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Symmetries and Parameterization of Abnormal Extremals in the Sub-Riemannian Problem with the Growth Vector $(2, 3, 5, 8)$

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The left-invariant sub-Riemannian problem with the growth vector $(2, 3, 5, 8)$ is considered. A two-parameter group of infinitesimal symmetries consisting of rotations and dilations is described. The abnormal geodesic flow is factorized modulo the group of symmetries. A parameterization of the vertical part of abnormal geodesic flow is obtained.

Keywords: sub-Riemannian geometry, abnormal extremals, symmetries

1. Problem statement

Let L be the free nilpotent Lie algebra with 2 generators of step 4. There exists a basis $L = \operatorname{span}(X_1, \ldots, X_8)$ in which the product table in L reads as follows:

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \tag{1.1}$$

$$[X_1, X_4] = X_6, \quad [X_2, X_4] = [X_1, X_5] = X_7, \quad [X_2, X_5] = X_8. \tag{1.2}$$

Let G be the connected simply connected Lie group with the Lie algebra L. Consider the left-invariant sub-Riemannian structure [1, 2] on G defined by (X_1, X_2) as an orthonormal frame. The corresponding optimal control problem reads as follows:

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x), \quad x \in G, \; u = (u_1, u_2) \in \mathbb{R}^2, \tag{1.3}$$

$$x(0) = x_0 = \mathrm{Id}, \quad x(t_1) = x_1, \tag{1.4}$$

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min. \tag{1.5}$$

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A symmetric model of this problem is the following one [4]:

$$G \cong \mathbb{R}^8_{x_1 \dots x_8}, \tag{1.6}$$

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8}, \tag{1.7}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4}\frac{\partial}{\partial x_7}. \tag{1.8}$$

In this paper we continue the study of abnormal extremals in problem (1.3)–(1.5) started in [3, 5]. Notice that the normal geodesic flow in problem (1.3)–(1.5) is not Liouville integrable [6].

Denote by D the distribution spanned by the vector fields X_1, X_2, and by g the inner product in D determined by (X_1, X_2) as an orthonormal frame. Then (D, g) is the sub-Riemannian structure given by (X_1, X_2) as an orthonormal frame.

This work has the following structure. In Section 2 we describe some infinitesimal symmetries of the sub-Riemannian structure (D, g) and the distribution D — rotation X_0 and dilation Y. In Section 3 we lift these symmetries to the cotangent bundle T^*G. In Section 4 we describe the action of these lifted symmetries on the Hamiltonian system for abnormal extremals. In particular, we show that initial conditions for abnormal extremals can be factorized via the rotations to a fundamental domain $\{h_7 = 0\}$. In Section 5, we give an explicit parameterization of rotations. In Section 6 we present an explicit parameterization of the vertical part of abnormal extremals with initial conditions in the fundamental domain $\{h_7 = 0\}$. Finally, in Section 7 we conclude on final parametrization of abnormal extremals for arbitrary initial conditions.

2. Infinitesimal symmetries of (D, g) and D

Definition 1. *A vector field $V \in \operatorname{Vec} G$ is called an infinitesimal symmetry of a distribution D if its flow e^{tV} preserves D, i.e., $e^{tV}_* D = D$.*

A vector field $V \in \operatorname{Vec} G$ is called an infinitesimal symmetry of a sub-Riemannian structure (D, g) if its flow preserves both the distribution D and the inner product g, i.e., $e^{tV}_* D = D$ and $(e^{tV})^* g = g$.

The Lie algebras of symmetries of a distribution D (a sub-Riemannian structure (D, g)) will be denoted by $\operatorname{Sym}(D)$ (respectively $\operatorname{Sym}(D, g)$).

Symmetries of distributions and sub-Riemannian structures may be computed via the following

Proposition [7]. *Let $X \in \operatorname{Vec}(G)$.*

(1) *$X \in \operatorname{Sym}(D)$ iff $\operatorname{ad} X(D) \subset D$, or, equivalently, $\operatorname{ad} X \in gl(D_x)$ for all $x \in G$, i.e.,*

$$[X, X_i] = \sum_{i,j=1}^{2} a_{ij} X_j, \qquad a_{ij} \in C^\infty(G).$$

(2) *$X \in \operatorname{Sym}(D, g)$ iff $\operatorname{ad} X \in so(D_x)$ for all $x \in G$, i.e.,*

$$[X, X_i] = \sum_{i,j=1}^{2} a_{ij} X_j, \qquad a_{ji} = -a_{ij}, \quad a_{ij} \in C^\infty(G).$$

Theorem 1.

(1) *There exists a vector field* $Y \in \mathrm{Sym}(D)$ *such that*

$$[Y, X_1] = -X_1, \quad [Y, X_2] = -X_2, \quad Y(0) = 0.$$

In model (1.6)–(1.8) *this vector field reads*

$$Y = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3} + 3x_4\frac{\partial}{\partial x_4} + 3x_5\frac{\partial}{\partial x_5} + 4x_6\frac{\partial}{\partial x_6} + 4x_7\frac{\partial}{\partial x_7} + 4x_8\frac{\partial}{\partial x_8}.$$

(2) *There exists a vector field* $X_0 \in \mathrm{Sym}(D, g)$ *such that*

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1, \quad X_0(0) = 0.$$

In model (1.6)–(1.8) *this vector field reads*

$$X_0 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_5} + P\frac{\partial}{\partial x_6} + Q\frac{\partial}{\partial x_7} + R\frac{\partial}{\partial x_8},$$
$$P = \frac{x_1^4}{24} - \frac{x_1^2x_2^2}{8} - x_7,$$
$$Q = -\frac{x_1x_2^3}{12} - \frac{x_1^3x_2}{12} + 2x_6 - 2x_8,$$
$$R = -\frac{x_1^2x_2^2}{8} + \frac{x_2^4}{24} + x_7.$$

(3) *The vector fields* Y *and* X_0 *commute:* $[Y, X_0] = 0$.

Proof. Follows from Theorem 2 [4]. □

The product table given by Eqs. (1.1), (1.2) yields the following statement.

Corollary 1.

(1) *The symmetry* Y *has the following Lie brackets with the basis vector fields in Lie algebra* L:

$$[Y, X_3] = -2X_3, \quad [Y, X_4] = -3X_4, \quad [Y, X_5] = -3X_5,$$
$$[Y, X_6] = -4X_6, \quad [Y, X_7] = -4X_7, \quad [Y, X_8] = -4X_8.$$

(2) *The symmetry* X_0 *has the following Lie brackets with the basis vector fields in Lie algebra* L:

$$[X_0, X_3] = 0, \quad [X_0, X_4] = -X_5, \quad [X_0, X_5] = X_4,$$
$$[X_0, X_6] = 2X_7, \quad [X_0, X_7] = X_8 - X_6, \quad [X_0, X_8] = -2X_7.$$

3. Lift of symmetries to T^*G

Introduce Hamiltonians linear on fibers and corresponding to the vector fields X_i, Y:

$$h_i(\lambda) = \langle \lambda, X_i(x) \rangle, \qquad i = 0, \ldots, 8,$$
$$h_Y(\lambda) = \langle \lambda, Y(x) \rangle, \qquad x = \pi(\lambda), \qquad \lambda \in T^*G,$$

where $\pi\colon T^*G \to G$ is the canonical projection. Consider the corresponding Hamiltonian vector fields on T^*G

$$\vec{h}_i(\lambda), \quad i = 0, \ldots, 8, \qquad \vec{h}_Y(\lambda), \qquad \lambda \in T^*G.$$

The vertical part of these vector fields reads in the coordinates $(h_1, \ldots, h_8)$ as follows: the rotation $\vec{h}_0$

$$\begin{aligned} &\dot{h}_1 = -h_2, \quad \dot{h}_2 = h_1, \quad \dot{h}_3 = 0, \\ &\dot{h}_4 = -h_5, \quad \dot{h}_5 = h_4, \\ &\dot{h}_6 = -2h_7, \quad \dot{h}_7 = h_6 - h_8, \quad \dot{h}_8 = 2h_7, \end{aligned} \tag{3.1}$$

the dilation $\vec{h}_Y$

$$\begin{aligned} &\dot{h}_1 = -h_1, \quad \dot{h}_2 = -h_2, \quad \dot{h}_3 = -2h_3, \\ &\dot{h}_4 = -3h_4, \quad \dot{h}_5 = -3h_5, \\ &\dot{h}_6 = -4h_6, \quad \dot{h}_7 = -4h_7, \quad \dot{h}_8 = -4h_8. \end{aligned}$$

The phase flow of rotations is visible via the Casimir $\Delta = h_6 h_8 - h_7^2$: we have

$$\vec{h}_0 \Delta = \vec{h}_0 (h_6 + h_8) = 0.$$

The vertical part of the field $\vec{h}_0$ is tangent to the closed curves $\{\Delta = \mathrm{const},\ h_6 + h_8 = \mathrm{const}\}$, thus it is periodic. An explicit parameterization of the flow of ODE (3.1) is given in Section 7.

4. Canonical abnormal flow and its symmetries

We described in [5] the structure of abnormal extremals for the sub-Riemannian structure (D, g) in terms of the Casimir Δ and an integral of abnormal extremals $I = h_8 h_4^2 - 2h_7 h_4 h_5 + h_6 h_5^2$.

In the (asymptotic) case $\Delta < 0$, $I = 0$ projections of abnormal extremals to the plane (h_4, h_5) are straight lines or broken lines.

In the complementary (main) case $\Delta \geqslant 0$ or $I \neq 0$ projections of abnormal extremals to the plane (h_4, h_5) are first- or second-order curves (straight lines, ellipses, hyperbolas, parabolas). In this case extremals are reparameterization of trajectories of the canonical Hamiltonian system

$$\dot{\lambda} = -h_5 \vec{h}_1 + h_4 \vec{h}_2, \qquad \lambda \in (\Delta^2)^{\perp}, \tag{4.1}$$

where $(\Delta^2)^{\perp} = \{\lambda \in T^*G \mid h_1(\lambda) = h_2(\lambda) = h_3(\lambda) = 0\}$. The vertical part of system (4.1) reads as follows:

$$h_1 = h_2 = h_3 = 0, \tag{4.2}$$

$$\begin{pmatrix} \dot{h}_4 \\ \dot{h}_5 \end{pmatrix} = C \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}, \qquad C = \begin{pmatrix} h_7 & -h_6 \\ h_8 & -h_7 \end{pmatrix}, \tag{4.3}$$

$$\dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0. \tag{4.4}$$

Following [5], we call system (4.2)–(4.4) the canonical system for abnormal extremals.

The symmetries $\vec{h}_0$ and $\vec{h}_Y$ act on the canonical abnormal Hamiltonian vector field $\vec{A} = -h_5\vec{h}_1 + h_4\vec{h}_2$ defined by system (4.1) as follows:

$$[\vec{h}_0, \vec{A}] = 0, \tag{4.5}$$

$$[\vec{h}_Y, \vec{A}] = -4\vec{A}. \tag{4.6}$$

We get from the Lie brackets (4.5), (4.6) the following statement.

Proposition 1. *For any $t, s, r \in \mathbb{R}$ we have*

$$e^{t\vec{A}} \circ e^{s\vec{h}_0} = e^{s\vec{h}_0} \circ e^{t\vec{A}}, \tag{4.7}$$

$$e^{t\vec{A}} \circ e^{r\vec{h}_Y} = e^{r\vec{h}_Y} \circ e^{t'\vec{A}}, \qquad t' = te^{4r}. \tag{4.8}$$

Consequently, we can find the vertical part of canonical abnormal extremals as follows:

$$e^{t\vec{A}}(\lambda_0) = e^{-s\vec{h}_0} \circ e^{-r\vec{h}_Y}(\widetilde{\lambda}_{t'}), \qquad \widetilde{\lambda}_{t'} = e^{t'\vec{A}} \circ e^{s\vec{h}_0} \circ e^{r\vec{h}_Y}(\lambda_0), \quad t' = te^{4r}. \tag{4.9}$$

For $r = 0$ we get:

$$e^{t\vec{A}}(\lambda_0) = e^{-s\vec{h}_0}(\widetilde{\lambda}_t), \qquad \widetilde{\lambda}_t = e^{t\vec{A}} \circ e^{s\vec{h}_0}(\lambda_0). \tag{4.10}$$

It is obvious from (3.1) that the space $\mathbb{R}^3_{h_6,h_7,h_8}$ factorizes by the flow of the rotation $\vec{h}_0$ to the half-plane $\{(h_6, h_7, h_8) \in \mathbb{R}^3 \mid h_7 = 0,\ h_6 - h_8 \geqslant 0\}$. Thus, we can take in (4.10)

$$\widetilde{\lambda}_0 \in \Omega = \{h_1 = h_2 = h_3 = 0,\ h_7 = 0,\ h_6 - h_8 \geqslant 0\}.$$

We call the previous set the fundamental domain of the rotation $\vec{h}_0$.

5. Explicit parameterization of rotations

Denote $\chi = (h_6, h_7, h_8) \in \mathbb{R}^3_{h_6,h_7,h_8}$. Then ODE (3.1) defines in $\mathbb{R}^3_{h_6,h_7,h_8}$ a linear system

$$\dot{\chi} = B\chi, \qquad B = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}. \tag{5.1}$$

System (5.1) has the solutions $\chi(s) = e^{Bs}\chi^0$, explicitly

$$\begin{aligned} h_6(s) &= \frac{1}{2}((h_6^0 + h_8^0) + (h_6^0 - h_8^0)\cos 2s - 2h_7^0 \sin 2s), \\ h_7(s) &= \frac{1}{2}((h_6^0 - h_8^0)\sin 2s + 2h_7^0 \cos 2s), \\ h_8(s) &= \frac{1}{2}((h_6^0 + h_8^0) - (h_6^0 - h_8^0)\cos 2s + 2h_7^0 \sin 2s). \end{aligned} \tag{5.2}$$

In the coordinates

$$\begin{pmatrix} h_6^* \\ h_7^* \\ h_8^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} h_6 - h_8 \\ \sqrt{2}h_7 \\ h_6 + h_8 \end{pmatrix}$$

we have

$$\begin{aligned} h_6^*(s) &= \rho\cos(2s+\varphi_0),\\ h_7^*(s) &= \frac{\rho}{\sqrt{2}}\sin(2s+\varphi_0),\\ h_8^*(s) &= h_8^{*0}, \end{aligned} \tag{5.3}$$

where $\rho^2 = (h_6^{*0})^2 + 2(h_7^{*0})^2$, $\cos\varphi_0 = \dfrac{h_6^{*0}}{\rho}$, $\sin\varphi_0 = \dfrac{\sqrt{2}h_7^{*0}}{\rho}$. It is visible from formulas (5.3) that the flow of ODE (5.1) defines motion along ellipses

$$\{h_8^* = \text{const},\ 2(h_7^*)^2 + (h_6^*)^2 = \text{const}\} = \{\Delta = \text{const},\ h_6 + h_8 = \text{const}\}.$$

Consequently, the fundamental set of rotation is

$$F = \{\chi_f = (h_6, 0, h_8) \mid h_6 \geqslant h_8\}. \tag{5.4}$$

6. Solution to the canonical system (4.3), (4.4) in the fundamental case $h_7 = 0$

In this section we consider the case $h_7 = 0$ and describe a solution to the canonical system (4.3), (4.4) with an initial condition $(h_4, h_5, h_6, h_7, h_8)(0) = (h_4^0, h_5^0, h_6, 0, h_8)$.

If $h_7 = 0$, then $\Delta = h_6h_8$. Denote $\delta = \sqrt{|\Delta|}$.

6.1. Elliptic case $\Delta > 0$

6.1.1. Subcase $I \neq 0$

In this case the fundamental set of rotation is

$$F = \{\chi_f = (h_6, 0, h_8) \mid h_6 \geqslant h_8 > 0 \quad \text{and} \quad h_6 \geqslant h_8, h_6 < 0, h_8 < 0\}.$$

Then system (4.3) has solution for parameters χ_f as follows:

$$\begin{pmatrix} h_4(t)\\ h_5(t)\end{pmatrix} = \begin{pmatrix} h_4^0 & -h_6h_5^0\\ h_5^0 & h_8h_4^0\end{pmatrix}\begin{pmatrix}\cos\delta t\\ \dfrac{1}{\delta}\sin\delta t\end{pmatrix} = \begin{pmatrix} a\cos(\delta t+\varphi)\\ b\sin(\delta t+\varphi)\end{pmatrix}, \tag{6.1}$$

where

$$\begin{aligned} &a = \sqrt{\frac{I}{h_8}}, \quad b = \sqrt{\frac{I}{h_6}},\\ &\cos\varphi = h_4^0/a = \frac{h_8h_4^0}{\delta b}, \quad \sin\varphi = \frac{h_6h_5^0}{\delta a} = h_5^0/b, \quad \varphi\in(-\pi,\ \pi). \end{aligned}$$

6.1.2. Subcase $I = 0$

If $I = 0$, then system (4.3) has solutions

$$h_4 \equiv 0, \quad h_5 \equiv 0.$$

6.2. The hyperbolic case $\Delta < 0$

In this case the fundamental set of rotation is

$$F = \{\chi_f = (h_6, 0, h_8) \mid h_6 > 0, h_8 < 0\}.$$

6.2.1. The nonasymptotic subcase $I \neq 0$

In this subcase the initial point does not belong to eigenspaces of the matrix C:

$$\forall k \in \mathbb{R} \quad \mathbf{h}^0 = (h_4^0, h_5^0) \neq \begin{cases} k(\sqrt{|h_6|}, -\sqrt{|h_8|}), \\ k(-\sqrt{|h_6|}, -\sqrt{|h_8|}). \end{cases} \tag{6.2}$$

Introduce the next value: $\sigma = \operatorname{sign}\left(|h_4^0| - \sqrt{\left|\frac{h_6}{h_8}\right|}|h_5^0|\right)$. For hyperbolic χ_f it is easy to prove that $Ih_8 > 0$ if $\sigma = 1$ and $Ih_6 > 0$ if $\sigma = -1$. Introduce the next parameters for fundamental set of rotation in hyperbolic case:

$$a = \sqrt{\sigma\frac{I}{h_8}}, \quad b = \sqrt{-\sigma\frac{I}{h_6}}.$$

1. Let $\sigma = 1$. Then system (4.3) has solution for parameters χ_f as follows:

$$\begin{pmatrix} h_4(t) \\ h_5(t) \end{pmatrix} = \begin{pmatrix} h_4^0 & -h_6 h_5^0 \\ h_5^0 & h_8 h_4^0 \end{pmatrix} \begin{pmatrix} \operatorname{ch}\delta t \\ \frac{1}{\delta}\operatorname{sh}\delta t \end{pmatrix} = \begin{pmatrix} \operatorname{sign} h_4^0 a \operatorname{ch}(\delta t - \operatorname{sign} h_4^0 \alpha) \\ -\operatorname{sign} h_4^0 b \operatorname{sh}(\delta t - \operatorname{sign} h_4^0 \alpha) \end{pmatrix}, \tag{6.3}$$

where

$$\operatorname{ch}\alpha = |h_4^0|/a = -\frac{h_8|h_4^0|}{\delta b}, \quad \operatorname{sh}\alpha = h_5^0/b = \frac{h_6 h_5^0}{\delta a}, \quad \alpha \in \mathbb{R}.$$

2. Let $\sigma = -1$.
Then system (4.3) has solution for parameters χ_f as follows:

$$\begin{pmatrix} h_4(t) \\ h_5(t) \end{pmatrix} = \begin{pmatrix} h_4^0 & -h_6 h_5^0 \\ h_5^0 & h_8 h_4^0 \end{pmatrix} \begin{pmatrix} \operatorname{ch}\delta t \\ \frac{1}{\delta}\operatorname{sh}\delta t \end{pmatrix} = \begin{pmatrix} -\operatorname{sign} h_5^0 a \operatorname{sh}(\delta t - \operatorname{sign} h_5^0 \beta) \\ \operatorname{sign} h_5^0 b \operatorname{ch}(\delta t - \operatorname{sign} h_5^0 \beta) \end{pmatrix}, \tag{6.4}$$

where

$$\operatorname{ch}\beta = |h_5^0|/b = \frac{h_6|h_5^0|}{\delta a}, \quad \operatorname{sh}\beta = h_4^0/a = -\frac{h_8 h_4^0}{\delta b}, \quad \beta \in \mathbb{R}.$$

6.2.2. The asymptotic subcase $I = 0$

In this case the initial point belongs to eigenspaces of the matrix C, and we have an equality in (6.2).

Introduce the parameter $p = \sqrt{\left|\frac{h_8}{h_6}\right|}$. In the upper case (6.2) we have

$$h_4(t) = h_4^0 e^{\delta t}, \quad h_5(t) = -h_4^0 p e^{\delta t},$$

and in the lower case (6.2) we have

$$h_4(t) = \frac{h_5^0}{p} e^{-\delta t}, \quad h_5(t) = h_5^0 e^{-\delta t}.$$

6.3. The parabolic case $\Delta = 0$

6.3.1. Subcase $C \neq 0$

In this case the fundamental set of rotation is

$$F = \{\chi_f = (h_6, 0, h_8) \mid h_6 > 0, h_8 = 0 \,\text{and}\, h_6 = 0, h_8 < 0\}.$$

1. Let $C\mathbf{h}^0 \neq 0$. If $h_6 \neq 0$, then

$$h_4(t) = h_6 h_5^0 t + h_4^0, \quad h_5(t) = h_5^0.$$

If $h_8 \neq 0$, then

$$h_4(t) = h_4^0, \quad h_5(t) = h_8 h_4^0 t + h_5^0.$$

2. Let $C\mathbf{h}^0 = 0$. If $h_6 \neq 0$, then

$$h_4(t) = h_4^0, \quad h_5(t) = 0.$$

If $h_8 \neq 0$, then

$$h_4(t) = 0, \quad h_5(t) = h_5^0.$$

6.3.2. Subcase $C = 0$

We have

$$\mathbf{h}(t) \equiv \mathbf{h}^0.$$

7. Conclusion on parameterization of extremals

On the basis of results of the previous sections, we can get a parameterization of abnormal extremals (4.10), with explicit parameterization of $\widetilde{\lambda}_t$ for the case $\widetilde{\lambda}_0 \in \Omega \subset \{h_7 = 0\}$ given in Section 6, and explicit parameterization of the flow $e^{s\vec{h}_0}$ as follows:

$$\begin{pmatrix} h_4(s) \\ h_5(s) \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} h_4^0 \\ h_5^0 \end{pmatrix},$$

$$h_6(s) = \frac{1}{2}((h_6^0 + h_8^0) + (h_6^0 - h_8^0)\cos 2s - 2h_7^0 \sin 2s),$$

$$h_7(s) = \frac{1}{2}((h_6^0 - h_8^0)\sin 2s + 2h_7^0 \cos 2s),$$

$$h_8(s) = \frac{1}{2}((h_6^0 + h_8^0) - (h_6^0 - h_8^0)\cos 2s + 2h_7^0 \sin 2s).$$

An explicit parameterization of abnormal extremal trajectories will be performed similarly in a forthcoming paper.

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