l_∞ Sub-Finsler Problems on the Cartan and Engel Groups

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Sub-Riemannian Geometry

- (M, Δ, g)
- smooth manifold M,
- vector distribution $\Delta = \{\Delta_q \subset T_q M \mid q \in M\}$,
- inner product $g = \{g_q$ — inner product in $\Delta_q \mid q \in M\}$.
- sub-Riemannian (Carnot-Caratheodory) distance

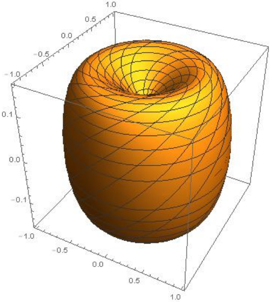
$$d_{SR}(q_0, q_1) = \inf \left\{ \int_0^T \sqrt{g(\dot{q}, \dot{q})} dt \mid q(t) \text{ a.e. tangent to } \Delta, \right.$$
$$\left. q(0) = q_0, \ q(T) = q_1 \right\},$$

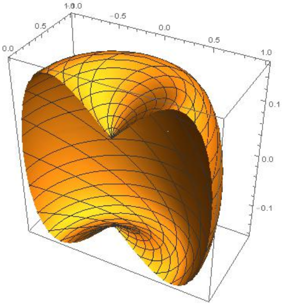
- sub-Riemannian sphere

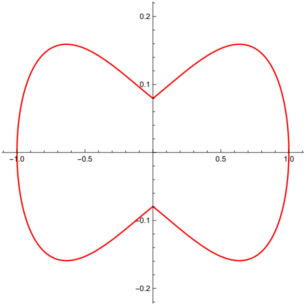
$$S_{q_0}(R) = \{q \in M \mid d_{SR}(q_0, q) = R\}.$$

Left-invariant SR problem on the Heisenberg group

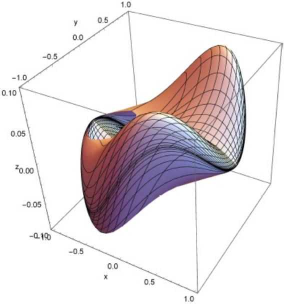
- $M = \mathbb{R}^3_{x,y,z}$, $\Delta = \text{span}(X_1, X_2)$, $g(X_i, X_j) = \delta_{ij}$,
- $X_1 = \partial_x - \frac{y}{2}\partial_z$, $X_2 = \partial_y + \frac{x}{2}\partial_z$.
- SR sphere is not smooth but sub-analytic:

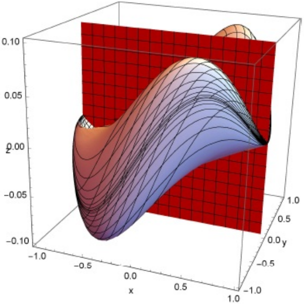


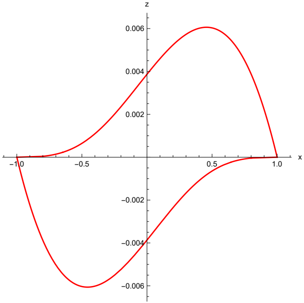




- $M = \mathbb{R}^3_{x,y,z}$, $\Delta = \text{span}(X_1, X_2)$, $g(X_i, X_j) = \delta_{ij}$,
- $X_1 = \partial_x$, $X_2 = \partial_y + \frac{x^2}{2}\partial_z$.
- SR sphere is neither smooth *nor* sub-analytic:







Sub-Finsler geometry as a generalization of sub-Riemannian one

- Sub-Riemannian geometry (M, Δ, g):
  - $g_q(v, v)$ — quadratic form in Δ_q,
  - $\sqrt{g_q(v, v)}$ — norm in Δ_q,
  - $\{v \in \Delta_q \mid g_q(v, v) = 1\}$ — ellipsoid.
- Sub-Finsler geometry $(M, \Delta, \| \cdot \|)$:
  - $\| \cdot \| = \{\| \cdot \|_q$ — norm in $\Delta_q \mid q \in M\}$,
  - $\{v \in \Delta_q \mid \| \cdot \|_q = 1\}$ — convex centrally symmetric surface with origin inside.

Left-invariant l_∞ sub-Finsler structures on nilpotent Lie groups

- M — nilpotent Lie group, $L_q : M \to M$, $L_q(q') = q \cdot q'$,
- $\Delta = L_{q*}\Delta$, $\| \cdot \| = L_q^* \| \cdot \|$,
- $X_1, \dots, X_k \in \operatorname{Vec} M$, $L_{q*}X_i = X_i$,
- $\Delta = \operatorname{span}(X_1, \dots, X_k)$,
- $\|v\|_q = \|u\|_\infty = \max\{|u_i|\}$, $v = \sum_{i=1}^k u_i X_i(q)$.
- Motivations:
  - geometric group theory (asymptotic cones of nilpotent finitely generated groups),
  - homogeneous manifolds with intrinsic metrics,
  - control theory (quantum systems).

Sub-Finsler minimizers of $(M, \Delta, \|\cdot\|)$ as solutions to time-optimal problem

- Sub-Finsler minimizer $q \in \mathrm{Lip}([0, T], M)$:

$$\begin{aligned} &\dot{q}(t) \in \Delta_{q(t)} \text{ a.e. } t \in [0, T], \\ &\|\dot{q}(t)\| \leq 1, \\ &q(0) = q_0, \quad q(T) = q_1, \quad T \to \min . \end{aligned}$$

- the case of l_∞ sub-Finsler structures:

$$\dot{q}(t) = \sum_{i=1}^{k} u_i X_i(q(t)), \qquad \|u\|_\infty \leq 1,$$
$$q(0) = q_0, \quad q(T) = q_1, \quad T \to \min .$$

- sub-Finsler distance:

$$d_{SF}(q_0, q_1) = \inf\{T > 0 \mid q(t) \text{ tangent to } \Delta, \ \|\dot{q}\| \leq 1, \\ q(0) = q_0, \ q(T) = q_1\}$$

Left-invariant l_∞ sub-Finsler problem on the Heisenberg group

- $M = \mathbb{R}^3_{x,y,z}$, $X_1 = \partial_x - \frac{y}{2}\partial_z$, $X_2 = \partial_y + \frac{x}{2}\partial_z$.

Theorem (Busemann 1947, Barilari, Boscain, Le Donne, Sigalotti 2017)

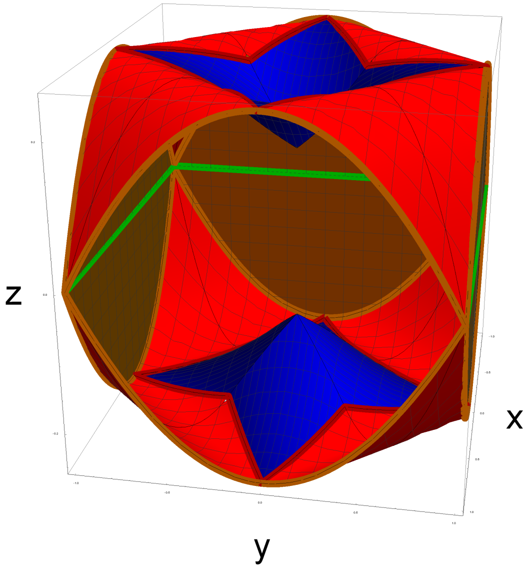
Sub-Finsler minimizers are curves of two types:

(1) *one component of controls is constantly equal to 1 or* -1,

(2) *controls are bang-bang (piecewise constant with values* ± 1*).*

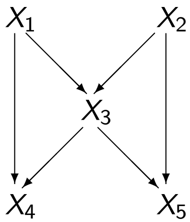
All curves of type (1) *are optimal.*
Optimal curves of type (2) *have* ≤ 4 *switchings.*
Sub-Finsler sphere is semi-analytic and homeomorphic to the Euclidean sphere.

l_∞ sub-Finsler sphere on the Heisenberg group



Cartan group

- Cartan algebra:
  - $L = \text{span}(X_1, \dots, X_5)$,
  - $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$,
  - growth vector $(2, 3, 5)$.

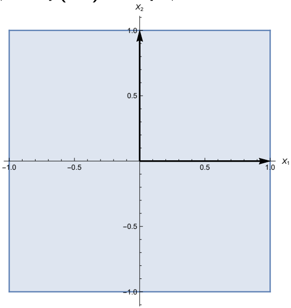


- Cartan group:
  - connected simply connected Lie group M with Lie algebra L,
  - $X_1, \dots, X_5$ — basis left-invariant vector fields on M.
- Model of the Cartan group:
  - $M = \mathbb{R}^5_{x,y,z,v,w}$,
  - $X_1 = \partial_x - \frac{y}{2}\partial_z - \frac{x^2+y^2}{2}\partial_w$, $X_2 = \partial_y + \frac{x}{2}\partial_z + \frac{x^2+y^2}{2}\partial_v$,
    $X_3 = \partial_z + x\partial_v + y\partial_w$, $X_4 = \partial_v$, $X_5 = \partial_w$.

l_∞ sub-Finsler problem on the Cartan group

- Problem statement:

$$\dot q = u_1 X_1 + u_2 X_2, \quad q \in M = \mathbb{R}^5_{x,y,z,v,w}, \quad \|u\|_\infty \le 1,$$
$$q(0) = q_0 = \mathrm{id}, \quad q(T) = q_1, \qquad T \to \min .$$



- Existence of sub-Finsler minimizers:
  - Rashevsky-Chow theorem: Complete controllability,
  - Filippov theorem: Existence of optimal controls.

Pontryagin Maximum Principle

- $\sqcup_{q \in M} T_q^* M = T^* M \ni \lambda$,
- $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, \dots, 5$.

Theorem (PMP)

If $q(t)$, $u(t)$ are optimal, then there exists $\lambda_t \in T^*_{q(t)} M$, $\lambda_t \neq 0$:

(1) $\dot{\lambda}_t = u_1(t)\vec{h}_1 + u_2(t)\vec{h}_2$,

(2) $u_1(t)h_1(\lambda_t) + u_2(t)h_2(\lambda_t) = \max\limits_{\|v\|_\infty \leq 1} (v_1 h_1(\lambda_t) + v_2 h_2(\lambda_t)) =$
$H(\lambda_t) \equiv \text{const} \geq 0$,
$H := |h_1| + |h_2|$.

- Extremal trajectory $q(t)$,
- extremal control $u(t)$,
- extremal λ_t.

Abnormal trajectories

- $H(\lambda_t) \equiv 0$

Theorem

Optimal abnormal controls are

$$u(t) \equiv \text{const}, \qquad \|u\|_\infty = 1.$$

These controls determine optimal synthesis on the abnormal manifold

$$\begin{aligned} A &= \{\exp(u_1 X_1 + u_2 X_2) \mid u_1,\ u_2 \in \mathbb{R}\} \\ &= \{q \in M \mid z = 0,\ v = y(x^2 + y^2)/6,\ w = -x(x^2 + y^2)/6\}. \end{aligned}$$

Types of normal extremals

- $H(\lambda_t) > 0$.
- Singular extremals:

$$h_1(\lambda_t) \equiv 0 \text{ or } h_2(\lambda_t) \equiv 0.$$

- Bang-bang extremals:

$$\text{card}\{t \mid h_1 h_2(\lambda_t) = 0\} < \infty.$$

- Mixed extremals: ones consisting of finite number of singular and bang-bang arcs.

Singular extremals

h_1-singular extremal:

$$h_1(\lambda_t) \equiv 0, \qquad h_2(\lambda_t) \neq 0.$$

Theorem

h_1-singular controls have the form:

$$\forall |u_1(t)| \leq 1, \qquad u_2(t) \equiv \text{const} = \operatorname{sgn} h_2(\lambda_t) \in \{\pm 1\}.$$

All such controls are optimal.

h_2-singular extremals are considered similarly:

$$h_2(\lambda_t) \equiv 0, \qquad h_1(\lambda_t) \neq 0.$$

Attainable set via singular trajectories

Denote by $\mathcal{A}_{q_0}^{\text{sing}}(T)$ the attainable set of $\dot{q} = u_1 X_1 + u_2 X_2$ for time $T > 0$ along singular trajectories starting from the point q_0.

Definition

We call a control $u(t)$ and the corresponding trajectory $q(t)$ geometrically optimal for $t \in [0, T]$, if $q(T) \in \partial \mathcal{A}_{q_0}^{\text{sing}}(T)$.

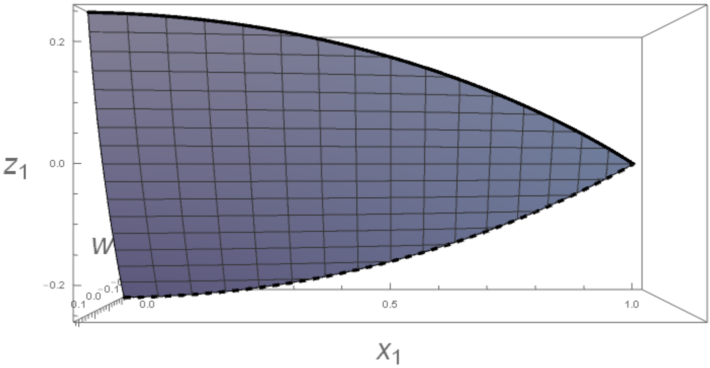
Geometrically optimal singular conrols

Theorem

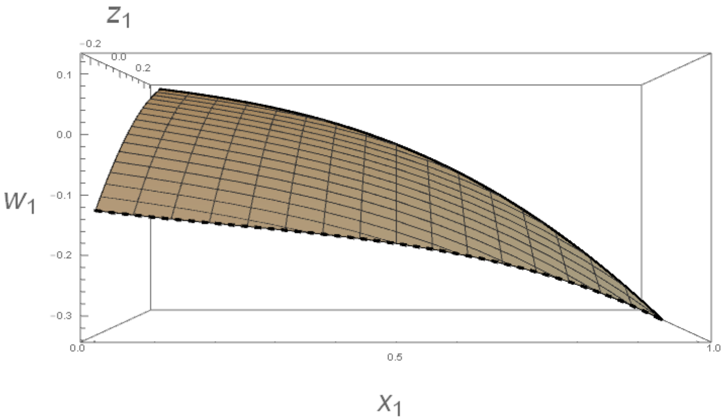
h_1-singular geometrically optimal trajectories with $u_2 \equiv 1$ have one of the following types of piecewise constant control u_1:

1. *with values $\pm 1, \mathfrak{h}_5, \pm 1$ or $\pm 1, \mathfrak{h}_5, \mp 1$ and no restrictions on time periods, where $\mathfrak{h}_5 \in [-1, 1]$ (up to 2 switchings);*
2. *with values $\pm 1, \mp 1, \pm 1, \mp 1$ and time periods T_b, T_1, T_2, T_e, s.t. $0 < T_b < T_2,\ 0 < T_e \leq \frac{T_2 - T_b}{T_2 + T_b} T_1$ (3 switchings).*

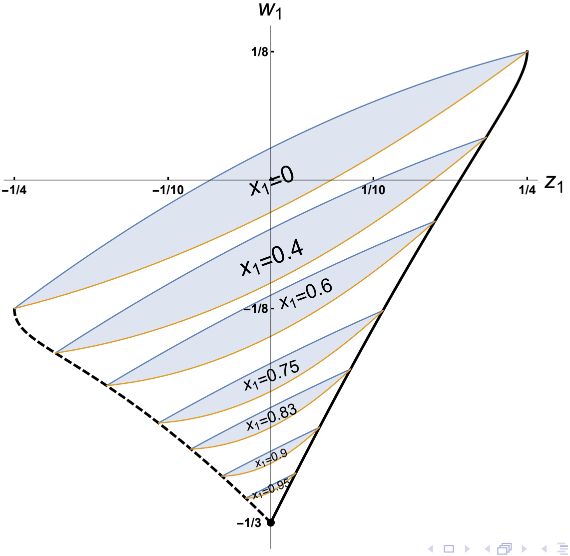
Projection of attainable set to (x_1, z_1, w_1)



Projection of attainable set to (x_1, z_1, w_1)



Projections of sections of attainable set to (z_1, w_1) with fixed values of x_1



$$v_{min}(x_1, z_1, w_1) \le v_1 \le v_{max}(x_1, z_1, w_1).$$

$$\begin{aligned}
w_{mm}(x_1, z_1) &= \frac{1}{6}\left(-x_1(1+x_1^2) + (3 + \operatorname{sgn} z_1) z_1 - \frac{4z_1^2}{1+x_1}\right), \\
v_{max}(x_1, z_1, w_1) &= \frac{1}{12}\left(1 + 3x_1^2 - 6(1-x_1)z_1\right) \\
&+ \frac{\sqrt{2}}{48}\sqrt{9\left(1 - x_1^2 + 4z_1\right)^3 + 8\left(12w_1 + 3x_1 + x_1^3 - 6(1-x_1)z_1\right)^2}, \\
v_{min+}(x_1, z_1, w_1) &= \frac{1}{12}\left(3 + 12w_1 + 3x_1^2 + 2x_1^3 - 6(1-x_1)z_1\right) \\
&- \frac{1}{12}(1-x_1)\sqrt{(1-x_1)(1+24w_1+3x_1+4x_1^3) - 12(1-x_1)z_1 - 12z_1^2}, \\
v_{min-}(x_1, z_1, w_1) &= \frac{1}{12}\Bigg(1 + 3x_1^2 + 6z_1 + 6x_1z_1 + \bigg(\left(1 - 12w_1 - 2x_1 - x_1^2 - 2x_1^3 + 2(1+x_1)z_1\right)^2 \\
&+ 4\left(1 - x_1^2 - 4z_1\right)\left((1+x_1)(6w_1 + x_1 + x_1^3) - 4(1+x_1)z_1 + 4z_1^2\right)\bigg)^{1/2}\Bigg), \\
v_{min}(x_1, z_1, w_1) &= \begin{cases} v_{min+}(-x_1, -z_1, -w_1) & w_1 \ge w_{mm}(x_1, z_1); \\ v_{min+}(x_1, z_1, w_1) & w_1 \le -w_{mm}(-x_1, -z_1); \\ v_{min-}(x_1 \operatorname{sgn} z_1, |z_1|, w_1 \operatorname{sgn} z_1) & -w_{mm}(-x_1, -z_1) \le w_1 \le w_{mm}(x_1, z_1). \end{cases}
\end{aligned}$$

Description of $\mathcal{A}_{q_0}^{\mathrm{sing}}(T)$

Symmetries: dilations and reflections

Theorem

The attainable set $\mathcal{A}_{q_0}^{\mathrm{sing}}(T)$ is described as follows:

$$
\begin{cases}
|x_1 y_1| \leq T, \\
|z_1| \leq T^2 z_{max}(x_1 y_1), \\
\left[\begin{array}{l}
\begin{cases}
|y_1| = T, \\
-T^3 w_{max}(-x_1, -z_1) \leq w_1 \leq T^3 w_{max}(x_1, z_1), \\
T^3 v_{min}(x_1, z_1, w_1) \leq y_1 v_1 \leq T^3 v_{max}(x_1, z_1, w_1);
\end{cases} \\
\begin{cases}
|x_1| = T, \\
-T^3 w_{max}(y_1, -z_1) \leq v_1 \leq T^3 w_{max}(-y_1, z_1), \\
-T^3 v_{max}(y_1, -z_1, -v_1) \leq x_1 w_1 \leq -T^3 v_{min}(y_1, -z_1, -v_1).
\end{cases}
\end{array} \right.
\end{cases}
$$

Thus $\mathcal{A}_{q_0}^{\mathrm{sing}}(T)$ is semi-algebraic.

Bang-bang trajectories

- $\text{card}\{t \mid h_1 h_2(\lambda_t) = 0\} < \infty$,
- $h_1 h_2(\lambda_t) \neq 0 \Rightarrow u_i(t) = \text{sgn}\, h_i(\lambda_t) =: s_i$,
- $u_i(t)$ piecewise constant, $u_i(t) \in \{\pm 1\}$.

$$\dot{h}_1 = -s_1 h_3, \ \dot{h}_2 = s_1 h_3, \ \dot{h}_3 = s_1 h_4 + s_2 h_5, \ \dot{h}_4 = \dot{h}_5 = 0,$$
$$\dot{q} = s_1 X_1 + s_2 X_2.$$

- $h_i(\lambda_t)$, $q(t)$ are piecewise polynomial.
- Factorization by homotheties $\lambda \mapsto k\lambda \qquad \Rightarrow \qquad H(\lambda_t) \equiv 1$.

Reduced Hamiltonian system of PMP for bang-bang trajectories and Casimir functions

- $H = |h_1| + |h_2| = 1$
- $h_1 = \operatorname{sgn}(\cos\theta)\cos^2\theta,\ h_2 = \operatorname{sgn}(\sin\theta)\sin^2\theta,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}$

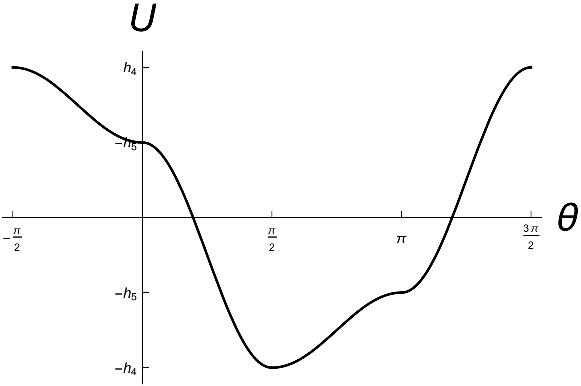
$$\dot{\theta} = \frac{h_3}{|\sin 2\theta|},$$
$$\dot{h}_3 = s_1 h_4 + s_2 h_5.$$

- Casimir functions $\Rightarrow$ integrals:

$$h_4, \quad h_5, \quad E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4.$$

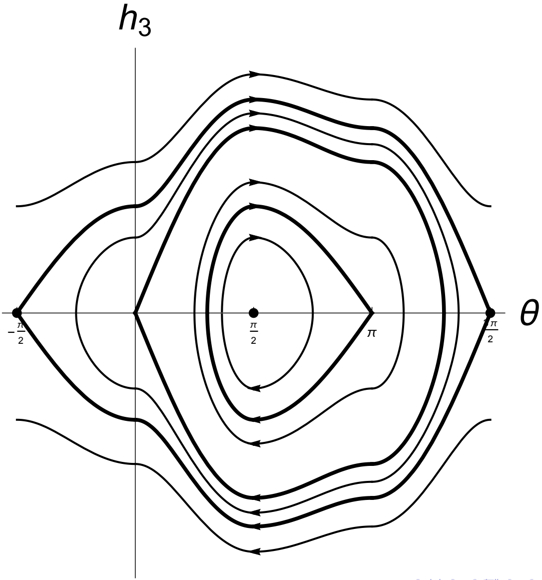
Full and potential energy

- Factorization modulo group of symmetries of the square $\{H = 1\} \Rightarrow$ $h_4 \geq h_5 \geq 0$.
- full energy $E = \frac{h_3^2}{2} + U(\theta)$,
- plot of potential energy $U(\theta) = s_1 h_5 \cos^2\theta - s_2 h_4 \sin^2\theta$ in the case 1) $h_4 > h_5 > 0$:



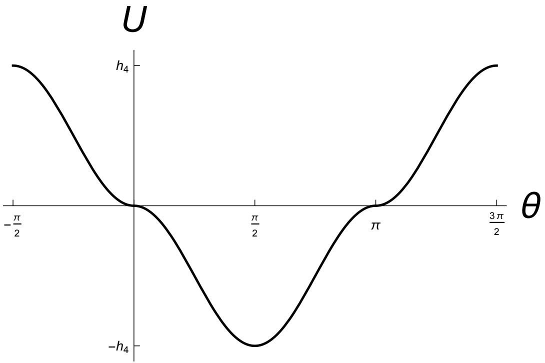
Phase portrait for bang-bang trajectories

Case 1) $h_4 > h_5 > 0$



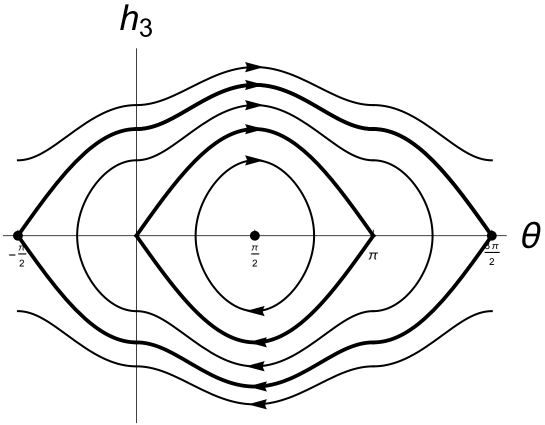
Potential energy for bang-bang trajectories

Case 2) $h_4 > h_5 = 0$



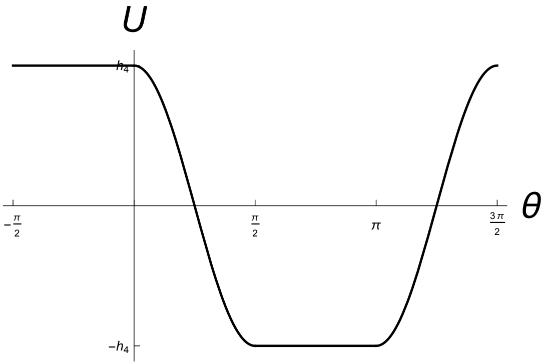
Phase portrait for bang-bang trajectories

Case 2) $h_4 > h_5 = 0$



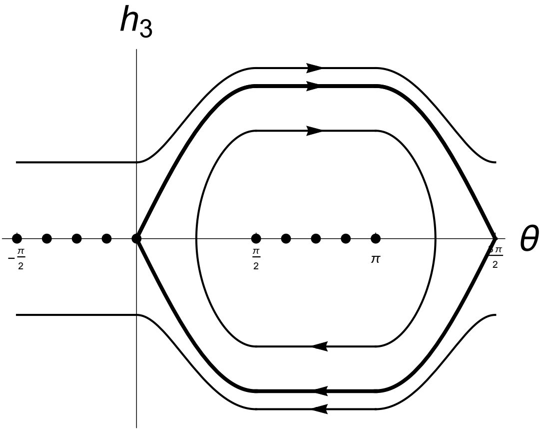
Potential energy for bang-bang trajectories

Case 3) $h_4 = h_5 > 0$



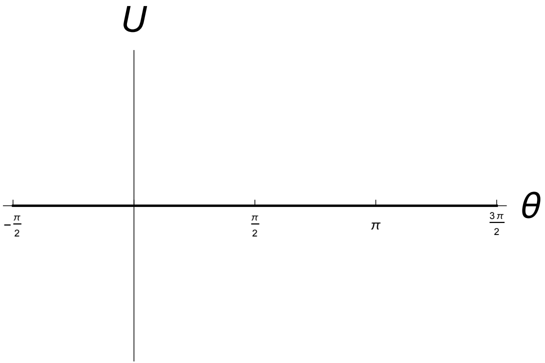
Phase portrait for bang-bang trajectories

Case 3) $h_4 = h_5 > 0$



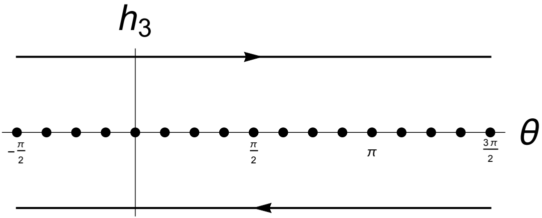
Potential energy for bang-bang trajectories

Case 4) $h_4 = h_5 = 0$



Phase portrait for bang-bang trajectories

Case 4) $h_4 = h_5 = 0$



Stratification of phase portrait for bang-bang trajectories in case 1) $h_4 > h_5 > 0$

- Critical level lines of energy E:

$$C_1 = E^{-1}(-h_4), \quad C_2 = E^{-1}(-h_5), \quad C_3 = E^{-1}(h_5), \quad C_4 = E^{-1}(h_4).$$

- Domains of regular values of energy E:

$$I_1 = E^{-1}(-h_4, -h_5), \quad I_2 = E^{-1}(-h_5, h_5), \quad I_3 = E^{-1}(h_5, h_4),$$
$$N = E^{-1}(h_4, +\infty).$$

$$C = T^*_{q_0} \cap \{H = 1\} = \left(\cup_{i=1}^4 C_i\right) \cup \left(\cup_{i=1}^3 I_i\right) \cup N.$$

Bang-bang flow and its cut time

- Bang-bang flow:

$$\lambda \in C \setminus C_4 \Rightarrow \forall t > 0 \ \exists! \ \lambda_t, \ q(t) =: \mathrm{Exp}(\lambda, t).$$

- Splitting of bang-bang flow:

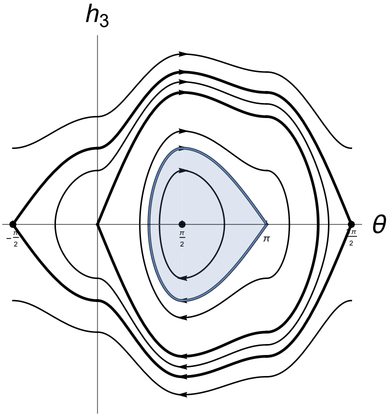
$$\lambda \in C_4 \Rightarrow \forall t > 0 \ \exists \ \{\lambda_t^1, \dots, \lambda_t^N\}, \ \{q^1(t), \dots, q^N(t)\} =: \mathrm{Exp}(\lambda, t).$$

- Cut time along bang-bang extremals:

$$t_{\mathrm{cut}}(\lambda) := \sup\{t > 0 \mid \text{ at least one trajectory } \mathrm{Exp}(\lambda, t) \text{ is optimal}\},$$
$$t_{\mathrm{cut}} \ : \ C \to (0, +\infty] =?$$

Optimality of low-energy bang-bang trajectories

$E \in (-h_4, -h_5] \Leftrightarrow \lambda \in I_1 \cup C_2$



Theorem

If $E \in (-h_4, -h_5]$, then the trajectory $q(t) = \mathrm{Exp}(\lambda, t)$ is singular, thus optimal for $t \in (0, +\infty)$, i.e., $t_{\mathrm{cut}}(\lambda) = +\infty$.

Second order optimality condition

Theorem (A.A.Agrachev, R.V.Gamkrelidze)

Let $(q(t), u(t))$ be an extremal pair and let λ_t be the corresponding extremal, of corank one. Assume that there exist $0 = t_0 < t_1 < \cdots < t_K < t_{K+1} = T$ and $u^0, \ldots, u^K \in \mathbb{R}^2$ such that $u(t)$ is constantly equal to u^j on (t_j, t_{j+1}) for $j = 0, \ldots, K$.
Fix $j = 1, \ldots, K$. For $i = 0, \ldots, K$ let $Y_i = u_1^i X_1 + u_2^i X_2 \in \operatorname{Vec} M$ and define recursively the operators $P_j = P_{j-1} = \mathrm{id}_{\operatorname{Vec} M}$,

$$P_i = P_{i-1} \circ e^{(t_i - t_{i-1}) \operatorname{ad}(Y_{i-1})}, \qquad i = j+1, \ldots, K,$$
$$P_i = P_{i+1} \circ e^{-(t_{i+2} - t_{i+1}) \operatorname{ad}(Y_{i+1})}, \qquad i = 0, \ldots, j-2.$$

Define the vector fields

$$Z_i = P_i(Y_i), \qquad i = 0, \ldots, K.$$

Second order optimality condition (continuation)

Theorem (A.A.Agrachev, R.V.Gamkrelidze)

Let Q be the quadratic form

$$Q(\alpha) = \sum_{0 \leq i < l \leq K} \alpha_i \alpha_l \langle \lambda_{t_j}, [Z_i, Z_l](q(t_j)) \rangle,$$

defined on the space

$$W = \left\{ \alpha = (\alpha_0, \dots, \alpha_K) \in \mathbb{R}^{K+1} \mid \sum_{i=0}^{K} \alpha_i = 0, \ \sum_{i=0}^{K} \alpha_i Z_i(q(t_j)) = 0 \right\}.$$

If Q is not negative semi-definite, then $q(t)$ is not optimal.

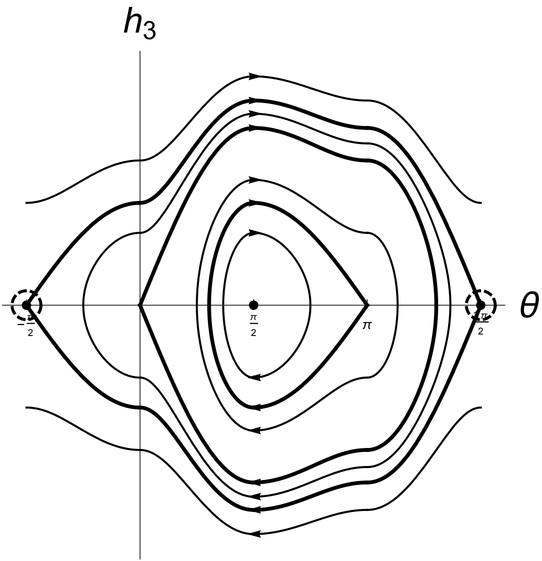
Optimality of high-energy bang-bang trajectories

Theorem

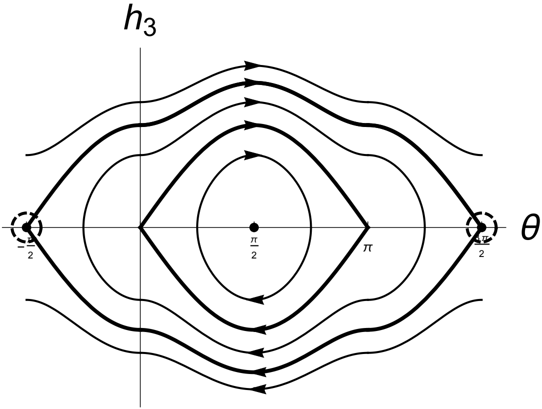
If $E > -h_5$, then:

- *optimal trajectories contain not more than 11 switchings,*
- $t_{\text{cut}}(\lambda) < \infty$.

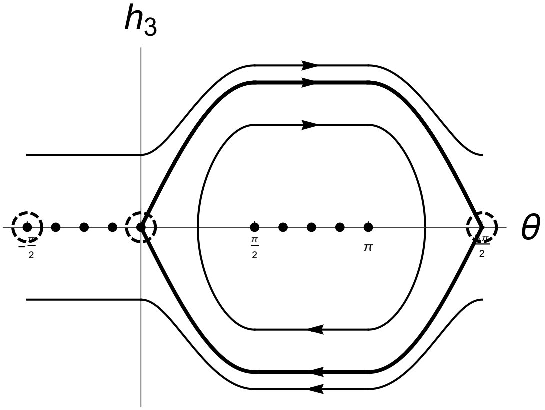
Mixed trajectories in case 1) $h_4 > h_5 > 0$



Mixed trajectories in case 2) $h_4 > h_5 = 0$



Mixed trajectories in case 3) $h_4 = h_5 > 0$



General description of optimal trajectories in the sub-Finsler problem on the Cartan group

Theorem

Sub-Finsler minimizers are curves of two types:

(1) *one component of controls is constantly equal to* 1 *or* -1,

(2) *controls are bang-bang (piecewise constant with values* ± 1*).*

All curves of type (1) *are optimal.*

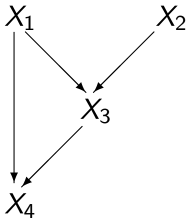
Optimal curves of type (2) *have* ≤ 13 *switchings.*

Corollary

Any two points in the Cartan group can be connected by a piecewise smooth minimizer with up to 14 *smooth pieces.*

Engel group

- Engel algebra:
  - $L = \text{span}(X_1, \ldots, X_4)$,
  - $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$,
  - growth vector $(2, 3, 4)$.

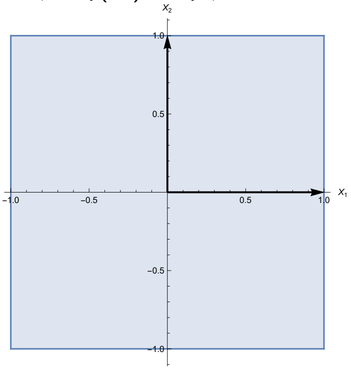


- Engel group:
  - connected simply connected Lie group M with Lie algebra L,
  - $X_1, \ldots, X_4$ — basis left-invariant vector fields on M.
- Model of the Engel group:
  - $M = \mathbb{R}^4_{x,y,z,v}$,
  - $X_1 = \partial_x - \frac{y}{2}\partial_z$, $X_2 = \partial_y + \frac{x}{2}\partial_z + \frac{x^2+y^2}{2}\partial_v$, $X_3 = \partial_z + x\partial_v$, $X_4 = \partial_v$.

l_∞ sub-Finsler problem on the Engel group

- Problem statement:

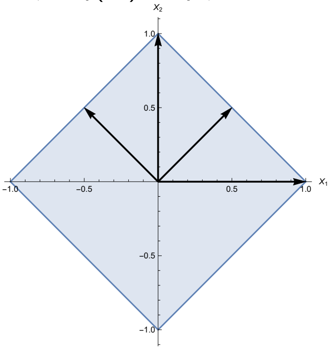
$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M = \mathbb{R}^4_{x,y,z,v}, \quad \|u\|_\infty \leq 1,$$
$$q(0) = q_0 = \text{id}, \quad q(T) = q_1, \qquad T \to \min .$$



l_1 sub-Finsler problem on the Engel group

- Problem statement:

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M = \mathbb{R}^4_{x,y,z,v}, \quad \|u\|_1 \leq 1,$$
$$q(0) = q_0 = \mathrm{id}, \quad q(T) = q_1, \qquad T \to \min .$$



- Similar results for the square rotated by arbitrary angle.

Questions and plans

- Cut time, sub-Finsler sphere and distance,
- More general sub-Finsler problems (polygon or convex set instead of square $\|u\|_{1,\infty} \leq 1$).
- Other Lie groups and homogeneous spaces.