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MATHEMATICS

Sub-Riemannian Geodesics in SO(3) with Application to Vessel Tracking in Spherical Images of Retina[1]

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Abstract— In order to detect vessel locations in spherical images of retina we consider the problem of minimizing the functional $\int_0^l \mathfrak{C}(\gamma(s))\sqrt{\xi^2 + k_g^2(s)}ds$ for a curve γ on a sphere with fixed boundary points and directions. The total length l is free, s denotes the spherical arclength, and k_g denotes the geodesic curvature of γ. Here the smooth external cost $\mathfrak{C} \geq \delta > 0$ is obtained from spherical data. We lift this problem to the sub-Riemannian (SR) problem in Lie group SO(3) and propose numerical solution to this problem with consequent comparison to exact solution in the case $\mathfrak{C} = 1$. An experiment of vessel tracking in a spherical image of the retina shows a benefit of using SO(3) geodesics.

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In computer vision, it is common to extract salient curves in images via data-driven minimal paths or geodesics [1]. The minimizing geodesic is defined as the curve that minimizes the length functional, which is typically weighted by a cost function with high values at image locations with low curve saliency. Another set of geodesic methods, inspired by the psychology of vision, was developed in [3, 4]. In these articles sub-Riemannian (SR) geodesics in respectively the Heisenberg H(3) and the Euclidean motion group SE(2) are proposed as a model for contour perception and contour integration.

The combination of such contour perception models with data-driven geodesic methods has been presented in [5]. There, a computational framework for tracking of salient curves via globally optimal data-driven sub-Riemannian geodesics on the Euclidean motion group SE(2) has been presented. In [5] the framework was used for tracking of retinal vessels in flat images of retina.

In this work we extend the framework for tracking of vessels in *spherical* images of the retina with reduced distortion. This requires a sub-Riemannian manifold structure in the group SO(3) acting transitively on the 2-sphere S^2.

Let $S^2 = \{\mathbf{n} \in \mathbb{R}^3 \mid \|\mathbf{n}\| = 1\}$ be a sphere of unit radius. We consider the problem $\mathbf{P_{curve}}$ (see Fig. 1), which is for given boundary points $\mathbf{n}_0, \mathbf{n}_1 \in S^2$ and directions $\mathbf{n}'_0 \in T_{\mathbf{n}_0}(S^2)$, $\mathbf{n}'_1 \in T_{\mathbf{n}_1}(S^2)$, $\|\mathbf{n}'_0\| = \|\mathbf{n}'_1\| = 1$ to find a smooth curve $\mathbf{n}(\cdot): [0, l] \to S^2$ that satisfies the boundary conditions

$$\mathbf{n}(0) = \mathbf{n}_0, \quad \mathbf{n}(l) = \mathbf{n}_1, \quad \mathbf{n}'(0) = \mathbf{n}'_0, \quad \mathbf{n}'(l) = \mathbf{n}'_1,$$

and for given $\xi > 0$ minimizes the functional

$$\mathcal{L}(\mathbf{n}(\cdot)) := \int_0^l \mathfrak{C}(\mathbf{n}(s))\sqrt{\xi^2 + k_g^2(s)}ds,$$

where $k_g(s)$ denotes the geodesic curvature on S^2 of $\mathbf{n}(\cdot)$ evaluated in time s, the total length l is free, s denotes the spherical arclength, and $\mathfrak{C}: S^2 \to [\delta, +\infty)$, $\delta > 0$, is a smooth function "external cost."

We call $\mathbf{P_{mec}}$ the following SR problem in SO(3):

$$\dot{R} = u_1 X_1 + u_2 X_2, \quad R(0) = \mathrm{Id}, \quad R(T) = R_{\mathrm{fin}},$$

$$\mathcal{L}(R(\cdot)) := \int_0^T C(R(t))\sqrt{\xi^2 u_1^2(t) + u_2^2(t)}dt \to \min,$$

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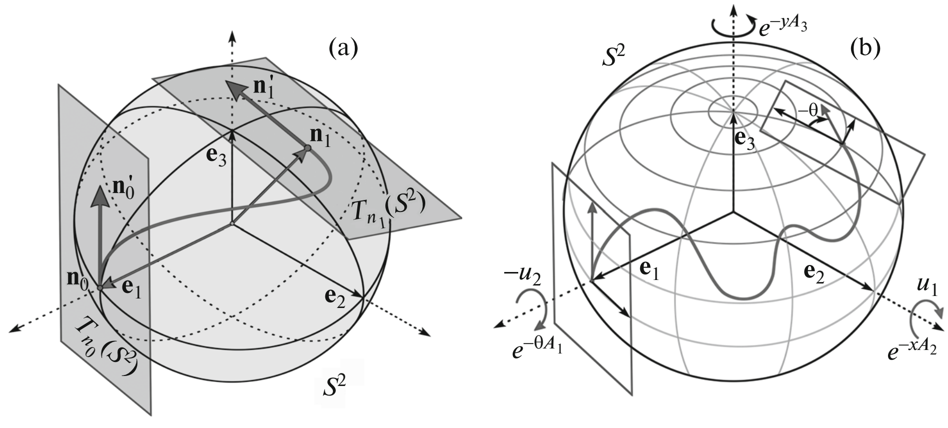

Fig. 1. Problems $\mathbf{P_{curve}}$ (a) on a sphere and (b) $\mathbf{P_{mec}}$ in SO(3).

where $R \in SO(3)$, X_i are basis left-invariant vector fields in SO(3), $\xi > 0$ is a constant and terminal time $T > 0$ is free.

The external cost C: $SO(3) \to [\delta, +\infty)$, $\delta > 0$, is a smooth function that is typically obtained by lifting the external cost $\mathfrak{C}$ from the sphere S^2 to the group SO(3), i.e., $C(R) = \mathfrak{C}(R\mathbf{e}_1)$.

We call a spherical projection the projection map $SO(3) \ni R \mapsto R\mathbf{e}_1 \in S^2$. We show that the spherical projection of certain minimizers of $\mathbf{P_{mec}}$ provides solution of problem $\mathbf{P_{curve}}$. More precisely this only holds for the minimizers whose spherical projection does not have a cusp. The spherical projection of a minimizer of $\mathbf{P_{mec}}$ is said to have a cusp at $t = t^n_{\text{cusp}}$ if the corresponding control $u_1(t)$ changes a sign locally, i.e., there exists $\epsilon > 0$, s.t. $u_1(a)u_1(b) < 0$ for all $a \in (t^n_{\text{cusp}} - \epsilon, t^n_{\text{cusp}})$ and $b \in (t^n_{\text{cusp}}, t^n_{\text{cusp}} + \epsilon)$. Before the first cusp a minimizer of $\mathbf{P_{mec}}$ can be parameterized by spherical arclength $s \in [0, s_{\max})$.

Theorem 1. *Let* $R(t)$, $t \in [0, T]$ *be a minimizer of* $\mathbf{P_{mec}}$ *parameterized by* SR-*arclength*, *and let the corresponding optimal control* $(u_1(t), u_2(t))$ *satisfy the inequality* $u_1(t) > 0$ *for all* $t \in [0, T]$.

Set $\mathbf{n}_0 = \mathbf{e}_1$, $\quad \mathbf{n}_1 = R(T)\mathbf{e}_1$, $\quad \mathbf{n}_0' = \mathbf{e}_3$, $\quad \mathbf{n}_1' = R(T)\mathbf{e}_3$.

For such boundary conditions $\mathbf{P_{curve}}$ *has a minimizer* $\mathbf{n}(s)$,

$$\mathbf{n}(s) = R(t(s))\mathbf{e}_1, \qquad \begin{cases} u_1(t) = \dfrac{ds}{dt}(t), \\ u_2(t) = k_g(s(t))\dfrac{ds}{dt}(t), \end{cases}$$

$$t(s) = \int_0^s \mathfrak{C}(\mathbf{n}(\sigma))\sqrt{\xi^2 + k_g^2(\sigma)}d\sigma,$$

for $0 \le s \le l < s_{\max}$, *and* $T = t(l)$.

We perform an analytical study of $\mathbf{P_{mec}}$ in case $C = 1$. The Lie group SO(3) is the group of all rotations about the origin in $\mathbb{R}^3$. We denote a counterclockwise rotation around axis $\mathbf{a} \in S^2$ with angle ϕ via $R_{\mathbf{a},\phi}$, and parameterize SO(3) by 3 angles x, y, θ as $R(x, y, \theta) = R_{\mathbf{e}_3,y}R_{\mathbf{e}_2,-x}R_{\mathbf{e}_1,\theta}$. Then we apply Pontryagin maximum principle (PMP), which is the first order necessary condition for optimality [9], and we derive the explicit formulas for the sub-Riemannian geodesics. Also, we present new formulas for SR geodesics parameterized by spherical arclength before the first cusp in their spherical projection. Such "cuspless" geodesics are determined by the Hamiltonian system

$$\begin{cases} h_1(s) = \xi^2 \dfrac{ds}{dt} \ge 0, \\ h_2'(s) = h_3(s), \\ h_3'(s) = (\xi^2 - 1)h_2(s), \end{cases}$$

– vertical part,

$$\begin{cases} x'(s) = \cos\theta(s), \\ y'(s) = -\sec x(s)\sin(s), \\ \theta'(s) = \sin\theta(s)\tan x(s) + \dfrac{\xi h_2(s)}{\sqrt{1 - h_2^2(s)}}, \end{cases}$$

–horizontal part,

with the boundary conditions

$$h_2(0) = h_2^0, \quad h_3(0) = h_3^0, \quad x(0) = 0,$$
$$y(0) = 0, \quad \theta(0) = 0.$$

The formulas for "cuspless" geodesics are simpler than the general formulas for SR geodesics in SO(3). In contrast to the general case, where the Jacobi elliptic functions appear, here only single elliptic integral is involved.

Theorem 2. *The unique solution of the Hamiltonian system for "cuspless" geodesics is defined for* $s \in [0, s_{max}(h(0)))$, *where*

$$s_{\max}(h(0)) = \begin{cases} \dfrac{\operatorname{sgn}(h_3^0) - h_2^0}{h_3^0} & \text{for} \quad \chi = 0, \quad h_3^0 \neq 0, \\ \dfrac{1}{\chi}\log\left(\dfrac{s_1(\sqrt{\kappa} + \chi)}{h_2^0\chi + h_3^0}\right) & \text{for} \quad \chi \neq 0, \\ \kappa \geq 0, \quad h_2^0\chi + h_3^0 \neq 0, & \\ +\infty \quad \text{otherwise}, & \end{cases}$$

with $\chi = \sqrt{\xi^2 - 1}$ (*where one should take principal square root*), $s_1 = \operatorname{sgn}(\operatorname{Re}(h_2^0\chi + h_3^0))$ *and* $\kappa = (h_3^0)^2 + (1 - (h_2^0)^2)\chi^2 \in \mathbb{R}$.

The solution to the vertical part is given by

for $\chi \neq 0$:

$$\begin{cases} h_2(s) = h_2^0 \cosh s\chi + \dfrac{h_3^0}{\chi}\sinh s\chi, \\ h_3(s) = h_3^0 \cosh s\chi + \chi h_2^0 \sinh s\chi, \\ h_1(s) = \xi\sqrt{1 - h_2^2(s)}, \end{cases}$$

for $\chi = 0$:

$$\begin{cases} h_2(s) = h_2^0 + h_3^0 s, \\ h_3(s) = h_3^0, \\ h_1(s) = \sqrt{1 - h_2^2(s)}. \end{cases}$$

The solution to the horizontal part is given by

$$x(s) = \arg(\sqrt{R_{11}^2(s) + R_{21}^2(s)} + iR_{31}(s)),$$

$$y(s) = \arg(R_{11}(s) + iR_{21}(s)),$$

$$\theta(s) = \arg(R_{33}(s) + iR_{32}(s)),$$

where $R(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) & R_{13}(s) \\ R_{21}(s) & R_{22}(s) & R_{23}(s) \\ R_{31}(s) & R_{32}(s) & R_{33}(s) \end{pmatrix} = D_0^T \mathbf{e}^{\tilde{y}(s)A_3}\mathbf{e}^{-\tilde{x}(s)A_2}\mathbf{e}^{\tilde{\theta}(s)A_1}$,

$$D_0 = \frac{1}{M}\begin{pmatrix} \mu & \dfrac{\xi h_2^0\sqrt{1 - (h_2^0)^2}}{\mu} & -\dfrac{h_2^0 h_3^0}{\mu} \\ 0 & \dfrac{Mh_3^0}{\mu} & \dfrac{\xi M\sqrt{1 - (h_2^0)^2}}{\mu} \\ h_2^0 & -\xi\sqrt{1 - (h_2^0)^2} & h_3^0 \end{pmatrix}, \text{ and}$$

$$\tilde{x}(s) = \arg(\sqrt{M^2 - h_2^2(s)} + ih_2(s)),$$

$$\tilde{y}(s) = \xi M^2\left(\int_0^s \frac{\sqrt{1 - h_2^2(\sigma)}}{M^2 - h_2^2(\sigma)}d\sigma\right),$$

$$\tilde{\theta}(s) = \arg(h_3(s) - i\xi\sqrt{1 - h_2^2(s)}),$$

with $\mu = \sqrt{M^2 - (h_2^0)^2}$, $M = \sqrt{\xi^2(1 - (h_2^0)^2) + (h_2^0)^2 + (h_3^0)^2}$.

We use these exact formulas to verify our PDE-approach to solve the problem $\mathbf{P_{mec}}$ for general external cost case $C \neq 1$. The approach is summarized by the following theorem.

Theorem 3. *Let* $\mathcal{W}(R)$ *be a viscosity solution of the eikonal system*

$$\begin{cases} \sqrt{\dfrac{(X_1|_R(\mathcal{W}))^2}{\xi^2} + (X_2|_R(\mathcal{W}))^2} = C(R), \\ \text{for} \quad \mathrm{Id} \neq R \in SO(3), \\ \mathcal{W}(\mathrm{Id}) = 0. \end{cases}$$

A SR length minimizer $R(t) = R_b(\mathcal{W}(R_{\text{fin}}) - t)$ *ending at* R_{fin} *is found by backward integration for* $t \in [0, \mathcal{W}(R_{\text{fin}})]$

$$\dot{R}_b(t) = -u_1(t)X_1|_{R_b(t)} - u_2(t)X_2|_{R_b(t)}, \quad R_b(0) = R_{\text{fin}},$$

where $u_1(t) = \dfrac{X_1|_{R_b(t)}(\mathcal{W})}{(\xi C(R_b(t)))^2}$ *and* $u_2(t) = \dfrac{X_2|_{R_b(t)}(\mathcal{W})}{(C(R_b(t)))^2}$.

Solving numerically this system for external cost C induced by a spherical image of the retina we obtain an effective method for vessel detection. Comparison with tracking via SE(2)-geodesics (see [5]) shows, that in general the SO(3)-geodesics have a slower variation in curvature, and are less eager to take short-cuts. Furthermore, there are visible differences between geodesic curvature of data-driven SR geodesics on the sphere and the curvature of their planar projections. As in retinal imaging applications curvature is considered as a relevant biomarker [2] for detection of diabetic retinopathy and other systemic diseases, the data-driven SR geodesic model in SO(3) is a relevant extension of data-driven geodesic model in SE(2) [5].

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