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Extremal Controls in the Sub-Riemannian Problem on the Group of Motions of Euclidean Space

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Abstract—For the sub-Riemannian problem on the group of motions of Euclidean space we present explicit formulas for extremal controls in the special case where one of the initial momenta is fixed.

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1. INTRODUCTION

In this paper, we consider a sub-Riemannian (SR) problem on the group of motions of Euclidean space SE(3). It can be interpreted as a problem of optimal motion of a rigid body in $\mathbb{R}^3$ with nonintegrable constraints [7]. Solution curves to the problem have applications in image processing (tracking of neural fibers and blood vessels in DW-MRI images of a human brain); and in robotics (motion planing problem for an aircraft that can move forward/backward).

The sub-Riemannian problem on SE(3) can be seen as follows. Given two orthonormal frames $N_0 = \{v_0^1, v_0^2, v_0^3\}$ and $N_1 = \{v_1^1, v_1^2, v_1^3\}$ attached, respectively, to two given points $q_0 = (x_0, y_0, z_0)$ and $q_1 = (x_1, y_1, z_1)$ in space $\mathbb{R}^3$, find an optimal motion that transfers q_0 to q_1 such that the frame N_0 is transferred to the frame N_1. The frame can move forward or backward along one of the vector in the frame and rotate simultaneously via two (of three) remaining vectors. The required motion should be optimal in the sense of minimal length in the space $\mathrm{SE}(3) \cong \mathbb{R}^3 \times \mathrm{SO}(3)$.

The two-dimensional analog of this problem was studied as a possible model of the mechanism used by the visual cortex V1 of the human brain to reconstruct curves that are partially corrupted or hidden from observation. The two-dimensional model was initially presented in [10] and subsequently refined in [11], where the authors recognized the sub-Riemannian Euclidean motion group structure of the problem. The related SR problem in SE(2) was solved in [12], where, in particular, explicit formulas for the geodesics were derived in SR arclength parameterization. Later, an alternative expression in spatial arclength parameterization for cuspless SR geodesics was derived in [13]. Application to contour completion in corrupted images was studied in [14]. The problem was also studied in [15]. However, many imaging applications such as diffusion weighted magnetic resonance imaging (DW-MRI) require an extension to three dimensions [16–18], which motivates us to study the problem on SE(3).

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The Lie group SE(3) of Euclidean motions of space $\mathbb{R}^3$ is generated by translations and rotations about coordinate axes. It is parameterized by the matrices

$$\begin{pmatrix} \cos\alpha\cos\beta & -\cos\beta\sin\alpha & \sin\beta & x \\ \cos\theta\sin\alpha+\cos\alpha\sin\beta\sin\theta & \cos\alpha\cos\theta-\sin\alpha\sin\beta\sin\theta & -\cos\beta\sin\theta & y \\ \sin\alpha\sin\theta-\cos\alpha\cos\theta\sin\beta & \cos\theta\sin\alpha\sin\beta+\cos\alpha\sin\theta & \cos\beta\cos\theta & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\beta \in [-\pi, \pi)$, $\alpha \in [0, 2\pi)$ are the angles of rotation about the axes OX, OY, OZ; and $(x, y, z) \in \mathbb{R}^3$ are coordinates with respect to the axes.

Let us choose

$$\begin{aligned}
\mathcal{A}_1 &= \cos\alpha\cos\beta\,\partial_x + (\sin\alpha\cos\theta + \cos\alpha\sin\beta\sin\theta)\,\partial_y + (\sin\alpha\sin\theta - \cos\alpha\sin\beta\cos\theta)\,\partial_z,\\
\mathcal{A}_2 &= -\sin\alpha\cos\beta\,\partial_x + (\cos\alpha\cos\theta - \sin\alpha\sin\beta\sin\theta)\,\partial_y + (\cos\alpha\sin\theta + \sin\alpha\sin\beta\cos\theta)\,\partial_z,\\
\mathcal{A}_3 &= \sin\beta\,\partial_x - \cos\beta\sin\theta\,\partial_y + \cos\beta\cos\theta\,\partial_z,\\
\mathcal{A}_4 &= -\cos\alpha\tan\beta\,\partial_\alpha + \sin\alpha\,\partial_\beta + \cos\alpha\sec\beta\,\partial_\theta,\\
\mathcal{A}_5 &= \sin\alpha\tan\beta\,\partial_\alpha + \cos\alpha\,\partial_\beta - \sin\alpha\sec\beta\,\partial_\theta,\\
\mathcal{A}_6 &= \partial_\alpha
\end{aligned}$$

as the basis left-invariant vector fields according to parameterization (1.1).

We consider the sub-Riemannian (SR) manifold $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$, see [1]. Here Δ is a left-invariant distribution generated by the vector fields $\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$; $\mathcal{G}_\xi$ is an inner product on Δ defined by

$$\mathcal{G}_\xi = \xi^2\omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5,$$

where $\xi > 0$ is a constant and ω^i are basis one forms satisfying

$$\langle \omega^i, \mathcal{A}^j \rangle = \delta^i_j, \quad \delta^j_i = 0, \text{ if } i \neq j, \quad \delta^i_i = 1.$$

We study the problem of finding sub-Riemannian length minimizers: given boundary conditions, find a Lipschitz curve $\gamma : [0, t_1] \to \mathrm{SE}(3)$ such that $\dot\gamma(t) \in \Delta$ for almost all $t \in (0, t_1)$ and γ minimizes a functional of sub-Riemannian length

$$l(\gamma) = \int_0^{t_1} \sqrt{\mathcal{G}_\xi\big(\dot\gamma(t), \dot\gamma(t)\big)}\,\mathrm{d}\,t.$$

SR geodesics are curves in SE(3) whose sufficiently short arcs are SR minimizers. They satisfy the Pontryagin maximum principle, and the corresponding controls are called extremal controls.

Due to left-invariance of the problem one can fix the initial value $\gamma(0) = e$, where e is the identical transformation of $\mathbb{R}^3$. Then the sub-Riemannian problem is equivalent to the following optimal control problem [1, 2]:

$$\begin{gathered}
\dot\gamma = u_3\mathcal{A}_3 + u_4\mathcal{A}_4 + u_5\mathcal{A}_5,\\
\gamma(0) = e, \qquad \gamma(t_1) = q,\\
l(\gamma) = \textstyle\int_0^{t_1} \sqrt{\xi^2 u_3^2(t) + u_4^2(t) + u_5^2(t)}\,\mathrm{d}\,t \to \min,
\end{gathered}$$

where the controls u_3, u_4, u_5 are real-valued functions from $L_\infty(0, t_1)$.

The Cauchy – Schwarz inequality implies that the minimization problem for the sub-Riemannian length functional l is equivalent to the minimization problem for the action functional

$$J(\gamma) = \frac{1}{2}\int_0^{t_1} \big(\xi^2 u_3^2(t) + u_4^2(t) + u_5^2(t)\big)\,\mathrm{d}\,t \to \min,$$

with fixed $t_1 > 0$.

In [19], the authors show that the problem can be reduced to the case $\xi = 1$ and that the application of the Pontryagin maximum principle leads to the following Hamiltonian system:

$$\begin{cases} \dot{u}_1 = -u_3 u_5, \\ \dot{u}_2 = u_3 u_4, \\ \dot{u}_3 = u_1 u_5 - u_2 u_4, \\ \dot{u}_4 = u_2 u_3 - u_5 u_6, \\ \dot{u}_5 = u_4 u_6 - u_1 u_3, \\ \dot{u}_6 = 0, \end{cases} \qquad \begin{cases} \dot{x} = u_3 \sin\beta, \\ \dot{y} = -u_3 \cos\beta \sin\theta, \\ \dot{z} = u_3 \cos\beta \cos\theta, \\ \dot{\theta} = \sec\beta (u_4 \cos\alpha - u_5 \sin\alpha), \\ \dot{\beta} = u_4 \sin\alpha + u_5 \cos\alpha, \\ \dot{\alpha} = -(u_4 \cos\alpha - u_5 \sin\alpha) \tan\beta, \end{cases} \tag{1.2}$$

— the vertical part (for extremal controls), — the horizontal part (for geodesics).

The vertical part describes the dynamics of the extremal controls u_3, u_4, u_5 together with the remaining momentum components u_1, u_2, u_6. SR geodesics are solutions to the horizontal part.

In this paper we focus on the simplest case $u_6 = 0$ as the most important for applications, in particular, for tracking of neural fibers and blood vessels in MRI and CT images of a human brain [19]. In this case, the system on extremal controls becomes

$$\dot{u}_1 = -u_3 u_5, \quad \dot{u}_2 = u_3 u_4, \quad \dot{u}_3 = u_1 u_5 - u_2 u_4, \quad \dot{u}_4 = u_2 u_3, \quad \dot{u}_5 = -u_1 u_3. \tag{1.3}$$

We generalize results of [19], where, in particular, the extremal controls are found in the case when the geodesics do not have cusps in their spatial projection. Such geodesics admit parameterization by spatial arclength, which leads to an expression for the extremal controls in elementary functions. Now we relax the 'cuspless' assumption and derive an explicit expression for $u_1, \ldots, u_5$ in terms of Jacobi elliptic functions.

In Section 2 we show that if the function u_3 is known, then the first, the second, the fourth and the fifth equations of system (1.3) allow us to express u_k, $k \in \{1, 2, 4, 5\}$ via the initial values $u_k(0)$. Then by substitution of u_k in the third equation of system (1.3) we obtain an ordinary differential equation on u_3. A solution to this equation is presented in Section 3.

Remark 1. Finding a parameterization of SR geodesics is a nontrivial problem. First, a natural question arises as to a theoretical possibility of such parameterization in some reasonable sense — the question of integrability of the Hamiltonian system, see, e. g., [8, 9]. It was shown in [19, Thm. 2] that (1.2) is Liouville integrable, since it has a complete set of functionally independent first integrals in involution: u_6, the Hamiltonian $H = \frac{1}{2}\left(u_3^2 + u_4^2 + u_5^2\right)$, a Casimir function $W = u_1 u_4 + u_2 u_5 + u_3 u_6$, and the right-invariant Hamiltonians

$$\rho_1 = -u_1 \cos\alpha \cos\beta + u_2 \cos\beta \sin\alpha - u_3 \sin\beta,$$
$$\rho_2 = -\cos\theta (u_2 \cos\alpha + u_1 \sin\alpha) + \big(u_3 \cos\beta + (-u_1 \cos\alpha + u_2 \sin\alpha) \sin\beta\big) \sin\theta,$$
$$\rho_3 = -u_3 \cos\beta \cos\theta + \cos\theta (u_1 \cos\alpha - u_2 \sin\alpha) \sin\beta - (u_2 \cos\alpha + u_1 \sin\alpha) \sin\theta.$$

The question of integrability of Hamiltonian systems was actively studied by V.I. Arnold [3]. Our research continues his study and examines an important example of an integrable system.

2. EXPRESSION FOR u_k, $k \neq 3$ VIA u_3 AND THE INITIAL VALUES

Let $T > 0$, $g \in C(0, T)$. If g is unbounded, assume the existence of the integral $\int_0^T g(t)\,\mathrm{d}\,t$. Denote

$$G(t) = \int_0^t g(\tau)\,\mathrm{d}\,\tau.$$

It is known [5, Ch. 1, par. 3] that under such assumptions the Cauchy problem $\dot{y}(t) = g(t) y(t)$, $y(0) = y_0$ has a unique solution $y \in C[0, T] \cup \mathcal{D}(0, T)$ given by $y(t) = y_0 \exp\big(G(t)\big)$.

Similarly, under the same assumptions the Cauchy problem

$$\begin{cases} \dot v(t) = g(t)w(t), & v(0) = v_0, \\ \dot w(t) = g(t)v(t), & w(0) = w_0 \end{cases} \tag{2.1}$$

has a unique solution (v, w) given by

$$\begin{aligned} v(t) &= \frac{v_0 + w_0}{2} \exp\big(G(t)\big) + \frac{v_0 - w_0}{2} \exp\big(-G(t)\big), \\ w(t) &= \frac{v_0 + w_0}{2} \exp\big(G(t)\big) - \frac{v_0 - w_0}{2} \exp\big(-G(t)\big). \end{aligned} \tag{2.2}$$

Notice that the first and the fifth equations of system (1.3) can be written in the form (2.1), where $g(t) = -u_3(t)$, and the second and the fourth equations of system (1.3) can be written in the form (2.1), where $g(t) = u_3(t)$. Thus, denoting

$$U(t) = \int_0^t u_3(\tau)\, \mathrm{d}\,\tau \tag{2.3}$$

and using (2.2), we express u_1, u_2, u_4, u_5 via integral (2.3) and the initial values

$$\begin{aligned} u_1(t) &= \frac{u_1(0) + u_5(0)}{2} \exp\big(-U(t)\big) + \frac{u_1(0) - u_5(0)}{2} \exp\big(U(t)\big), \\ u_2(t) &= \frac{u_2(0) + u_4(0)}{2} \exp\big(U(t)\big) + \frac{u_2(0) - u_4(0)}{2} \exp\big(-U(t)\big), \\ u_4(t) &= \frac{u_2(0) + u_4(0)}{2} \exp\big(U(t)\big) - \frac{u_2(0) - u_4(0)}{2} \exp\big(-U(t)\big), \\ u_5(t) &= \frac{u_1(0) + u_5(0)}{2} \exp\big(-U(t)\big) - \frac{u_1(0) - u_5(0)}{2} \exp\big(U(t)\big). \end{aligned} \tag{2.4}$$

3. EXPRESSION FOR THE FUNCTION u_3

It follows from (2.4) that

$$\begin{aligned} u_1(t)u_5(t) &= \left(\frac{u_1(0) + u_5(0)}{2}\right)^2 \exp\big(-2U(t)\big) - \left(\frac{u_1(0) - u_5(0)}{2}\right)^2 \exp\big(2U(t)\big), \\ u_2(t)u_4(t) &= \left(\frac{u_2(0) + u_4(0)}{2}\right)^2 \exp\big(2U(t)\big) - \left(\frac{u_2(0) - u_4(0)}{2}\right)^2 \exp\big(-2U(t)\big). \end{aligned}$$

Therefore,

$$u_1(t)u_5(t) - u_2(t)u_4(t) = \frac{1}{4}\Big(A\exp\big(-2U(t)\big) - B\exp\big(2U(t)\big)\Big), \tag{3.1}$$

where $A = \big(u_1(0) + u_5(0)\big)^2 + \big(u_2(0) - u_4(0)\big)^2$, $B = \big(u_1(0) - u_5(0)\big)^2 + \big(u_2(0) + u_4(0)\big)^2$.

Substitution of (3.1) in the third equation of system (1.3) gives the following second-order autonomous differential equation on integral (2.3):

$$\dot u_3(t) = \ddot U(t) = \frac{A}{4} \exp\big(-2U(t)\big) - \frac{B}{4} \exp\big(2U(t)\big). \tag{3.2}$$

There are three possible cases: two special cases ($A = 0$ or $B = 0$) and the general case $AB \neq 0$ (in this case A and B both are positive). Next we study these 3 cases.

I. $A = 0 \Leftrightarrow u_1(0) = -u_5(0),\ u_2(0) = u_4(0)$. Equation (3.2) becomes

$$\ddot U(t) = -B_1 \exp\big(2U(t)\big), \qquad \text{where } B_1 = u_4^2(0) + u_5^2(0). \tag{3.3}$$

We aim for a solution that satisfies the initial conditions

$$U(0) = 0, \quad \dot U(0) = u_3(0). \tag{3.4}$$

The initial value problem (3.3), (3.4) can be solved by standard methods. A solution is given by

$$U(t) = -\ln\left(\frac{1}{2}\left[\left(1+\tfrac{u_3(0)}{b}\right)\mathrm{e}^{-bt}+\left(1-\tfrac{u_3(0)}{b}\right)\mathrm{e}^{bt}\right]\right), \quad \text{where } b=\sqrt{u_3^2(0)+u_4^2(0)+u_5^2(0)}.$$

Therefore, we find

$$u_3(t)=\dot{U}(t)=\frac{\big(b+u_3(0)\big)\,\mathrm{e}^{-bt}-\big(b-u_3(0)\big)\,\mathrm{e}^{bt}}{\left(1+\frac{u_3(0)}{b}\right)\mathrm{e}^{-bt}+\left(1-\frac{u_3(0)}{b}\right)\mathrm{e}^{bt}}. \tag{3.5}$$

II. $B=0 \Leftrightarrow u_1(0)=u_5(0),\ u_2(0)=-u_4(0)$. Equation (3.2) becomes

$$\ddot{U}(t)=B_1\exp\big(-2U(t)\big).$$

A solution that satisfies the initial conditions (3.4) is given by

$$U(t)=\ln\left(\frac{1}{2}\left[\left(1+\tfrac{u_3(0)}{b}\right)\mathrm{e}^{bt}+\left(1-\tfrac{u_3(0)}{b}\right)\mathrm{e}^{-bt}\right]\right).$$

Therefore, we find

$$u_3(t)=\dot{U}(t)=\frac{\big(b+u_3(0)\big)\,\mathrm{e}^{bt}-\big(b-u_3(0)\big)\,\mathrm{e}^{-bt}}{\left(1+\frac{u_3(0)}{b}\right)\mathrm{e}^{bt}+\left(1-\frac{u_3(0)}{b}\right)\mathrm{e}^{-bt}}. \tag{3.6}$$

III. $AB\neq 0 \Rightarrow A>0,\ B>0$. Denote $V=2U$, $V_0=\dfrac{1}{2}\ln\left(\dfrac{B}{A}\right)$ and rewrite Eq. (3.2) as

$$\ddot{V}=\sqrt{AB}\,\frac{\sqrt{A/B}\,\mathrm{e}^{-V}-\sqrt{B/A}\,\mathrm{e}^{V}}{2}=\sqrt{AB}\,\frac{\mathrm{e}^{-V-V_0}-\mathrm{e}^{V+V_0}}{2}=-\sqrt{AB}\,\mathrm{sh}(V+V_0).$$

Next, denoting $y=V+V_0$, we obtain the following Cauchy problem:

$$\ddot{y}=-\sqrt{AB}\,\mathrm{sh}\,y,\quad y(0)=\frac{1}{2}\ln\left(\frac{B}{A}\right),\quad \dot{y}(0)=2u_3(0). \tag{3.7}$$

In [20], the authors find a solution to problem (3.7). It leads to

$$\begin{aligned}
y(t)&=\ln\left(1+\frac{P^2}{2\sqrt{AB}}\left(\mathrm{cn}^2(\psi_t,k)+\frac{1}{k}\,\mathrm{cn}(\psi_t,k)\,\mathrm{dn}(\psi_t,k)\right)\right),\\
\dot{y}(t)&=-P\ \mathrm{sn}(\psi_t,k),
\end{aligned}$$

where $\psi_t=F(p_0,k)+\dfrac{Q}{2}t,\quad k=\dfrac{P}{Q},\quad p_0=\begin{cases}-\arcsin\left(\dfrac{2u_3(0)}{P}\right), & \text{if } B\geqslant A,\\[2mm] \pi+\arcsin\left(\dfrac{2u_3(0)}{P}\right), & \text{if } B<A,\end{cases}$

with $P=\sqrt{4u_3^2(0)+\left(\sqrt{A}-\sqrt{B}\right)^2}$, $Q=\sqrt{4u_3^2(0)+\left(\sqrt{A}+\sqrt{B}\right)^2}$.

Here, the Jacobi functions sn, cn, dn and the elliptic integral of the first kind F are used, see [6].

Finally, by backward substitutions we express

$$U(t)=\frac{y(t)}{2}-\frac{1}{4}\ln\left(\frac{B}{A}\right),\quad u_3(t)=\frac{\dot{y}(t)}{2}. \tag{3.8}$$

4. CONCLUSION

Let us summarize the results of Sections 1–3. The following theorem is proved.

Theorem 1. *Consider the SR problem in* SE(3). *Suppose* $u_6(0) = 0$*; then the vertical part* (*on extremal controls*) *of the Hamiltonian system of PMP is given by* (1.3).
The extremal controls u_4, u_5 *are expressed via* $U(t) = \int_0^t u_3(\tau)\,\mathrm{d}\,\tau$ *and the initial values in* (2.4). *The extremal control* u_3 *is given in terms of the initial values depending on several cases. For the cases* $u_1(0) = \pm u_5(0)$, $u_2(0) = \mp u_4(0)$, *we have* (3.5) *and* (3.6). *Otherwise, we have* (3.8).

In future work, we plan to perform an explicit integration of the geodesic equation $\dot{\gamma}(t) = \sum_{i=3}^{5} u_i(\tau)\mathcal{A}_i$ and to carry out a study of the general case $u_6(0) \neq 0$.

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