Optimal bang-bang trajectories in sub-Finsler problem on the Cartan group*

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Abstract

The Cartan group is the free nilpotent Lie group of step 3, with 2 generators. This paper studies the Cartan group endowed with the left-invariant sub-Finsler ℓ_∞ norm. We adopt the viewpoint of time-optimal control theory. By Pontryagin maximum principle, all sub-Finsler length minimizers belong to one of the following types: abnormal, bang-bang, singular, and mixed. Bang-bang controls are piecewise controls with values in the vertices of the set of control parameter.

In a previous work, it was shown that bang-bang trajectories have a finite number of patterns determined by values of the Casimir functions on the dual of the Cartan algebra. In this paper we consider, case by case, all patterns of bang-bang trajectories, and obtain detailed upper bounds on the number of switchings of optimal control.

For bang-bang trajectories with low values of the energy integral, we show optimality for arbitrarily large times.

The bang-bang trajectories with high values of the energy integral are studied via a second order necessary optimality condition due to A.Agrachev and R.Gamkrelidze. This optimality condition provides a quadratic form, whose sign-definiteness is related to optimality of bang-bang trajectories. For each pattern of these trajectories, we compute the maximum number of switchings of optimal control. We show that optimal bang-bang controls may have not more than 11 switchings. For particular patterns of bang-bang controls, we obtain better bounds. In such a way we improve the bounds obtained in previous works.

On the basis of results of this work we can start to study the cut time along bang-bang trajectories, i.e., the time when these trajectories lose their optimality. This question will be considered in subsequent works.

Keywords: Sub-Finsler geometry, optimal control, switchings, bang-bang trajectories.
Mathematics Subject Classification 2010: 49K30

1 Introduction

Sub-Finsler geometry on Lie groups has received considerable attention during last years due to its applications, especially in geometric group theory and in harmonic analysis, see articles [10, 6, 4] and introductions of [16, 18] for a broad explanation of the reasons and for several references of the state-of-the-art. To our knowledge the term sub-Finsler appears for the first time in paper [11].

In the case of step two nilpotent Lie groups and homogeneous spaces there is a good understanding of sub-Finsler structures (Heisenberg group, flat Martinet case, Grushin plane) after work [16]. On the other hand, a detailed study of the left-invariant sub-Finsler structure on the free nilpotent Lie group of step 3 with 2 generators (called the Cartan group) began in works [18, 19]. This paper continues those works.

We adopt the viewpoint of time-optimal control theory. Pontryagin maximum principle [13] implies that sub-Finsler length minimizers are of one of the following types: abnormal, bang-bang, singular, or mixed (concatenations of finite number of bang and singular arcs). In this work we study optimality of bang-bang trajectories. There is a finite number of patterns of these trajectories described in [18, 19], and for each pattern we prove an upper bound on the number of switchings of bang-bang optimal control. The main tool here is a second order necessary optimality condition due to A.Agrachev and R. Gamkrelidze [15].

This work has the following structure. In Section 2 we recall the problem statement and some previously obtained results from [18, 19]. In Section 3 the second order optimality condition by Agrachev-Gamkrelidze [15]

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is stated. In Section 4 we prove the main results of this paper: we consider all patterns of bang-bang trajectories, and obtain upper bounds on the number of switchings of the optimal control. Results of Sec. 4 improve Th. 6 [18] by giving detailed bounds on the number of switchings for all patterns of bang-bang optimal control. Finally, some concluding remarks are given in Sec. 5.

2 Problem statement and previous results

Consider the 5-dimensional free nilpotent Lie algebra with 2 generators, of step 3. There exists a basis $L = \mathrm{span}(X_1, \dots, X_5)$ in which the product rule in L takes the form

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad \mathrm{ad}\, X_4 = \mathrm{ad}\, X_5 = 0.$$

The Lie algebra L is called the Cartan algebra, and the corresponding connected simply connected Lie group M is called the Cartan group. We will use the following model:

$$M = \mathbb{R}^5_{x,y,z,v,w},$$

with the Lie algebra L modeled by left-invariant vector fields on $\mathbb{R}^5$

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2+y^2}{2}\frac{\partial}{\partial w},\\
X_2 &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v},\\
X_3 &= \frac{\partial}{\partial z} + x\frac{\partial}{\partial v} + y\frac{\partial}{\partial w},\\
X_4 &= \frac{\partial}{\partial v},\\
X_5 &= \frac{\partial}{\partial w}.
\end{aligned}$$

The product rule in the Cartan group M in this model is given in [12].

Left-invariant ℓ_∞ sub-Finsler problem on the Cartan group is stated as the following time-optimal problem:

$$\dot q = u_1 X_1 + u_2 X_2, \quad q \in M, \quad u \in U = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \le 1\}, \tag{2.1}$$
$$\|u\|_\infty = \max(|u_1|, |u_2|),$$
$$q(0) = q_0 = \mathrm{Id} = (0, \dots, 0), \quad q(T) = q_1, \tag{2.2}$$
$$T \to \min. \tag{2.3}$$

Problem (2.1)–(2.3) was considered first in papers [18, 19]. We recall some results of those papers.

Existence of optimal controls follows from Rashevsky-Chow and Filippov theorem [13].

Pontryagin Maximum Principle implies that optimal abnormal controls are constant.

Introduce linear-on-fibers Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$, $i = 1, \dots, 5$. A normal extremal arc $\lambda_t, t \in I = (\alpha, \beta) \subset [0, T]$ is called:

- a bang-bang arc if
$$\mathrm{card}\{t \in I \mid h_1 h_2(\lambda_t) = 0\} < \infty,$$
- a singular arc if one of the condition holds:
$$\begin{aligned}
h_1(\lambda_t) &\equiv 0, \quad t \in I \quad (h_1\text{-singular arc}), \text{ or}\\
h_2(\lambda_t) &\equiv 0, \quad t \in I \quad (h_2\text{-singular arc}),
\end{aligned}$$
- a mixed arc if it consists of a finite number of bang-bang and singular arcs.

Singular controls have one of components constantly equal to 1 or -1, thus they are optimal. The fix-time attainable set along singular trajectories was explicitly described and was shown to be semi-algebraic.

Bang-bang extremal trajectories satisfy the Hamiltonian system with the Hamiltonian function $H = |h_1| + |h_2|$:

$$\begin{cases} \dot{h}_1 = -s_2 h_3, \\ \dot{h}_2 = s_1 h_3, \\ \dot{h}_3 = s_1 h_4 + s_2 h_5, \\ \dot{h}_4 = \dot{h}_5 = 0, \\ \dot{q} = s_1 X_1 + s_2 X_2. \end{cases} \tag{2.4}$$

The dual of the Lie algebra $L^* = T^*_{\mathrm{Id}} M$ has Casimir functions h_4, h_5, $E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$, thus Hamiltonian system (2.4) has integrals h_4, h_5, E, and H.

The mapping $(\lambda, q) \mapsto (k\lambda, q)$, $k > 0$, preserves extremal trajectories, thus we can consider only the reduced case

$$H(\lambda) \equiv 1.$$

With the use of the coordinate $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$:

$$h_1 = \operatorname{sgn}(\cos\theta)\cos^2\theta, \quad h_2 = \operatorname{sgn}(\sin\theta)\sin^2\theta,$$

the vertical part of Hamiltonian system (2.4) reduces to the following system:

$$\begin{cases} \dot{\theta} = \frac{h_3}{|\sin 2\theta|}, \quad \theta \neq \frac{\pi n}{2}, \\ \dot{h}_3 = s_1 h_4 + s_2 h_5, \quad s_1 = \operatorname{sgn}\cos\theta, \quad s_2 = \operatorname{sgn}\sin\theta. \end{cases} \tag{2.5}$$

Consider the cylinder

$$C = T^*_{q_0} M \cap \{H = 1\}.$$

In work [18] it was shown that bang-bang trajectories can be represented as images of an exponential mapping: $\{q(t)\} = \mathrm{Exp}(\lambda, t)$, $\lambda \in C$, $t > 0$. The exponential mapping is single-valued for generic $\lambda \in C$, and is multi-valued for certain special subsets of C, see [18].

System (2.5) is preserved by the group of symmetries of the square $\{(h_1, h_2) \in \mathbb{R}^2 \mid |h_1| + |h_2| = 1\}$. Thus in the study of system (2.5) we can restrict ourselves by the case $h_4 \geq h_5 \geq 0$. This group of symmetries reduces the cylinder C to the fundamental domain of the group $\{\lambda \in C \mid h_4 \geq h_5 \geq 0\}$. Further, this fundamental domain admits a stratification by invariant subsets of the Hamiltonian system (2.5):

$$\begin{aligned} &\{\lambda \in C \mid h_4 \geq h_5 \geq 0\} = \cup_{i=1}^4 C^i, \\ &C^1 = \{\lambda \in C \mid h_4 > h_5 > 0\}, \\ &C^2 = \{\lambda \in C \mid h_4 > h_5 = 0\}, \\ &C^3 = \{\lambda \in C \mid h_4 = h_5 > 0\}, \\ &C^4 = \{\lambda \in C \mid h_4 = h_5 = 0\}. \end{aligned}$$

Further, we have the following stratifications:

$$\begin{aligned} &C^1 = \cup_{i=1}^8 C^1_i, \\ &C^1_1 = E^{-1}(-h_4), \quad C^1_2 = E^{-1}(-h_4, -h_5), \quad C^1_3 = E^{-1}(-h_5), \quad C^1_4 = E^{-1}(-h_5, h_5), \\ &C^1_5 = E^{-1}(h_5), \quad C^1_6 = E^{-1}(h_5, h_4), \quad C^1_7 = E^{-1}(h_4), \quad C^1_8 = E^{-1}(h_4, +\infty), \\ &C^2 = \cup_{i=1}^6 C^2_i, \\ &C^2_1 = E^{-1}(-h_4), \quad C^2_2 = E^{-1}(-h_4, 0), \quad C^2_3 = E^{-1}(0), \\ &C^2_4 = E^{-1}(0, h_4), \quad C^2_5 = E^{-1}(h_4), \quad C^2_6 = E^{-1}(h_4, +\infty), \\ &C^3 = \cup_{i=1}^4 C^3_i, \\ &C^3_1 = E^{-1}(-h_4), \quad C^3_2 = E^{-1}(-h_4, 0), \quad C^3_3 = E^{-1}(h_4), \quad C^3_4 = E^{-1}(h_4, +\infty), \\ &C^4 = C^4_1 \cup C^4_2, \\ &C^4_1 = E^{-1}(0), \quad C^4_2 = E^{-1}(0, +\infty). \end{aligned}$$

In paper [19] was obtained the following optimality result for bang-bang trajectories with low energy E.

Theorem 1 (Th. 2 [19]). *If a bang-bang extremal λ_t, $t \in [0, +\infty)$, satisfies the inequality*

$$\min(-|h_4|, -|h_5|) < E \leq \max(-|h_4|, -|h_5|) \tag{2.6}$$

then it is optimal.

3 Theorem by Agrachev-Gamkrelidze

We obtain an upper bound on the number of switchings on optimal bang-bang trajectories via the following theorem due to A. Agrachev and R. Gamkrelidze.

Theorem 2 ([15, 16])**.** *Let $(q(\cdot), u(\cdot))$ be an extremal pair for problem (2.1)–(2.3) and let $\lambda.$ be an extremal lift of $q(\cdot)$. Assume that $\lambda.$ is the unique extremal lift of $q(\cdot)$, up to multiplication by a positive scalar. Assume that there exist $0 = t_0 < t_1 < t_2 < \cdots < t_k < \tau_{k+1} = T$ and $u^0, \dots, u^k \in U$ such that $u(\cdot)$ is constantly equal to u^j on (τ_j, τ_{j+1}) for $j = 0, \dots, k$.*

Fix $j = 1, \dots, k$. For $i = 0, \dots, k$ let $Y_i = u_1^i X_1 + u_2^i X_2$ and define recursively the operators

$$\begin{aligned} P_j &= P_{j-1} = \mathrm{Id}_{\mathrm{Vec}(M)}, \\ P_i &= P_{i-1} \circ e^{(t_i - t_{i-1})\,\mathrm{ad}\,Y_{i-1}}, \qquad i = j+1, \dots, k, \\ P_i &= P_{i+1} \circ e^{-(t_{i+2} - t_{i+1})\,\mathrm{ad}\,Y_{i+1}}, \qquad i = 0, \dots, j-2. \end{aligned}$$

Define the vector fields

$$Z_i = P_i(Y_i), \quad i = 0, \dots, k.$$

Let Q be the quadratic form

$$Q(\alpha) = \sum_{0 \le i < l \le k} \alpha_i \alpha_l \langle \lambda_{t_j}, [Z_i, Z_l](q(t_j)) \rangle,$$

defined on the space

$$W = \left\{ \alpha = (\alpha_0, \dots, \alpha_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^{k} \alpha_i = 0, \quad \sum_{i=0}^{k} \alpha_i Z_i(q(t_j)) = 0 \right\}.$$

If Q is not negative-semidefinite, then $q(\cdot)$ is not optimal.

4 Bounds on the number of switchings

Now we obtain bounds on the number of switchings for bang-bang optimal trajectories $\mathrm{Exp}(\lambda, t)$ with $\lambda \in \cup_{i=1}^{4} C^i$, case by case.

4.1 Case $\lambda \in C^1$

In the case $\lambda \in C^1$ system (2.5) has phase portrait given in Fig. 1.

4.1.1 Low values of integral E

Theorem 1 implies the following statement.

Corollary 1. *If $\lambda \in C_1^1 \cup C_2^1 \cup C_3^1$, then the trajectory $\mathrm{Exp}(\lambda, t)$, $t \in [0, +\infty)$, is optimal.*

4.1.2 High values of integral E

We apply Th. 2 and obtain the following upper bounds on the number of switchings on optimal bang-bang trajectories. An example of detailed computation on the basis of Th. 2 is given in the proof of Th. 5 [19].

Theorem 3. *Let $\lambda \in \cup_{i=4}^{8} C_i^1$. Then the bang-bang trajectory $\mathrm{Exp}(\lambda, t)$ with k switchings is not optimal, where k is given by the following tables:*

- $\lambda \in C_4^1 \cup C_5^1 \Rightarrow$ *Table* 1,
- $\lambda \in C_6^1 \Rightarrow$ *Table* 2,
- $\lambda \in C_7^1 \Rightarrow$ *Table* 3,
- $\lambda \in C_8^1 \Rightarrow$ *Table* 4.

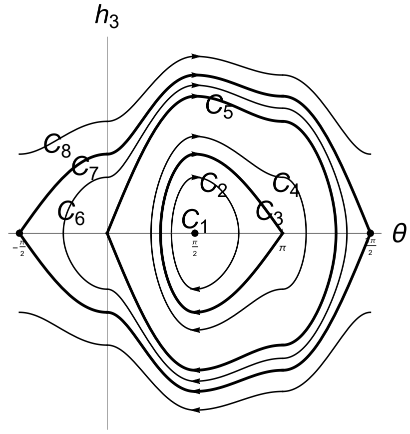

Figure 1: Phase portrait of system (2.5) in case $\lambda \in C^1$

Start	$(+,+)$	$(-,+)_+$	$(-,-)$	$(-,+)_-$
k	8	9	9	8

Table 1: $\lambda \in C^1_4 \cup C^1_5$

Start	$(+,+)_+$	$(-,+)_+$	$(-,-)$	$(-,+)_-$	$(+,+)_-$	$(+,-)$
k	9	10	10	10	9	11

Table 2: $\lambda \in C^1_6$

N	Start	k	N	Start	k	N	Start	k	N	Start	k
1	$(+,+)+++$	9	9	$(+,-)+++$	8	17	$(-,+)+++$	8	25	$(-,-)+++$	8
2	$(+,+)++-$	9	10	$(+,-)++-$	9	18	$(-,+)++-$	9	26	$(-,-)++-$	8
3	$(+,+)+-+$	8	11	$(+,-)+-+$	11	19	$(-,+)+-+$	9	27a	$(-,-)+-++$	10
4	$(+,+)+--$	8	12	$(+,-)+--$	9	20	$(-,+)+--$	7	27b	$(-,-)+-+-$	11
5	$(+,+)-++$	9	13	$(+,-)-++$	8	21	$(-,+)-++$	10	28	$(-,-)+--$	8
6	$(+,+)-+-$	9	14	$(+,-)-+-$	12	22a	$(-,+)-+-+$	12	29	$(-,-)-++$	7
7	$(+,+)--+$	10	15a	$(+,-)--++$	9	22b	$(-,+)-+--$	11	30	$(-,-)-+-$	7
8	$(+,+)---$	9	15b	$(+,-)--+-$	10	23	$(-,+)--+$	10	31	$(-,-)--+$	7
			16	$(+,-)---$	8	24	$(-,+)---$	8	32	$(-,-)---$	7

Table 3: $\lambda \in C^1_7$

Start	$(+,-)$	$(+,+)$	$(-,+)$	$(-,-)$
k	8	8	6	6

Table 4: $\lambda \in C^1_8$

Remark. *We explain now how Tables 1–4 should be read.*

Consider Table 1. The first line — Start — gives the values of $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0))$ *and, if necessary, the signs of* $h_3(0)$ *as a lower index. For example, the first column of Table 1 corresponds to* $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0)) = (+1, +1)$*. The second column of Table 1 corresponds to the initial values* $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0)) = (-1, +1)$ *and* $\operatorname{sgn} h_3(0) = +1$*. The second line of Table 1 gives the number of switchings* k *for the corresponding* $\lambda \in C_4^1 \cup C_5^1$ *such that the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *is not optimal. Similar agreement is applied for Tables 2, 4.*

Table 3 should be read as follows. Consider, e.g., entry $N = 10$ *of Table 3. The sequence of signs* $(+,-)++-$ *has the following meaning:*

- *the signs* $(+,-)$ *determine the initial control* $(u_1(0), u_2(0)) = (\operatorname{sgn} h_1(0), \operatorname{sgn} h_2(0)) = (+1, -1)$,
- *the subsequent signs* $++-$ *determine the signs of* $h_3(t)$ *between switchings of control, i.e.,*
  - $\operatorname{sgn} h_3(t) = +1,\ t \in [0, t_1]$;
  - $\operatorname{sgn} h_3(t) = +1,\ t \in [t_1, t_2]$;
  - $\operatorname{sgn} h_3(t) = -1,\ t \in [t_2, T]$,

  where t_1, t_2 *are switching times at which* $h_3(t)$ *vanishes.*

The number 9 for entry $N = 10$ *of Table 3 gives the number of switchings of a non-optimal bang-bang control.*

The same agreement on reading similar tables is used in subsequent subsections.

The below cases $\lambda \in_{i=2}^{4} C^i$ are considered similarly to the above case $\lambda \in C^1$.

4.2 Case $\lambda \in C^2$

In the case $\lambda \in C^2$ system (2.5) has phase portrait given in Fig. 2.

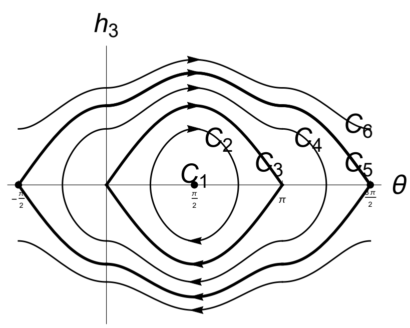

Figure 2: Phase portrait of system (2.5) in case $\lambda \in C^2$

4.2.1 Low values of integral E

Theorem 1 implies the following statement.

Corollary 2. *If* $\lambda \in C_1^2 \cup C_2^2 \cup C_3^2$*, then the trajectory* $\operatorname{Exp}(\lambda, t)$*,* $t \in [0, +\infty)$*, is optimal.*

4.2.2 High values of integral E

Theorem 4. *Let* $\lambda \in \cup_{i=4}^{2} C_i^2$*. Then the bang-bang trajectory* $\operatorname{Exp}(\lambda, t)$ *with* k *switchings is not optimal, where* k *is given by the following tables:*

- $\lambda \in C_4^2 \Rightarrow$ *Table 5,*
- $\lambda \in C_6^2 \Rightarrow$ *Table 6.*

Start	$(+,+)_+$	$(-,+)_+$	$(-,-)$	$(-,+)_-$	$(+,+)_-$	$(+,-)$
k	8	9	8	8	9	8

Table 5: $\lambda \in C_4^2$

Start	$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$
k	7	6	6	7

Table 6: $\lambda \in C_6^2$

4.3 Case $\lambda \in C^3$

In the case $\lambda \in C^3$ system (2.5) has phase portrait given in Fig. 3.

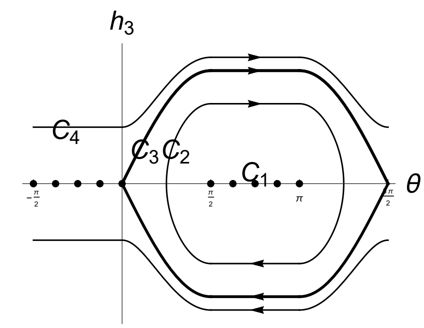

Figure 3: Phase portrait of system (2.5) in case $\lambda \in C^3$

4.3.1 Low values of integral E

Theorem 1 implies the following statement.

Corollary 3. *If $\lambda \in C_1^3$, then the trajectory* $\mathrm{Exp}(\lambda, t)$, $t \in [0, +\infty)$, *is optimal.*

4.3.2 High values of integral E

Theorem 5. *Let $\lambda \in \cup_{i=2}^4 C_i^3$. Then the bang-bang trajectory* $\mathrm{Exp}(\lambda, t)$ *with k switchings is not optimal, where k is given by the following tables:*

- $\lambda \in C_2^3 \cup C_3^3 \Rightarrow$ *Table* 7,
- $\lambda \in C_4^3 \Rightarrow$ *Table* 8.

4.4 Case $\lambda \in C^4$

In the case $\lambda \in C^4$ system (2.5) has phase portrait given in Fig. 4.

4.4.1 Low values of integral E

Theorem 1 implies the following statement.

Corollary 4. *If $\lambda \in C_1^4$, then the trajectory* $\mathrm{Exp}(\lambda, t)$, $t \in [0, +\infty)$, *is optimal.*

4.4.2 High values of integral E

Theorem 6. *Let $\lambda \in C_2^4$. Then the bang-bang trajectory* $\mathrm{Exp}(\lambda, t)$ *with $k = 7$ switchings is not optimal.*

Start	$(+,+)$	$(-,+)_+$	$(-,-)$	$(-,+)_-$
k	7	6	7	6

Table 7: $\lambda \in C_2^3 \cup C_3^3$

Start	$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$
k	7	6	7	7

Table 8: $\lambda \in C_3^4$

5 Conclusion

An obvious next question that arises after the upper bounds on the number of switchings of optimal bang-bang control is the following one: when exactly do the bang-bang trajectories lose their optimality? That is, we would like to describe the cut time along bang-bang trajectories. We hope that this is possible by (extension of) the symmetry method applied successfully for description of cut time in several sub-Riemannian and Riemannian problems [20, 21, 22, 23, 24]. This question will be studied in forthcoming papers.

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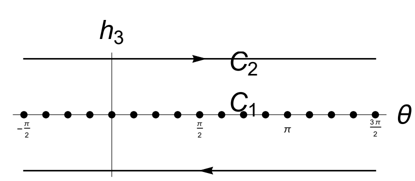

Figure 4: Phase portrait of system (2.5) in case $\lambda \in C^4$

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