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# Existence of Global Fundamental Solution to a Class of Fokker–Planck Equations

Abstract. In this paper, we investigate global solvability of the Fokker–Planck equations of a special type. Such equations arise in models of the primary visual cortex of the human brain and describe a process of anisotropic blurring of the image of the visual field on the retina of the eye. By modifying the Folland lifting technique for linear hypoelliptic differential operators satisfying the Hormander condition, we propose a method to saturate the system of vector fields in the equation to a basis of the tangent space at every point. We present the conditions that guarantee existence of a global fundamental solution to the considered equations.

*Key words and phrases:* nilpotent and stratified Lie groups, lifting of operator, saturation, fundamental solution.

## 1. Introduction

In this paper, we study a class of Fokker–Planck equations given by

$$\frac{\partial u(x,t)}{\partial t} = \Big(\sum_{i=1}^{m} X_i^2 - Y\Big)u(x,t), \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$ and the vector fields

$$X_i = \sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}, \quad i = 1, 2, ..., m, \qquad Y = \sum_{j=1}^{n} a_{(m+1)j}(x)\frac{\partial}{\partial x_j} \tag{2}$$

satisfy the following conditions:

(*1*) $X_i$ and $Y$ are $C^\infty$ vector fields on $\mathbb{R}^n$, i.e. their coefficients are smooth functions $a_{ij}(x) \in C^\infty(\mathbb{R}^n)$;

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(2) $X_1, \dots, X_m$ are homogeneous of degree 1 and $Y$ is homogeneous of degree 2 w.r.t. $\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \lambda^{\sigma_2} x_2, ..., \lambda^{\sigma_n} x_n)$, $1 \leq \sigma_1 \ldots \leq \sigma_n$, i.e. for any smooth on $\mathbb{R}^n$ test function $f$

$$X_i(f \circ \delta_\lambda(x)) = \lambda(X_i f(x)) \circ \delta_\lambda(x), \quad Y(f \circ \delta_\lambda(x)) = \lambda^2(Y f(x)) \circ \delta_\lambda(x);$$

(3) $X_1, \dots, X_m, Y$ are linearly independent almost everywhere w.r.t. standard Lebesgue measure on $\mathbb{R}^n$

$$\operatorname{rank}(X_1, \dots, X_m, Y) = m + 1;$$

(4) $X_1, \dots, X_m, Y$ satisfy Hormander hypoellipticity condition

$$\operatorname{rank} \operatorname{Lie}(X_1, \dots, X_m, Y) = n.$$

The question of solvability of equation (1) is equivalent to the question of existence of a global fundamental solution to the corresponding partial differential operator of the second order

$$\frac{\partial}{\partial t} - \sum_{i=1}^{m} X_i^2 + Y. \tag{3}$$

The paper has the following structure. It starts from motivation that comes from modelling of the primary visual cortex of the human brain, where equations (1) describe a process of anisotropic blurring (diffusion) of an image of the visual field on the retina of the eye. Then, in Section 3, we prepare a necessary mathematical background. Afterwards, in Section 4, we present the main result, the conditions that guarantee existence of a global fundamental solution of (3), followed by its proof in Section 5. Finally, we summarize the work in Conclusion.

## 2. Motivation

Our motivation to study Fokker–Planck equations (1) comes from modelling of the primary visual cortex V1 of the human brain, see e.g. [1], where such equations describe anisotropic diffusion of the image transmitted from the retina of the eye to the visual cortex V1. Such a diffusion underlies a mechanism of contour completion. According to the Petitot–Citti–Sarti model [2, 3], the primary visual cortex lifts the image from the retina $\mathbb{R}^2$ to the extended space of positions and directions $\mathbb{R}^2 \times S^1 \cong \mathrm{SE}(2)$ (the group of Euclidean motions of the plane [4]):

$$\mathcal{F} : I = \big((x_1, x_2) \in \mathbb{R}^2 \to [0,1]\big) \to \big((x_1, x_2, x_3) \in \mathrm{SE}(2) \to [0,1]\big) = \hat{I},$$

where $x_3 \in S^1$ is the direction angle. Thus, the original image of $\hat{I}(x)$ on $\mathrm{SE}(2)$ has the form $\hat{I}(x) = \mathcal{F}(I)(x)$.

Denote by $X_i$ the basis left invariant vector fields on SE(2):

$$X_1 = \frac{\partial}{\partial x_3},\ X_2 = \cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2},\ X_3 = -\sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}.$$

The Fokker–Planck equation that simulates the contour completion mechanism has the following form, see [5],

$$\frac{\partial u(x,t)}{\partial t} = \Big(BX_1^2 - X_2\Big)u(x,t), \tag{4}$$

where $B > 0$ is the diffusion coefficient.

In the paper [6], the authors show that such a diffusion process can be modelled by the following Fokker–Planck equation of type (1) in $\mathbb{R}^2$:

$$\frac{\partial u(x,t)}{\partial t} = \Big(BY_1^2 - Y_2\Big)u(x,t), \tag{5}$$

where $(x, y) \in \mathbb{R}^2$, $t \in \mathbb{R}$ and the vector fields $Y_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, $Y_2 = \frac{\partial}{\partial x}$.

## 3. Preliminaries

Let $P$ be a linear partial differential operator of an arbitrary order with smooth on $\mathbb{R}^n$ real-valued coefficients. We say that a function

$$\Gamma : \{(x; y) \in \mathbb{R}^n \times \mathbb{R}^n\ :\ x \neq y\} \to \mathbb{R},$$

is a (global) fundamental solution for $P$ if it satisfies the following assumptions:

(1) for every fixed $x \in \mathbb{R}^n$ the function $\Gamma(x; \cdot)$ is locally integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} \Gamma(x; y) P'\phi(y) dy = -\phi(x) \text{ for every } \ \phi \in C^\infty(\mathbb{R}^n),$$

where $P'$ denotes the usual formal adjoint of $P$ (this condition can be rewritten as $P\Gamma_x = -Dir_x$ in $\mathcal{D}'(\mathbb{R}^n)$);

(2) $\Gamma(x, y) > 0$ whenever $x \neq y$;

(3) $\Gamma(x; y) \in L_{1,loc}(\mathbb{R}^p \times \mathbb{R}^p)$ for every fixed $y \in \mathbb{R}^n$ the function $\Gamma(\cdot; y)$ is locally integrable on $\mathbb{R}^n$;

(4) for every fixed $x \in \mathbb{R}^n$ the function $y \mapsto \Gamma(x; y)$ vanishes as $y \to \infty$;

(5) for every fixed $x \in \mathbb{R}^n$ the function $y \mapsto \Gamma(x; y)$ tends to $\infty$ as $y \to x$.

Let $P$ be a smooth linear partial differential operator on $\mathbb{R}^n$. We say that a linear partial differential operator $\tilde{P}$, defined on a higher-dimensional space $\mathbb{R}^n \times \mathbb{R}^p$, is a lifting of $P$ if the following conditions are fulfilled:

(1) $\tilde{P}$ has smooth coefficients, possibly depending on $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^p$,

(2) for every fixed $f \in C^\infty(\mathbb{R}^n)$, one has

$$\tilde{P}(f \circ \pi)(x,\xi) = Pf(x), \text{ for every}(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^p \equiv \mathbb{R}^N, \tag{6}$$

where $N = n + p$ and $\pi(x,\xi) = x$ is the canonical projection.

It is obvious that (6) holds if and only if

$$\tilde{P} = P + R \text{ with } R = \sum_{\beta \neq 0} r_{\alpha,\beta}(x,\xi) D_x^\alpha D_\xi^\beta,$$

for a finite number of coefficients $r_{\alpha,\beta} \in C^\infty(\mathbb{R}^N)$, possibly identically vanishing on $\mathbb{R}^N$. The use of the term 'lifting' here is more specific than commonly accepted in differential geometry.

Let $P$ be a smooth linear partial differential equation on $\mathbb{R}^n$, and $\tilde{P} = P + R$ be a lifting of $P$ on $\mathbb{R}^N$. We say that $\tilde{P}$ is saturable lifting of $P$ if the following conditions hold:

(1) Every summand of the formal adjoint $R'$ to a given operator $P$ collect as least one derivative along some $\xi$, i.e., $R'$ has a form

$$R' = \sum_{\beta \neq 0} {r'}_{\alpha,\beta}(x,\xi) D_x^\alpha D_\xi^\beta,$$

for a finite number of possibly vanishing smooth coefficients ${r'}_{\alpha,\beta}$.

(2) There exists a sequence $\{\theta_j(\xi)\}_{j=1}^\infty : \mathbb{R}^p \mapsto [0,1]$ of smooth function with compact supports such that

$$\bigcup_{j \in \mathbb{N}} \Omega_j = \mathbb{R}^p \text{ where } \Omega_j = \{\xi \in \mathbb{R}^p : \ \theta_j(\xi) = 1\} \text{ and for } \ \forall j \ \Omega_j \subset \Omega_{j+1}.$$

Moreover, for every compact set $K \subseteq \mathbb{R}^n$ and for any coefficient function ${r'}_{\alpha,\beta}(x,\xi)$ of $R'$ there are exist constants $C_{\alpha,\beta}(K)$ such that

$$|{r'}_{\alpha,\beta}(x,\xi) \frac{\partial^{|\beta|}}{\partial \xi^\beta} \theta_j(\xi)| \leq C_{\alpha,\beta}(K) \text{ for every } x \in K, \xi \in \mathbb{R}^p, j \in \mathbb{N}.$$

In the paper of Bonfiglioli-Biagi [7] one can find some sufficient conditions for a lifting operator to be saturable. In particular, for any smooth second order operator on $\mathbb{R}^2$ the associated operator $\tilde{P} = \partial_t - P$ is a saturable lifting of $P$.

THEOREM 1. (*Bonfiglioli-Biagi*) *Let $P$ be a smooth linear partial differential equation on $\mathbb{R}^n$ and let $\tilde{P}$ be a saturable lifting of $P$ on $\mathbb{R}^N$. Assume that there exists a fundamental solution $\tilde{\Gamma}$ to $\tilde{P}$ on the whole $\mathbb{R}^N$ which satisfies the following conditions*

(1) *for every fixed $x, y \in \mathbb{R}^n$ such that $x \neq y$*

$$\eta \mapsto \tilde{\Gamma}(x, 0; y, \eta) \in L_1(\mathbb{R}^p);$$

(2) *for every fixed $x \in \mathbb{R}^n$ and for any compact $K \subset \mathbb{R}^n$*

$$(y, \eta) \mapsto \tilde{\Gamma}(x, 0; y, \eta) \in L_1(K \times \mathbb{R}^p).$$

*Then the function $\Gamma : \{(x; y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ x \neq y\} \mapsto \mathbb{R}$, defined by*

$$\Gamma(x; y) = \int_{\mathbb{R}^p} \tilde{\Gamma}(x, 0; y, \eta) d\eta$$

*is a global fundamental solution of $P$.*

## 4. Main result

As a basic example, let us consider Grushin vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y},$$

which are smooth on a plane $\mathbb{R}^2 \ni (x, y)$. The vector fields $X_1, X_2$ homogeneous of degree 1 w.r.t. $\delta_\lambda^G(x) = (\lambda x, \lambda^2 y)$. Conditions (1)–(4) hold. $\mathbb{R}^2$ is a Lie group homogeneous w.r.t. $\delta_\lambda^G(x)$. Vector fields $X_1, X_2$ satisfy the Hōrmander rank condition (4), hence Hōrmander operator $X_1^2 + X_2^2$ as well as Kolmogorov operator $X_1^2 + X_2$ are both hypoelliptic but there is no Lie group structure on $\mathbb{R}^2$ making these operators left-invariant on it.

In general case in such situation we need to use a special modification of Folland-Bonfiglioli-Biagi technique built in this paper. In the considered example we can use the Folland-Bonfiglioli-Biagi saturation-lifting technique without any modifications, which leads to a new set of generating vector fields (let us call them Kolmogorov vector fields)

$$X_1 = \frac{\partial}{\partial x}, \ X_2 = -\frac{\partial}{\partial t} + x\frac{\partial}{\partial y}.$$

Now, $\mathbb{R}^3$ is a saturated Lie group with a group law $\bullet$

$$(x, y, t) \bullet (x', y', t') = (x + x', y + y' + t'x, t + t').$$

Let us construct Hōrmander and Kolmogorov operators $H$ and $K$ on these vector fields

$$H = X_1^2 + X_2^2 = \frac{\partial^2}{\partial x^2} + \left(-\frac{\partial}{\partial t} + x\frac{\partial}{\partial y}\right)^2,$$
$$K = X_1^2 + X_2 = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} + x\frac{\partial}{\partial y}.$$

Notice that the well-known Kolmogorov operator $K$ on $\mathbb{R}^3$ coincides with the Fokker–Planck operator on Grushin vector fields $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y}$ on $\mathbb{R}^2 \times \mathbb{R}$ which we would like to solve, cf. (3).

It is easy to check that $K$ is invariant w.r.t. the left translations on $\mathbb{R}^3$ and commutes with the following dilations:

$$\delta_\lambda^K(x) = (\lambda x, \lambda^3 y, \lambda^2 t). \tag{7}$$

Kolmogorov vector fields are homogeneous for these dilations family. Therefore, Lie group $\mathbb{R}^3$ is homogeneous w.r.t. (7). But this time $X_1$ is 1-homogeneous whereas $X_2$ is 2-homogeneous w.r.t. (7). We can see that $H$ is also invariant w.r.t. left translations on $\mathbb{R}^3$ while $H$ commutes with another family of dilations $\delta_\lambda^H(x)$:

$$\delta_\lambda^H(x) = (\lambda x, \lambda^2 y, \lambda^2 t). \tag{8}$$

The homogeneity for hypoellipic operators guarantees that these operators have global fundamental solutions. Thus, by lifting technique we have proved the existence of a global solution to the Fokker–Planck equation on Grushin vector fields.

Let us formulate the main result of this paper

THEOREM 2. *For any set of vector fields that satisfy conditions* (1)-(4) *there exists a global fundamental solution to Fokker–Planck differential operator* $F = \frac{\partial}{\partial t} - \sum_{i=1}^m X_i^2 + Y$.

## 5. Proof of Theorem

### 5.1. Lifting construction

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let us consider the family (by $\lambda \geq 0$) of non-isotropic diagonal maps on $\mathbb{R}^n$

$$\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \lambda^{\sigma_2} x_2, \ldots, \lambda^{\sigma_n} x_n),\ 1 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n,$$

and a set of vector fields $X_i$ and $Y$ such that 4 main conditions hold:

(1) Coefficients are smooth functions $a_{ij}(x) \in C^\infty(\mathbb{R}^n)$. So, $X_1, \ldots, X_m$ and $Y$ are $C^\infty$ vector fields in $\mathbb{R}^n$;

(2) $X_1, \dots, X_m$ are homogeneous of degree 1 and $Y$ is homogeneous of degree 2 w.r.t. $\delta_\lambda(x)$, i.e. for an arbitrary vector filed $X$ homogeneous in degree $l$ and for any smooth on $\mathbb{R}^n$ test function $f$

$$X(f \circ \delta_\lambda(x)) = \lambda^l (Xf(x)) \circ \delta_\lambda(x);$$

(3) $X_i$, $Y$ are linearly independent (as linear differential operators);

(4) $X_i$, $Y$ satisfy Hōrmander hypoellipticity condition:

$$\text{rank Lie}\{X_1, \dots, X_m, Y\} = n.$$

Condition (4) means that at any point of $\mathbb{R}^n$ one can find $n$ linearly independent differential operators among $X_1, \dots, X_m, Y$ and their non-zero commutators.

We recall that a typical example of vector fields that satisfy conditions (1) − (4) is given by Grushin vector fields $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial y}$, which are smooth in $\mathbb{R}^2$ and homogeneous of degree 1 w.r.t. $\delta_\lambda(x) = (\lambda x, \lambda^2 y)$.

The innovative modification is that now we can extend $x \in \mathbb{R}^n$ to $(x,t) \in \mathbb{R}^{n+1}$. The dilations can be extended as well

$$\delta^+_\lambda(x,t) = (\delta_\lambda(x), \lambda^2 t). \tag{9}$$

Using the commutativity operation, let us construct Lie algebra $\mathbf{a} = \text{Lie}\{X_1, \dots, X_m, Y\}$. We denote

$$\dim\ \mathbf{a} := N \geq n. \tag{10}$$

This algebra must be extended from $\mathbf{a}$ to $\mathbf{a}^+$ with one more vector filed $\frac{\partial}{\partial t} = T$, linearly independent from $X_1, X_2, \dots, X_m, Y$. We have changed the dilations for $T$ to be homogeneous with degree 2. This field is independent from others in sense that $[T, X_i] = [T, Y] = 0$ and, thus, it gives an additional dimension $\dim\ \mathbf{a}^+ = N + 1$.

From homogeneity of the extended set of generators $X_1, \dots, X_m, Y, T$ (property (2)) one can conclude that $\mathbf{a}^+$ is nilpotent of step $r$ and stratified $\mathbf{a}^+ = \mathbf{a}^+_1 \oplus ... \oplus \mathbf{a}^+_r$.

Let us denote by $A$ the set of vector fields which is a basis of algebra $\mathbf{a}^+ = \text{Lie}\{X_1, X_2, .., X_m, Y, T\}$.

As far as $X_1, X_2, \dots, X_m, Y, T$ are linearly independent in $\mathbb{R}^{n+1}$ (condition (3)) to fulfil condition (10) (to find $N+1$ linearly independent differential operators and construct the basis $\mathbf{a}^+$) we can choose first $X_1$, $X_2, \dots$, $X_m$,$Y$,$T$, and then some additional operators among commutators of $X_1, \dots, X_m, Y$ or their linear combinations. Let us denote them $X_{m+2}, \dots, X_N$. We will call this set the additional part of the basis $A^\perp$.

Let us notice that each vector field among $A^{\perp}$ belongs to some layer $\mathbf{a}_k^+$, $1 \leq k \leq r$, consequently, it is $\delta_\lambda(x)$-homogeneous of degree $k$ while $X_1, X_2, \ldots, X_m$ having 1-homogeneity belong to $\mathbf{a}_k^+$ and $Y, T \in \mathbf{a}_2^+$ for the same reason.

Thus, Campbell-Hausdorff formula for arbitrary $X, Y \in \mathbf{a}^+$ is finite $X \diamond Y = X + Y + \frac{1}{2}[X, Y] ... + \text{const}[X, [, [...[..., ...]]]]$.

The last commutator has degree $r$ (and we have $r > n$).

Each element from $\mathbf{a}^+$ can be written in a unique way as a linear combination of vector fields from $A$.

Thus, we can rewrite Campbell-Hausdorff formula as a linear composition of the basis $A$ and coefficients of linear combinations of vector fields $X$ and $Y$

$$X \diamond Y = X + Y + \sum_{i=1}^{n} p_i X_i.$$

Here $p_i$ are in our case finite polynomials of degree $\leq k$ (in general, infinite polynomials) of coefficients of decompositions of $X$ and $Y$ on the basis vector fields. So $\mathbf{a}^+$ can be considered as a homogeneous Lie group.

Observe that since among vector fields $X_1, .., X_{N+1}$ there are $n+1$ linearly independent then to this set belong our generators $X_1, .., X_m, Y, T$. Let us denote them

$$B = (X_{i_1}, ..., X_{i_{n+1}}) \text{ and take } B(0) \text{ as a basis for } \mathbb{R}^{n+1}.$$

As a consequence, $B$ must be homogeneous with degrees $\sigma_1, \ldots, \sigma_n$, $(\sigma_{m+2} = 2)$. We rearrange them saving the same notation to

$$1 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{n+1}.$$

Now, let us reorder $A^{\perp}$ as well and continue the rearrangement to get the matrix of coefficients $(X_{i_1}, \ldots, X_{i_{n+1}}, \ldots, X_{i_{N+1}})$ with homogeneity degrees

$$\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_{n+1}, s_{n+2}, \ldots, s_{N+1}.$$

It was shown in the paper of Bofiglioli-Biagi [7] that by a smooth change of variables in $\mathbf{a}^+$ one can construct a basis $J_1, \ldots, J_{N+1}$ such that matrix $(X_{i_1}, \ldots, X_{i_{n+1}}, \ldots, X_{i_{N+1}})$ will be transformed to $(Z_{i_1}, \ldots, Z_{i_{n+1}}, \ldots, Z_{i_{N+1}})$ which at point 0 has the following form

$$\begin{pmatrix} B(0) & 0 \\ 0 & \mathbb{I} \end{pmatrix},$$

where $\mathbb{I}$ is a unit matrix (which corresponds to $\mathbb{R}^n$) and coincides with $(X_{i_1} + R_{i_1}, ..., X_{i_{n+1}} + R_{i_{n+1}}, ..., X_{i_{N+1}} + R_{i_{N+1}})$, where each $R_i$ satisfies the following conditions:

(1) $R_i$ is a vector field of $\mathbf{a}^+$ and it consists only from $\xi$-derivatives with coefficients possibly depending from $(x, \xi)$;

(2) $Z_i = X_i + R_i$ holds the same homogeneity as $X_i$.

Now, following the Folland technique [8], there is the possibility to construct 1-to-1 smooth map $\pi$

$$\begin{aligned} \mathbf{a}^+ \simeq \mathbb{R}^{N+1} &\mapsto \mathbb{R}^{n+1}, \\ \operatorname{Exp}(sX) = \operatorname{Exp}(s_1 J_1) \circ \ldots \circ &\operatorname{Exp}(s_{N+1} J_{N+1}), \\ \pi(X) = &\operatorname{Exp}(sX)\big|_{s=1}. \end{aligned}$$

Here $\operatorname{Exp}(sX)$ is a smooth integral curve (we can call it a flow) which starts from the origin at time $s = 0$ and moves always in the direction $X$ with unit speed. This curve is a unique solution of the system of smooth ordinary differential equations $\dot{\gamma} = X(\gamma(s))$. Thus, for any smooth function $f$

$$\frac{\partial}{\partial s} f(x_1(s), x_2(s), \ldots, x_n(s)) = \frac{\partial}{\partial s} f(\gamma(s)) = X f(x_1, x_2, \ldots, x_n). \tag{11}$$

In the paper of Folland [8] and in the consequent article of Bonfilgioli and Biagi [7] there was proved that if $B(0)$ is a basis for $\mathbb{R}^{n+1}$ then Jacobi matrix of the projection $\pi$ at point 0 coincides with $B(0)$ and one can find the neighborhood in which $\pi$ is surjective. Moreover, it is polynomial map which preserves the dilations, $\pi(x, t, \xi) = (x, t)$ and, the most important, $d\pi(J_i)_a = (X_i)_{\pi(a)}, \ \ \forall\, a \in \mathbf{a}^+ \simeq \mathbb{R}^{N+1}, \ \forall i$. Thus, $Z_i$ are liftings of $X_i$, $\forall i = 1, .., m$, $Z_{m+1}$ is a lift of $Y$. It easy to see also that $Z_{m+2} = T$ is left without changes.

Let us consider the operator $\tilde{F} = \frac{\partial}{\partial t} - \sum_{i=1}^{m} Z_i^2 + Z_{m+1}$. By the construction it is a saturable lifting of $F$. From Folland results one can conclude that $\tilde{F}$ is homogeneous of degree 2 w.r.t. $\delta_\lambda^+(x, t)$.

### 5.2. Solvability which depends from homogeneity of an operator

Proof of solvability for homogeneous hypoelliptic operators is based on two classical theorems. The first states the local solvability for hypoelliptic operators. It belongs to Fr. Treves

Theorem 3. (*Treves*) (*see* [9] *Theorem 52.2*) *If $D$ is a hypoelliptic differential operator on an open domain $\Omega \subseteq \mathbb{R}^n$, then every point in $\Omega$ has an open neighborhood in which formal adjoint operator $D'$ has a fundamental kernel. If $D'$ is also hypoelliptic, then every point of $\Omega$ has neighborhood in which $D$ has a two-sided fundamental kernel, which is very regular* (*belongs to Frechet space*).

The second theorem belongs to F. Folland [10]

Theorem 4. (*Folland*) *Let $D$ be a homogeneous of degree $\alpha$ differential operator on the homogeneous Lie group $G$ ($0 < \alpha < Q$, $Q = \sum_{i=1}^{n} \sigma_i$ is a homogeneous dimension of $G$) such that $D$ and his adjoint $D'$ are both hypoelliptic on $G$. Then there is a unique kernel $K_0$ of type $\alpha$ which is a fundamental solution for $D$ at point $0$, i.e. satisfies in distributional meaning the equation $DK_0 = Dir_x$. Here $Dir_x$ is a Dirac distribution.*

To prove Theorem 4 Folland has used the so called "local-to-global" or blow up argument to construct from local solution the global one.

According to these theorems operator $\tilde{F}$ has a global fundamental solution with properties (1)-(4) from definition of a fundamental solution, and following the Bofiglioli-Biagi theorem we can construct the fundamental solution to $F$. Thus, the proof is complete.

## 6. Conclusion

In this paper, we have stated global solvability of the Fokker–Planck equations of a special type. Our motivation comes from modelling of the primary visual cortex of the human brain, where equations (1) describe a process of anisotropic blurring (diffusion) of an image of the visual field on the retina of the eye. By modifying the Folland lifting technique for linear hypoelliptic differential operators satisfying the Hormander condition, we have obtained a method to saturate the system of vector fields in the equation to a basis of the tangent space at every point. Finally, in Theorem 2 we have presented the conditions that guarantee existence of a global fundamental solution to the considered equations.

Sections 1, 2 and 6 of the paper are written by A. Mashtakov, and Sections 3, 4 and 5 are written by V. Markasheva.

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В. А. Маркашева, А. П. Маштаков. *Существование глобального фундаментального Решения для Класса Уравнений Фоккера–Планка.*

Аннотация. В статье исследован вопрос глобальной разрешимости уравнений Фоккера–Планка специального вида. Уравнения такого вида возникают в моделях первичной зрительной коры головного мозга человека и описывают процесс анизотропного размытия изображения, поступающего на сетчатку глаза. Модифицируя технику лифтинга Фолланда для линейных гипоэллиптических дифференциальных операторов, удовлетворяющих условию Хермандера, был предложен метод насыщения системы векторных полей в уравнении до базиса касательного пространства в каждой точке. Найдены условия, гарантирующие существование глобального фундаментального решения для уравнений рассматриваемого вида.

*Ключевые слова и фразы:* Уравнение Фоккера–Планка, группа Ли, фундаментальное решение, насыщение.

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