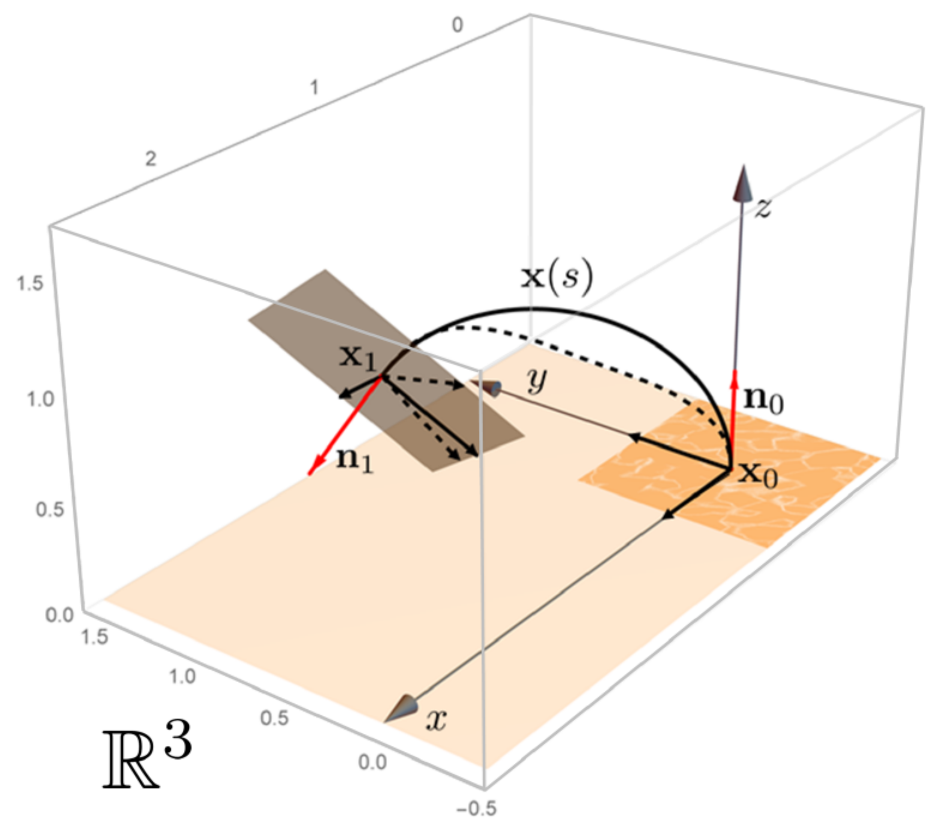


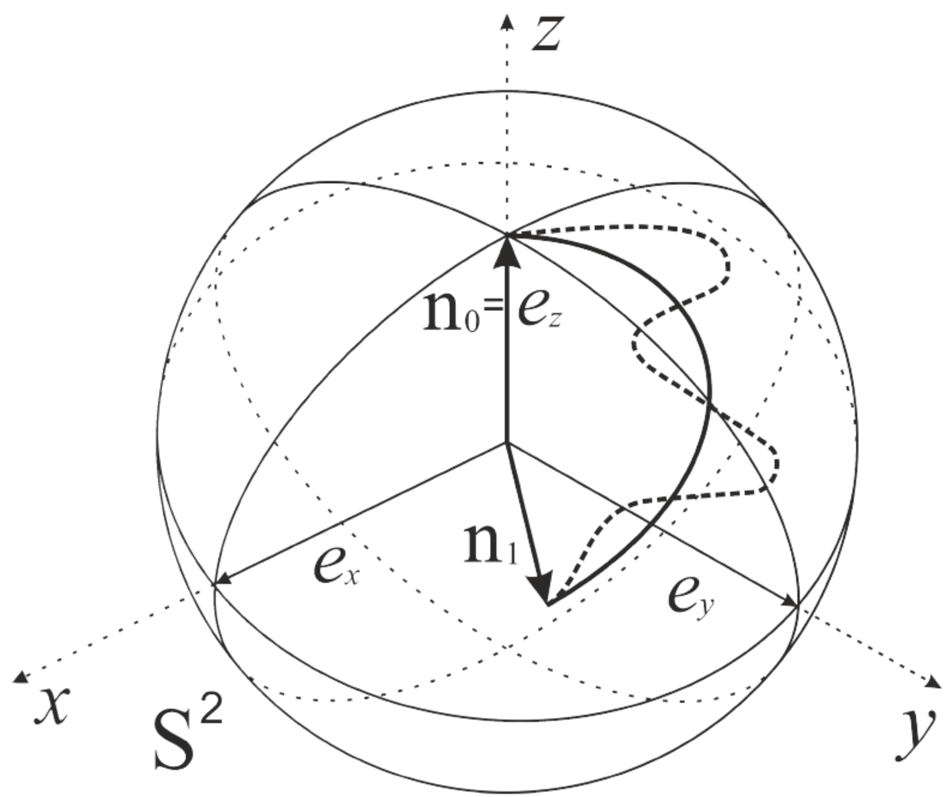
Problem **Pcurve(R^3)**: Shortest Path on $R^3 \times S^2$

Given $\xi > 0, \quad \mathbf{x}_i \in \mathbb{R}^3, \quad \mathbf{n}_i \in S^2, \; i \in \{0,1\}.$

Find a smooth curve $\mathbf{x} \in C^\infty([0,L],\mathbb{R}^3)$ s.t. $\mathbf{x}(0) = \mathbf{x}_0, \; \mathbf{x}(L) = \mathbf{x}_1 \in \mathbb{R}^3,$
$\mathbf{x}'(0) = \mathbf{n}_0, \; \mathbf{x}'(L) = \mathbf{n}_1 \in \mathrm{S}^2,$

and $E(\mathbf{x}) := \int_0^L \sqrt{\xi^2 + \kappa^2(s)}\,\mathrm{d}s \to \min,$ where $\kappa(s) = \|\mathbf{x}''(s)\|.$

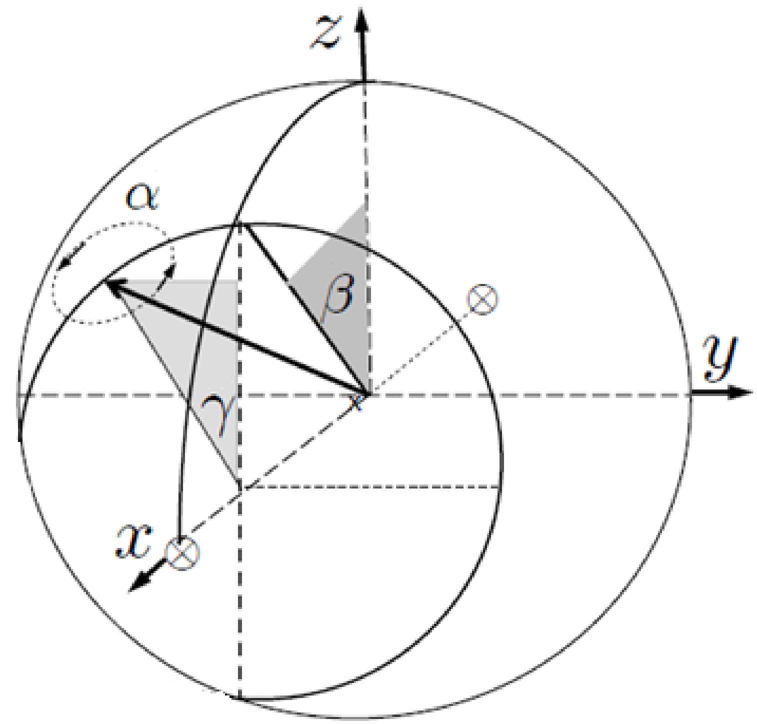




Lie group SE(3)

The group of Euclidean motions of 3-dimensional space

$$g = (\mathbf{x}, R) \in \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$$



Group operations

$$g_1 g_2 = (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)$$
$$= (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1 R_2),$$
$$g^{-1} = (-R^T\mathbf{x}, R^T).$$

We use the parameterization of SE(3)

$$\mathbf{x} = (x, y, z) \in \mathbb{R}^3,$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{where } \alpha \in (-\pi, \pi], \ \ \beta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \ \ \gamma \in (-\pi, \pi]$$

Left-invariant Vector Fields on SE(3)

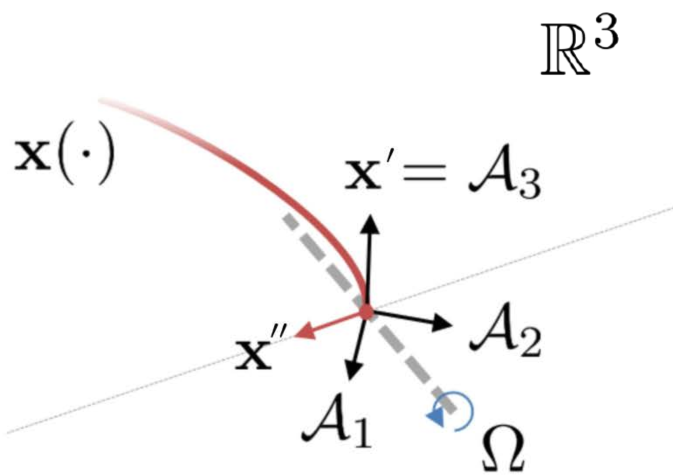
$$(x, R) \in \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$$

$$\mathrm{se}(3) = T_e\,\mathrm{SE}(3) = \mathrm{span}\{A_1, A_2, A_3, A_4, A_5, A_6\}$$

$$\mathcal{A}_i|_g = (L_g)_* A_i, \quad i \in \{1, \dots, 6\}, \quad L_g h = gh$$

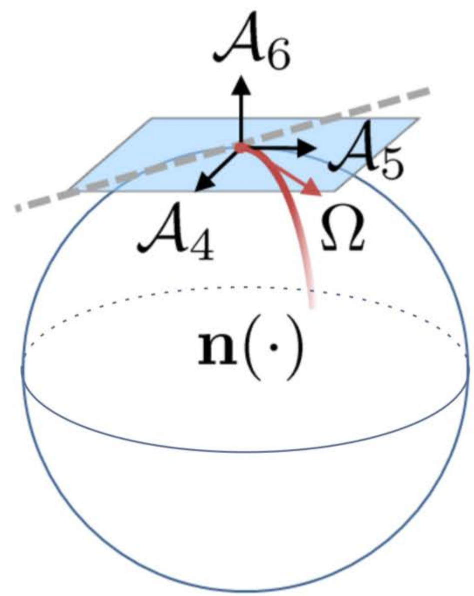
$$\text{Co-frame } \{\omega^1, \dots, \omega^6\}\text{: } \langle \omega^i, \mathcal{A}_j \rangle = \delta^i_j, \quad i, j \in \{1, \dots, 6\}.$$

$$\mathbb{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$$



$\mathbf{x}'$ spatial velocity:
$\mathbf{x}' = \langle \omega^3|_\gamma, \gamma' \rangle \, \mathcal{A}_3|_\gamma = \mathcal{A}_3|_\gamma$

$\mathbf{x}''$ spatial curvature:
$\mathbf{x}'' = \langle \omega^5|_\gamma, \gamma' \rangle \, \mathcal{A}_1|_\gamma - \langle \omega^4|_\gamma, \gamma' \rangle \, \mathcal{A}_2|_\gamma$



Ω angular velocity:
$\Omega = \langle \omega^4, \gamma' \rangle \mathcal{A}_4 + \langle \omega^5, \gamma' \rangle \mathcal{A}_5$

PMEC(SE(3)): Sub-Riemannian problem in SE(3)

SR structure (SR manifold):

$$(M, \Delta, \mathcal{G}_\xi) \qquad M = \mathrm{SE}(3), \qquad \Delta = \mathrm{span}\{\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\},$$

$$\mathcal{G}_\xi = \xi^2 \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5$$

SR distance (Carnot-Carathéodory distance):

$$d(g,h) = \min_{\substack{\gamma \in \mathrm{Lip}([0,T], \mathrm{SE}(3)), T \geq 0, \\ \dot{\gamma} \in \Delta, \ \gamma(0) = g, \gamma(T) = h}} \int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t.$$

$\mathbf{P_{MEC}}$(SE(3)): to find a Lipschitzian curve $\gamma : [0,T] \to \mathrm{SE}(3)$, s.t.

$$\gamma(0) = e := (\mathbf{0}, I), \quad \gamma(T) = (\mathbf{x}_1, R_1) \in \mathrm{SE}(3),$$

$$\dot{\gamma}(t) \in \Delta \text{ for a.e. } t \in [0,T],$$

and $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t \to \min$ (with free T).

Optimal Control Formulation of SR-Problem in SE(3)

Control system

$$\dot{\gamma}(t) = u^3(t)\mathcal{A}_3|_{\gamma(t)} + u^4(t)\mathcal{A}_4|_{\gamma(t)} + u^5(t)\mathcal{A}_5|_{\gamma(t)}$$

Boundary conditions

$$\gamma(0) = e, \quad \gamma(T) = g_1 \in \mathrm{SE}(3)$$

Minimizing functional (here action functional)

$$\int_0^T \frac{1}{2}\left(\xi^2(u^3(t))^2 + (u^4(t))^2 + (u^5(t))^2\right)\, dt \to \min .$$

In coordinates

$$\begin{cases} \dot{x} = u^3 \sin\beta, \\ \dot{y} = -u^3 \cos\beta \sin\gamma, \\ \dot{z} = u^3 \cos\beta \cos\gamma, \\ \dot{\gamma} = \sec\beta(u^4\cos\alpha - u^5\sin\alpha), \\ \dot{\beta} = u^4 \sin\alpha + u^5\cos\alpha, \\ \dot{\alpha} = -(u^4\cos\alpha - u^5\sin\alpha)\tan\beta, \end{cases}$$

$$(x(0), y(0), z(0), \gamma(0), \beta(0), \alpha(0)) = \mathbf{0}$$

$$(x(T), y(T), z(T), \gamma(T), \beta(T), \alpha(T)) = (x^1, y^1, z^1, \gamma^1, \beta^1, \alpha^1)$$

- Complete controllability (Chow-Rashevski)
- Existance of minimizers (Filippov)
- No abnormal extremals: $\dim\,[\Delta, \Delta] = \dim\,(\mathrm{SE}(3))$
- The minimizers are analytic

Pontryagin Maximum Principle

- Left Invariant Hamiltonians $\lambda_i = \langle p, \mathcal{A}_i \rangle$, $i = 1, \ldots, 6$, where $p = p_1 \mathrm{d}x|_g + p_2 \mathrm{d}y|_g + p_3 \mathrm{d}z|_g + p_4 \mathrm{d}\gamma|_g + p_5 \mathrm{d}\beta|_g + p_6 \mathrm{d}\alpha|_g$
- Control dependent Hamiltonian $H_u = u^3\lambda_3 + u^4\lambda_4 + u^5\lambda_5 - \frac{1}{2}\left(\xi^2(u^3)^2 + (u^4)^2 + (u^5)^2\right)$
- Maximality Condition $u^3 = \frac{\lambda_3}{\xi^2}, \quad u^4 = \lambda_4, \quad u^5 = \lambda_5.$
- The (maximized) Hamiltonian $H = \frac{1}{2}\left(\xi^{-2}\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- The Hamiltonian system of PMP (via Poisson brackets $\dot{\lambda}_i = \{H, \lambda_i\}$)

$$\begin{cases} \dot{\lambda}_1 = -\lambda_3\lambda_5, \\ \dot{\lambda}_2 = \lambda_3\lambda_4, \\ \dot{\lambda}_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \\ \dot{\lambda}_4 = \frac{\lambda_2\lambda_3}{\xi^2} - \lambda_5\lambda_6, \\ \dot{\lambda}_5 = \lambda_4\lambda_6 - \frac{\lambda_1\lambda_3}{\xi^2}, \\ \dot{\lambda}_6 = 0, \end{cases} \qquad \begin{cases} \dot{x} = \frac{\lambda_3}{\xi^2}\sin\beta, \\ \dot{y} = -\frac{\lambda_3}{\xi^2}\cos\beta\sin\gamma, \\ \dot{z} = \frac{\lambda_3}{\xi^2}\cos\beta\cos\gamma, \\ \dot{\gamma} = \sec\beta(\lambda_4\cos\alpha - \lambda_5\sin\alpha), \\ \dot{\beta} = \lambda_4\sin\alpha + \lambda_5\cos\alpha, \\ \dot{\alpha} = -(\lambda_4\cos\alpha - \lambda_5\sin\alpha)\tan\beta, \end{cases}$$

— vertical part, — horizontal part.

Liouville Integrability of the Hamiltonian System

First Integrals:

- the Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- Left-invariant basis Hamiltonian λ_6
- Casimir functions $W = -\lambda_1\lambda_4 - \lambda_2\lambda_5 - \lambda_3\lambda_6, \quad \mathfrak{c}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- Right-invariant Hamiltonians

$$
\begin{aligned}
\rho_1 &= -\lambda_1 \cos\alpha \cos\beta + \lambda_2 \cos\beta \sin\alpha - \lambda_3 \sin\beta, \\
\rho_2 &= -\cos\gamma(\lambda_2 \cos\alpha + \lambda_1 \sin\alpha) + (\lambda_3 \cos\beta + (-\lambda_1 \cos\alpha + \lambda_2 \sin\alpha)\sin\beta)\sin\gamma, \\
\rho_3 &= -\lambda_3 \cos\beta \cos\gamma + \cos\gamma(\lambda_1 \cos\alpha - \lambda_2 \sin\alpha)\sin\beta - (\lambda_2 \cos\alpha + \lambda_1 \sin\alpha)\sin\gamma, \\
&\rho_4, \quad \rho_5, \quad \rho_6.
\end{aligned}
$$

Complete system of first Integrals: $\mathrm{I} = (H,\ \lambda_6,\ W,\ \rho_1,\ \rho_2,\ \rho_3)$

$$
\{I_i, I_j\} = 0 \qquad \frac{\partial(\rho_1,\rho_2,\rho_3,W,H,\lambda_6)}{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)}(q,\lambda) = \quad -\lambda_2\lambda_4 + \lambda_1\lambda_5 \not\equiv 0
$$

Theorem *The Hamiltonian system of PMP for sub-Riemanian problem on* SE(3) *is Liouville integrable.*

Integration of the Vertical Part

Theorem *Suppose $\lambda_6(0) = 0$; then the vertical part is given by*

$$\dot\lambda_1 = -\lambda_3\lambda_5, \quad \dot\lambda_2 = \lambda_3\lambda_4, \quad \dot\lambda_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \quad \dot\lambda_4 = \lambda_2\lambda_3, \quad \dot\lambda_5 = -\lambda_1\lambda_3.$$

The momenta λ_4, λ_5 are expressed via $U(t) = \int_0^t \lambda_3(\tau)\,\mathrm{d}\,\tau$ and the initial values

$$\lambda_4(t) = \frac{\lambda_2(0) + \lambda_4(0)}{2} \exp\left(U(t)\right) - \frac{\lambda_2(0) - \lambda_4(0)}{2} \exp\left(-U(t)\right),$$
$$\lambda_5(t) = \frac{\lambda_1(0) + \lambda_5(0)}{2} \exp\left(-U(t)\right) - \frac{\lambda_1(0) - \lambda_5(0)}{2} \exp\left(U(t)\right).$$

The momentum λ_3 is expressed via the initial values depending on several cases.
For the cases $\lambda_1(0) = \pm\lambda_5(0)$, $\lambda_2(0) = \mp\lambda_4(0)$, we have

$$\lambda_3(t) = \frac{(b + \lambda_3(0))\,\mathrm{e}^{\pm bt} - (b - \lambda_3(0))\,\mathrm{e}^{\mp bt}}{\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}}, \qquad U(t) = -\ln\left(\frac{1}{2}\left[\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}\right]\right),$$

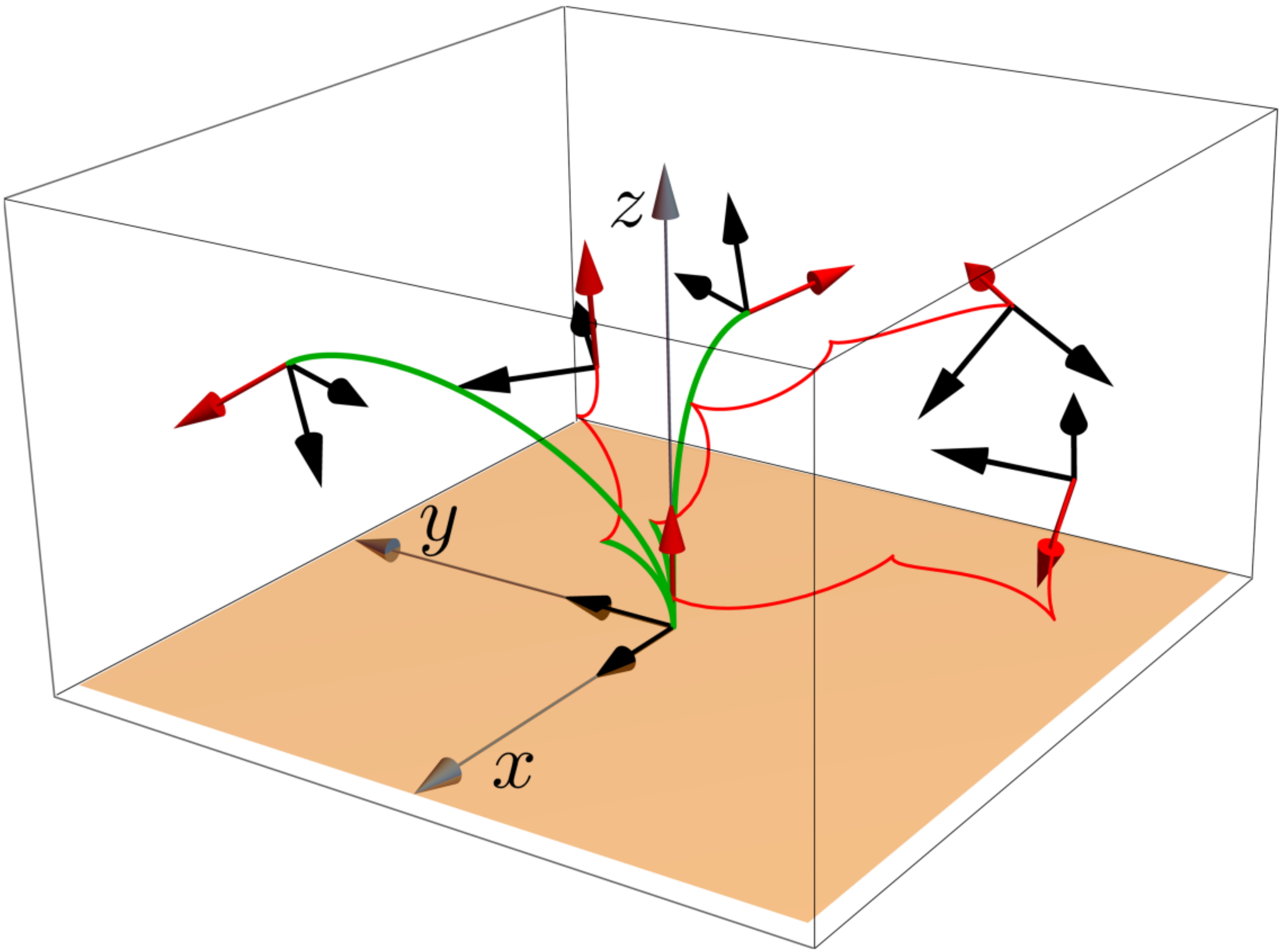
where $b = \sqrt{\lambda_3^2(0) + \lambda_4^2(0) + \lambda_5^2(0)}$. Otherwise, we have

$$\lambda_3(t) = -\frac{P}{2}\,\mathrm{sn}\left(\psi_t, k\right), \qquad U(t) = \frac{1}{2}\ln\left(\frac{A}{B} + \frac{P^2}{2B}\left(\mathrm{cn}^2\left(\psi_t, k\right) + \frac{1}{k}\,\mathrm{cn}\left(\psi_t, k\right)\mathrm{dn}\left(\psi_t, k\right)\right)\right),$$

where $A = (\lambda_1(0) + \lambda_5(0))^2 + (\lambda_2(0) - \lambda_4(0))^2$, $B = (\lambda_1(0) - \lambda_5(0))^2 + (\lambda_2(0) + \lambda_4(0))^2$,
$P = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} - \sqrt{B}\right)^2}$, $Q = \sqrt{4\lambda_3^2(0) + \left(\sqrt{A} + \sqrt{B}\right)^2}$,

$$\psi_t = F\left(p_0, k\right) + \frac{Q}{2}t, \quad k = \frac{P}{Q}, \qquad p_0 = \begin{cases} -\arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B \geq A, \\ \pi + \arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B < A. \end{cases}$$

Spatial projection of SR geodesics in SE(3) can have singularities (the cusp points)



SR problem Pmec($\mathbb{R}^3 \times S^2$) in Quotient SE(3)/({0}xSO(2))

Well-defined distance on the quotient $\mathbb{R}^3 \rtimes S^2$

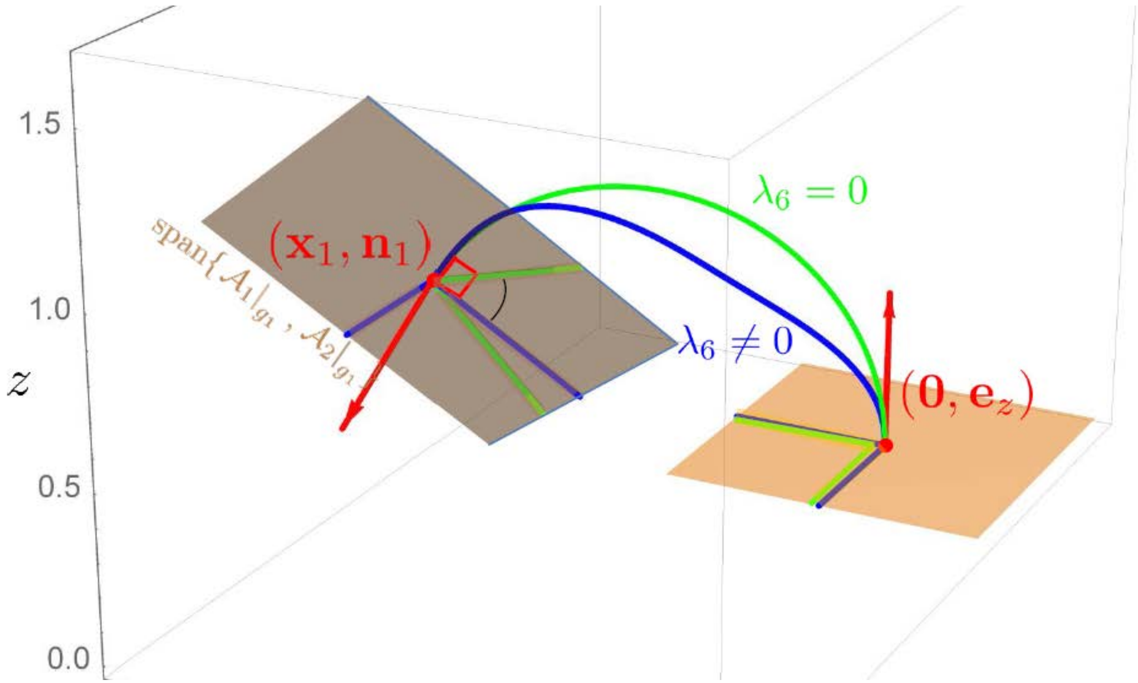
$$\begin{aligned} d_{\mathbb{R}^3 \rtimes S^2}((\mathbf{0}, \mathbf{e}_z), (\mathbf{y}_1, \mathbf{n}_1)) &= \min_{\alpha^1, \alpha^2 \in [0, 2\pi)} d(e h_{\alpha^1}, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2}) \\ &= \min_{\alpha^1, \alpha^2 \in [0, 2\pi)} d(e, h_{\alpha^1}^{-1} (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2 - \alpha^1} h_{\alpha^1}) \\ &= \min_{\alpha \in [0, 2\pi)} d(e, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_\alpha) \end{aligned}$$

$\mathbf{P}_{\mathbf{mec}}(\mathbb{R}^3 \times S^2)$: Let $(\mathbf{y}_1, \mathbf{n}_1) \in \mathbb{R}^3 \rtimes S^2$. Find

$$[0, T] \ni t \mapsto (\mathbf{x}(t), \mathbf{n}(t)) = \gamma(t) \odot (\mathbf{0}, \mathbf{e}_z) \in \mathbb{R}^3 \rtimes S^2,$$

with γ a Lipschitzian curve in SE(3) with velocity $\dot{\gamma} \in \Delta$, such that sub-Riemannian length $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} dt$ is minimal under boundary conditions $\gamma(0) = (\mathbf{0}, I)$ and $\gamma(T) = (\mathbf{y}_1, R_{\mathbf{n}_1} R_{\mathbf{e}_z, \alpha})$, where both $T \geq 0$ and $\alpha \in [0, 2\pi)$ are free variables in the optimization process.

Relation of Pcurve($\mathbb{R}^3$), Pmec($\mathbb{R}^3 \times S^2$) and PMEC(SE(3))



Theorem *If* $g_1 = (\mathbf{x}_1, R_1) \in \text{SE}(3)$ *is chosen s.t. a corresponding minimizer* γ^* *of* $\mathbf{P_{MEC}}$ *satisfies* $u^3(t) := \langle \omega^3|_{\gamma^*(t)}, \dot{\gamma}^*(t)\rangle > 0,\ t \in (0,T)$, *then* γ^* *can be parameterized by spatial arclength s, and its spatial projection does not exhibit a cusp. If moreover* g_1 *is chosen s.t.* γ^* *has* $\lambda_6(0) = 0$ *then this yields the required minimum choice of* α, *and* $\gamma^*(t)$ *provides the minimizer* $(\mathbf{x}^*(t), \mathbf{n}^*(t)) = \gamma^*(t) \odot (\mathbf{0}, \mathbf{e}_z)$ *of* $\mathbf{P_{mec}}$.

Under these two requirements the spatial projection $\mathbf{x}^*(\cdot)$ *of* $\gamma^*(\cdot) = (\mathbf{x}^*(\cdot), R^*(\cdot))$ *coincides with a minimizer of problem* $\mathbf{P_{curve}}$.

Partial Cartan Connection

Partial Cartan connection $\overline{\nabla}$ on the tangent bundle of $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$

$$\overline{\nabla}_{\dot{\gamma}} \mathcal{A} := \sum_{k=3}^{5} \left((\dot{a}^k) - \sum_{i,j=3}^{5} c^k_{i,j} \, (\dot{\gamma}^i) a^j \right) \mathcal{A}_k,$$

with $\dot{\gamma} = \sum_{i=3}^{5} \dot{\gamma}^i \mathcal{A}_i|_\gamma$, $\quad \mathcal{A} = \sum_{k=3}^{5} a^k \mathcal{A}_k$,

and Lie algebra structure constants $c^k_{i,j}$.

Horizontal exponential curves $t \mapsto g_0 \, e^{t \sum_{i=3}^{5} c^i A_i}$ with $\xi^2 (c^3)^2 + (c^4)^2 + (c^5)^2 = 1$, $g_0 \in \mathrm{SE}(3)$ are the auto-parallel curves, i.e. $\overline{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$

Partial Cartan connection $\overline{\nabla}^*$ on the cotangent bundle of $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$

$$\overline{\nabla}^*_{\dot{\gamma}} \lambda := \sum_{i=1}^{6} \left(\dot{\lambda}_i + \sum_{j=3}^{5} \sum_{k=1}^{6} c^k_{i,j} \lambda_k \, \dot{\gamma}^j \right) \omega^i$$

with $\dot{\gamma} = \sum_{i=3}^{5} \dot{\gamma}^i \mathcal{A}_i|_\gamma$, $\quad \lambda = \sum_{i=1}^{6} \lambda_i \omega^i|_\gamma$,

and Lie algebra structure constants $c^k_{i,j}$.

Along SR geodesics one has covariantly constant momentum

$\overline{\nabla}^*_{\dot{\gamma}} \lambda = 0$ and $\mathcal{G}_\xi^{-1} \left(\sum_{i=3}^{5} \lambda_i \omega^i \right) = \dot{\gamma}$

Group representation $m : \mathrm{SE}(3) \to \mathrm{Aut}(\mathbb{R}^6)$ visible in the Cartan-matrix

$$m(\mathbf{x}, R) := \begin{pmatrix} R & \sigma_{\mathbf{x}} R \\ 0 & R \end{pmatrix}, \text{ with } \mathbf{x} = \sum_{i=1}^{3} x^i \mathbf{e}_i, \; \sigma_{\mathbf{x}} = \sum_{i=1}^{3} x^i A_{3+i} \in \mathrm{so}(3), \text{ s.t. } \sigma_{\mathbf{x}} \mathbf{y} = \mathbf{x} \times \mathbf{y}$$

Theorem *Let m be our matrix group representation, s.t.*

$$\mathrm{d}\boldsymbol{\lambda}|_\gamma = \boldsymbol{\lambda}|_\gamma \; m(\gamma^{-1}) \mathrm{d}m(\gamma).$$

Then along the SR geodesics in $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$ the following relation holds:

$$\boldsymbol{\lambda}(t) m(\gamma(t))^{-1} = \boldsymbol{\lambda}(0) \, m(\gamma(0))^{-1} = \boldsymbol{\lambda}(0).$$

Explicit Expression for Geodesics

Theorem *The spatial part of the cuspless sub-Riemannian geodesics in* $\mathbf{P}_{\mathbf{mec}}$ *is given by*

$$\boldsymbol{x}(s) = \tilde{R}(0)^T(\tilde{\boldsymbol{x}}(s) - \tilde{\boldsymbol{x}}(0)),$$

where $\tilde{R}(0)$ *and* $\tilde{\boldsymbol{x}}(s) := (\tilde{x}(s), \tilde{y}(s), \tilde{z}(s))$ *are given in terms of* $\underline{\lambda}^{(1)}(0)$ *and* $\underline{\lambda}^{(2)}(0)$ *depending on several cases.*
For all cases with $\underline{\lambda}^{(1)}(0) \neq \underline{\lambda}^{(2)}(0)$ *we have* $\tilde{x}(s) = \frac{1}{\mathfrak{c}}\int_0^s \lambda_3(\tau)\,\mathrm{d}\tau = -\frac{i\sqrt{1-d}\sqrt{1+\mathfrak{c}^2}}{\mathfrak{c}\sqrt{2}}\left(E\left(\left(s+\frac{\varphi}{2}\right)i, M\right) - E\left(\frac{\varphi}{2}i, M\right)\right)$,
where $M := \frac{2d}{d-1}$, $d := \frac{\|\underline{\lambda}^{(2)}(0)+\underline{\lambda}^{(1)}(0)\|\|\underline{\lambda}^{(2)}(0)-\underline{\lambda}^{(1)}(0)\|}{1+\mathfrak{c}^2} \leq 1$, *and* $\varphi := \log\frac{\|\underline{\lambda}^{(2)}(0)+\underline{\lambda}^{(1)}(0)\|}{\|\underline{\lambda}^{(2)}(0)-\underline{\lambda}^{(1)}(0)\|}$.

For the case $\underline{\lambda}^{(1)}(0) = \mathbf{0}$, *we have* $\tilde{R}(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(3)$, $\quad \begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{-1}{\mathfrak{c}}\begin{pmatrix} \lambda_4(s) \\ \lambda_5(s) \end{pmatrix}$.

For the case $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$, *we have* $\tilde{R}(0) = \frac{1}{\mathfrak{c}}\begin{pmatrix} \lambda_1(0) & \lambda_2(0) & \lambda_3(0) \\ \mathfrak{c}\frac{-\lambda_2(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \mathfrak{c}\frac{\lambda_1(0)}{\|\underline{\lambda}^{(1)}(0)\|} & 0 \\ \frac{-\lambda_1(0)\lambda_3(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \frac{-\lambda_2(0)\lambda_3(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \|\underline{\lambda}^{(1)}(0)\| \end{pmatrix} \in \mathrm{SO}(3)$.

For the case $W = 0$ *along with* $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$, *we have* $\begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{\underline{\lambda}^{(2)}(s)\cdot\underline{\lambda}^{(1)}(0)}{\mathfrak{c}\|\underline{\lambda}^{(1)}(0)\|}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For $W \neq 0$ *along with* $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$ *we have*

$$\begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{\sqrt{\|\underline{\lambda}^{(2)}(s)\|^2 - W^2\mathfrak{c}^{-2}}}{\mathfrak{c}^2\|\underline{\lambda}^{(1)}(0)\|\sqrt{\|\underline{\lambda}^{(2)}(0)\|^2 - W^2\mathfrak{c}^{-2}}}\begin{pmatrix} \cos\tilde{\psi}(s) & -\sin\tilde{\psi}(s) \\ \sin\tilde{\psi}(s) & \cos\tilde{\psi}(s) \end{pmatrix}\begin{pmatrix} W\lambda_3(0) \\ \mathfrak{c}(\underline{\lambda}^{(2)}(0)\cdot\underline{\lambda}^{(1)}(0)) \end{pmatrix},$$ *where*

$$\tilde{\psi}(s) = \int_0^s \frac{W\mathfrak{c}^{-1}\lambda_3(\tau)}{\|\underline{\lambda}^{(2)}(\tau)\|^2 - W^2\mathfrak{c}^{-2}}\,\mathrm{d}\tau = -\frac{W}{\mathfrak{c}}\frac{\sqrt{2}}{\sqrt{1+\mathfrak{c}^2}\sqrt{1-d}}\frac{1}{i}\Big(F(i(s+\frac{\varphi}{2}), M) - F(\frac{i\varphi}{2}, M)$$
$$-(1-\frac{1}{D})(\Pi\left(\frac{M}{D}, i(s+\frac{\varphi}{2}), M\right) - \Pi\left(\frac{M}{D}, \frac{i\varphi}{2}, M\right))\Big),$$

with $D = 2(\frac{W^2}{\mathfrak{c}^2} - 1)(1+\mathfrak{c}^2)^{-1}(1-d)^{-1} + 1$ *and* $|\tilde{\psi}(s)| < \pi$, $\mathrm{sign}(\tilde{\psi}(s)) = \mathrm{sign}(W)$.

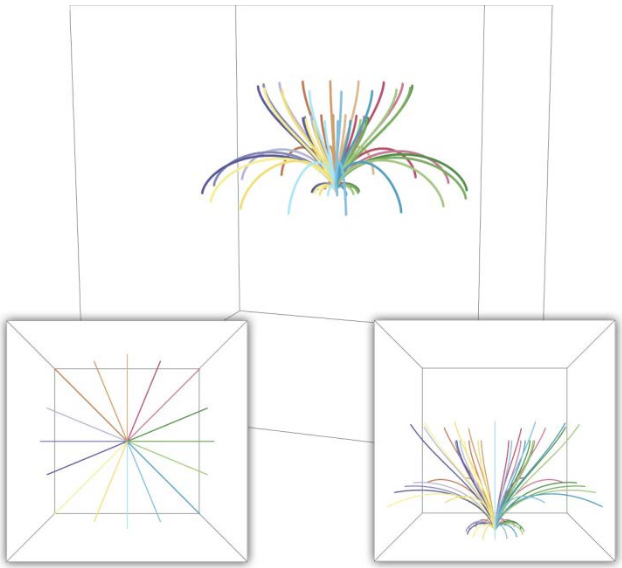
Geometric Properties of Geodesics

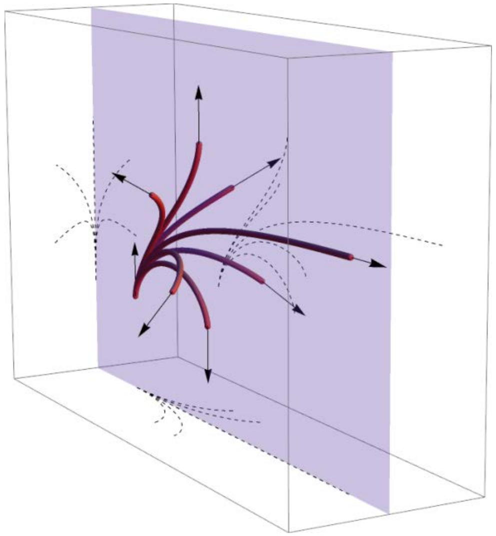
Corollary 1 *The absolute curvature and the signed torsion of a geodesic of* $\mathbf{P_{curve}}$ *are given by* $\kappa = \frac{\sqrt{\lambda_4^2+\lambda_5^2}}{\lambda_3} = \frac{\sqrt{1-\lambda_3^2}}{\lambda_3}$, $\tau = \frac{W}{\lambda_4^2+\lambda_5^2}$. *Thus, the torsion is bounded as* $|W| \leq |\tau(s)| \leq \frac{2|W|}{\sqrt{(1-\mathfrak{c}^2)^2+4W^2}+1-\mathfrak{c}^2}$ *for all* $0 \leq s \leq s_{max}$.

Corollary 2 *The cuspless spatial projections of sub-Riemannian geodesics of* $\mathbf{P_{mec}}$ *(i.e. geodesics of* $\mathbf{P_{curve}}$*) are planar if and only if* $W = 0$.

Corollary 3 *Given admissible coplanar end conditions for* $\mathbf{P_{curve}}$*, the unique cuspless geodesic connecting them is planar.*

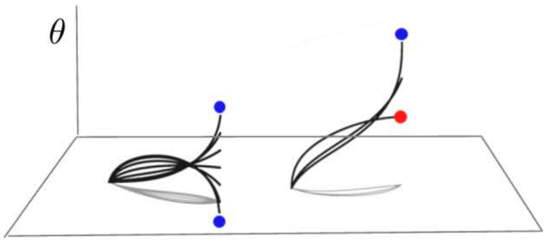
Corollary 4 *All cuspless sub-Riemannian geodesics in* $(\mathrm{SE}(3), \Delta, \mathcal{G}_1)$ *with* $\lambda_6 = 0$ *and* $\sum_{i=1}^{3} \lambda_i^2(0) \neq 0$, *departing from* $e = (\mathbf{0}, I)$ *stay in the upper half space* $z \geq 0$.

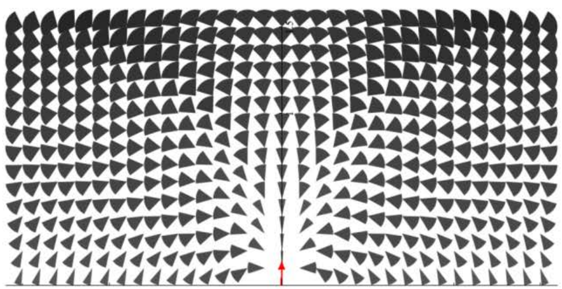


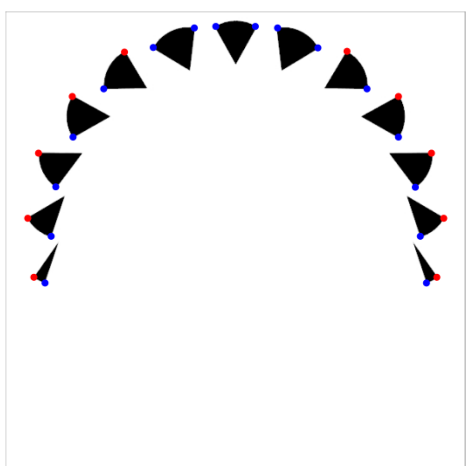


Range of Exponential Map of Pcurve

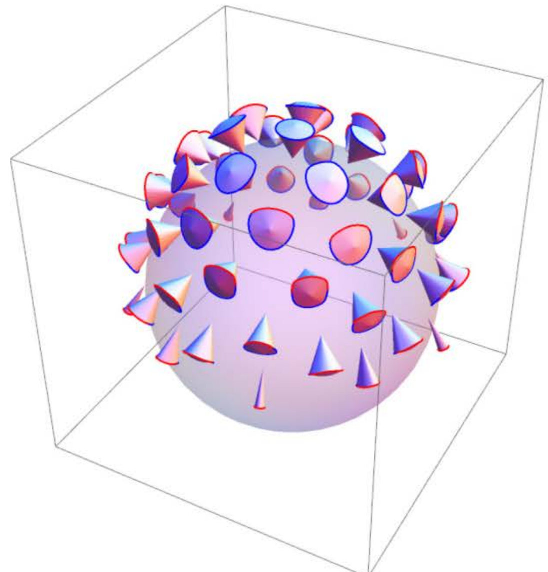
$\mathbb{R}^2 \rtimes S^1 = SE(2)$

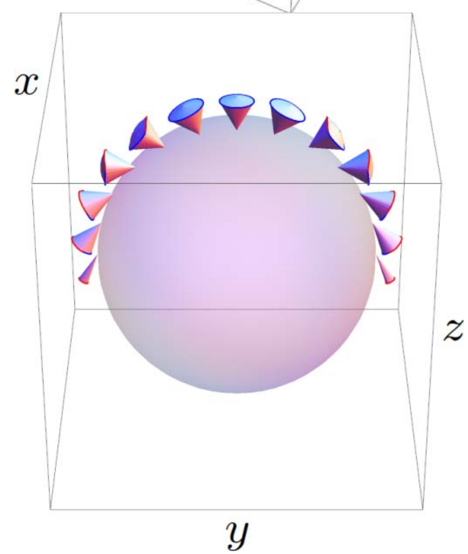






$\mathbb{R}^3 \rtimes S^2 = SE(3)/(\{0\} \times SO(2))$





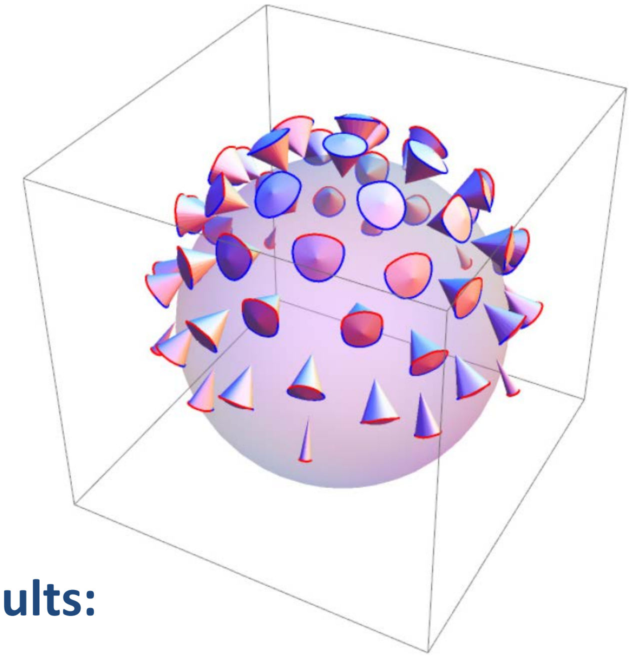
Let $\mathcal{R}$ denotes the range, and $\mathcal{D}_0$ the domain of exponential map $\mathbf{P_{curve}}$.

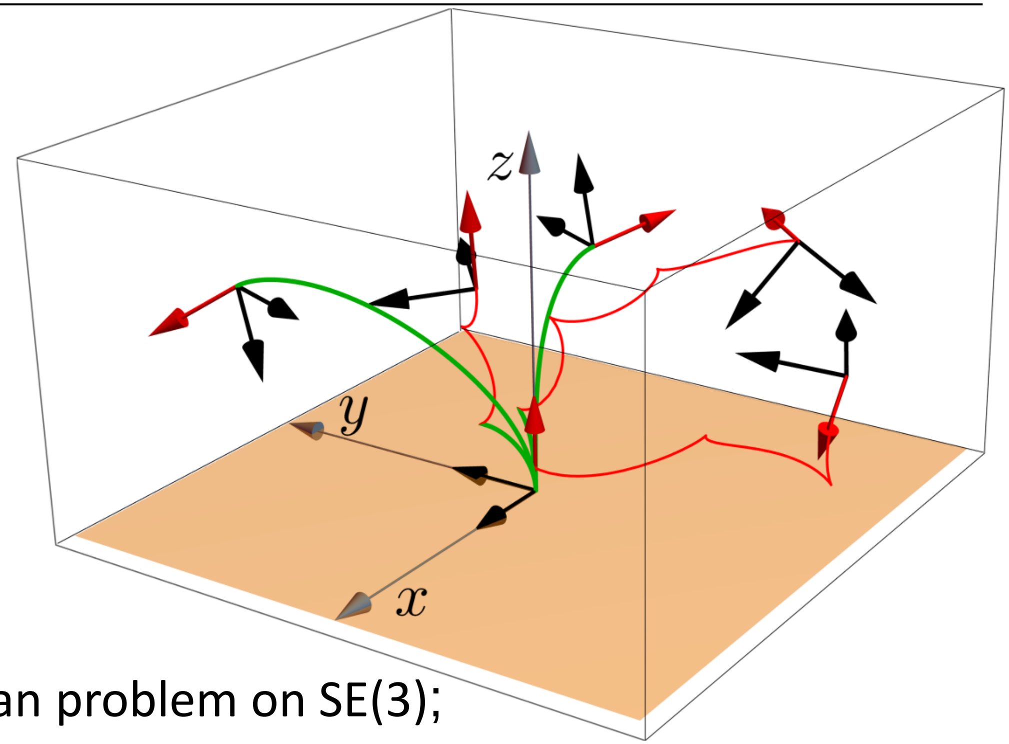
Conjecture:

$Exp : \mathcal{D}_0 \to \mathcal{R}$
is a homeomorphism

$Exp : int(\mathcal{D}_0) \to int(\mathcal{R})$
is a diffeomorphism

SR-geodesics on SE(3) with cuspless spatial projections





Results:

- Lift $P_{\text{curve}}(\mathbb{R}^3 \times S^2)$ to sub-Riemannian problem on SE(3);
- Hamiltonian system of PMP;
- Liouville integrability of the Hamiltonian system;
- Explicit expressions for SR-geodesics in spatial arclength parameterization;
- Evaluation of first cusp time;
- Admissible boundary conditions reachable by cuspless geodesics;
- Geometrical properties: bounds on torsion, planarity conditions, symmetries;
- Numerical investigation of absence of conjugate points;
- Numerical solution to the boundary value problem.

Thank you for your attention!