Hamilton-Jacobi PDEs and diffusion equations on Lie groups, metric spaces and homogeneous spaces and their implementation in Neurovision

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Plan of the talk:

1) Grushin metric spaces (heuristic description)

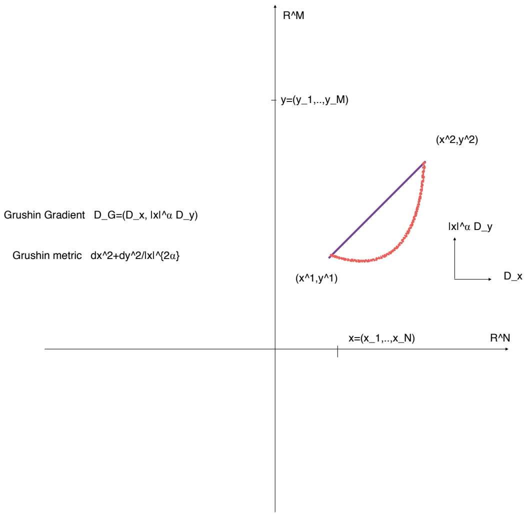
2) Motivation (Neurovision (image completion), Approximation and Stochastic Control Theory)

3) Hamilton-Jacobi equation (Existence, Regularity of solution and Total Mass Decay criterion)

4) Liouville type results for mean curvature operator and the related topic.

5) Grushin regular surfaces

plane.pdf


R^M
y=(y_1,..,y_M)
(x^2,y^2)
Grushin Gradient D_G=(D_x, |x|^α D_y)
|x|^α D_y
Grushin metric dx^2+dy^2/|x|^{2α}
(x^1,y^1)
D_x
x=(x_1,..,x_N)
R^N

$\mathbb{R}^{N+M}$ denotes $N+M$- dimensional Carnot-Carathéodory space $(N \geq 1, M \geq 1)$ with elements $z=(x,y)$ which is a manifold with degenerate Riemannian metrics $dx^2+\frac{dy^2}{|x|^{2\alpha}}$ agreed with a choice of generalized Baouendi-Grushin vector fields
$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_N}, |x|^\alpha \frac{\partial}{\partial y_1},\ |x|^\alpha \frac{\partial}{\partial y_2}, .., |x|^\alpha \frac{\partial}{\partial y_M},\ \ \alpha>0.$
The agreement mentioned above means that the shortest curves in this metric are the straightest ones according to the horizontal gradient $D_G=(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_N}, |x|^\alpha \frac{\partial}{\partial y_1},\ |x|^\alpha \frac{\partial}{\partial y_2}, .., |x|^\alpha \frac{\partial}{\partial y_M})$.
The shortest curves are geodesics and the straightest ones we will call horizontal.

Let us give a precise definition of horizontal curves. A piecewise C^1-curve $\gamma: [0, T] \to \mathbb{R}^{N+M}$ is called horizontal if whenever $\gamma'(t)$ exists, one has for every $\xi \in \mathbb{R}^{N+M}$

$$\langle \gamma'(t), \xi \rangle^2 \leq \langle D_G(\gamma(t)), \xi \rangle^2.$$

Using shortest curves, it is natural to introduce a metric distance (Carnot-Carathéodory or CC-distance). Given two points $z_1, \ z_2 \in \mathbb{R}^{N+M}$ one defines

$$d_{CC}(z_1, \ z_2) = inf\{ T \text{ among all horisontal curves } \gamma : [0, T] \to \mathbb{R}^{N+M},$$

$$\gamma(0) = z_1, \ \gamma(T) = z_2\}.$$

An appropriate curve γ in this definition, in general, can be not unique.

For the generalized Grushin settings Franchi and Lanconelli have proved the following universal estimate for the distance.

The distance estimate

There is a constant $C > 0$ such that

$$C^{-1}\left(|x-\xi| + min\ \left\{\frac{|y-\eta|}{|x|^\alpha},\ |y-\eta|^{1/(\alpha+1)}\right\}\right)$$
$$\leq d_{CC}((x,y),(\xi,\eta))$$
$$\leq C\left(|x-\xi| + min\ \left\{\frac{|y-\eta|}{|x|^\alpha},\ |y-\eta|^{1/(\alpha+1)}\right\}\right) \qquad (1)$$

for all points $z_1 = (x,y),\ z_2 = (\xi,\eta) \in \mathbb{R}^{M+N}$.

Thus, d_{CC} is well defined.

Let us consider $\mathbb{R}^2$, where vector fields $\frac{\partial}{\partial x}, |x|^\alpha \frac{\partial}{\partial y}$ correspond to Riemannian degenerate metrics $g_{ij}(z)dz_i dz_j = dx^2 + \frac{dy^2}{|x|^{2\alpha}}$. If $\alpha = 1$ the manifold $\mathbb{R}^2$ with the metrics $g_{ij}(z)dz_i dz_j = dx^2 + \frac{dy^2}{x^2}$ is called Grushin plane.

In the paper of Faízullin formulas of all locally possible geodesics on Grushin plane were obtained. Moreover,

Moreover, from now we can describe the CC-ball (metric ball). By definition, it is a ball $B_{CC}(z_0, r) = \{z \in \mathbb{R}^{N+M} : d_{CC}(z, z_0) < r\}$. Faízullin has found a vector parametric equation for CC-ball of radius R centered at the point (0,0) of Grushin plane

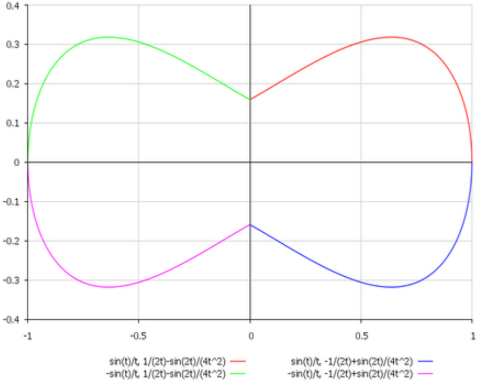


Figure: CC-ball of radius 1 on Grushin plane

Example. Grushin plane. $\alpha = 1$.

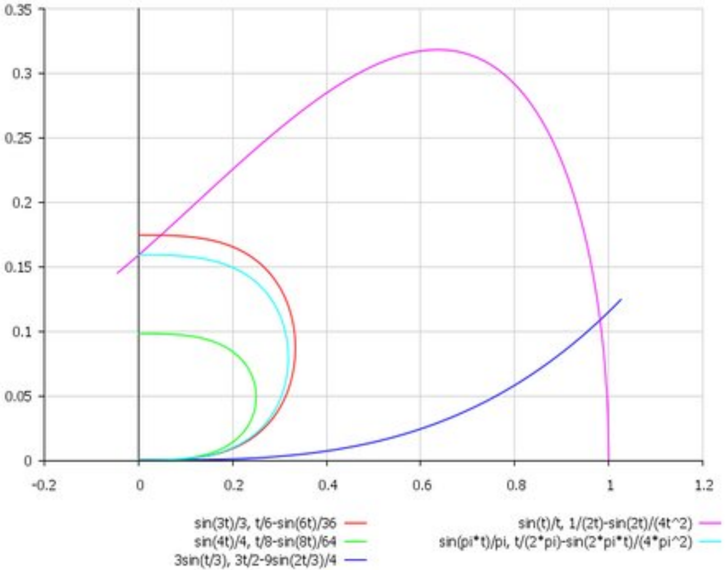


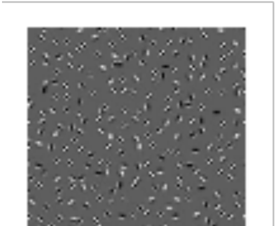
Figure: Geodesics cross a ball of radius 1

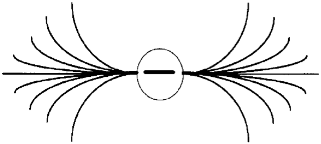
Motivation to the study of Grushin settings

Let us recall some backgrounds.

Gestalt laws have been proposed to explain several phenomena of visual perception (see for example **Wertheimer '38 , Kanizsa '80**). Among the local laws the law of good continuation plays a central role for perceptual completion.

The same principle of good continuation has been described through the notion of association fields by the psychophysical experiments of **Field, Hayes and Hess '93** . During the experiment researchers presented different stimuli to an observer (see Figure - left) in which it was possible to recognize perceptual units in a background of random Gabor patches. Then they experimentally studied when it is possible to recognize an unitary stimulus changing orientations of the patch. The results were summarized in the so-called association fields (Figure, right), which describes the complete set of possible subjective contour starting from a point with an horizontal orientation.





Integral curves of the Grushin structure provide natural models of association fields.

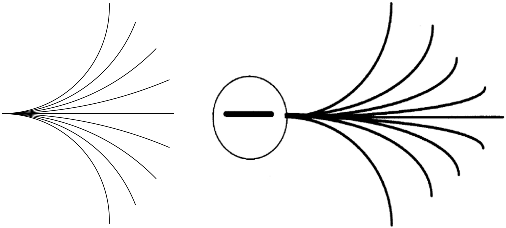


Figure: Grushin integral curves starting from a point (left), and association field experimentally found (right) .

The metric ball of the Grushin structure provides natural model of horizontal connectivity in the cortex.

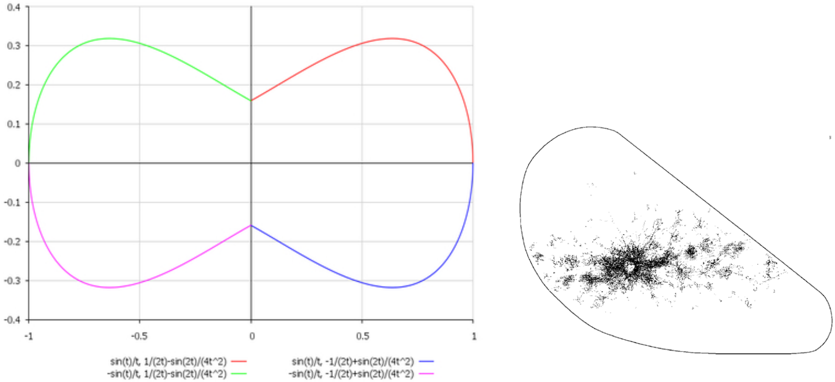


Figure: Grushin metric ball (left), and horizontal connectivity pattern experimentally found (right) .

Indeed, the model of cortical connectivity of **Citti and Sarti ’06** is expressed as the 2D projection of a $SE(2)$ structure, which is well approximated via the Grushin group. The 2D projection of integral curves for roto-translation settings is a good model of the association fields and the 2D projection of fundamental solutions provide the model of connectivity (see Fig.).

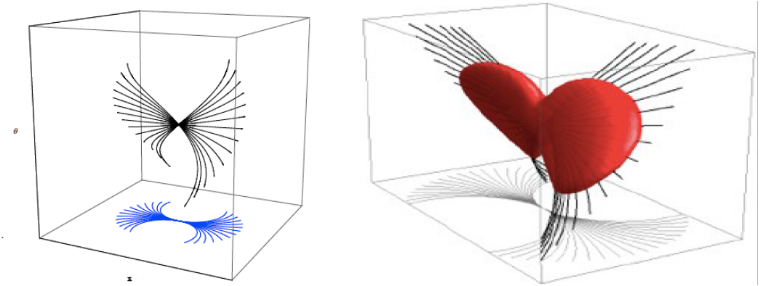


Figure: The integral curves starting from a point (left), and the fundamental solution of the FokkerPlanck in $SE(2)$ (right). Their 2D projection are model of the association fields of Fields, Heyes and Hess

Motivation to the study of Hamilton-Jacobi equations Let us discuss viscous Hamilton-Jacobi equation

$$u_t - \varepsilon div(|\nabla u|^{p-2}\nabla u) + \mu|\nabla u|^q = 0. \quad (2)$$

That kind of equations in Euclidean and in more general Carnot-Carathéodory metric settings appear as

- the viscosity approximation to the first order partial differential equations of Hamilton-Jacobi type,
- in the stochastic control theory (**Benachour, Roynette, Vallois '97**),
- in a number of interesting and different physical considerations, for example, to the study of ballistic deposition mechanism to describe the growth of surfaces of crystals, chemical deposits, flame or tumor (**Gilding, Guedda and Kersner' 03**).

Here we assume that $p \geq 2$ and $\mu, \varepsilon, q > 0$. The case of $p > 2$ corresponds to the slow p-Laplacian diffusion.

It is well-known that the behavior of solution to problem (2) changes dramatically because it strongly depends on the value of the parameter $q > 0$. Thus, the model is reach in unusual phenomena even in linear cases. It explains unfading investigative interest.

We would like to mention the papers of

- **Ben-Artzi '92 and his collaborators** (classical variational approach),
- **Weissler and his co-authors** (the analysis focused on the invariance properties of equations),
- the approach of **Benachour '97, Laurençot and collaborators** (approach based on Bernstein type estimates), in particular, for the linear case.
- **Balseiro and coauthors** have used Lie group algebraic approach to find the geometry underlying different dynamical systems. In particular, in example 4.3 they have constructed the equation of Hamilton-Jacobi type built on special smooth vector fields for a specified non-holonomic mechanical system.

The long-time behavior of solution for the evolution equation of problem (2) with linear diffusion and gradient absorption was studied by **Ben-Artzi '92, '99 , Benachour, Roynette, Vallois '97, Amour, Ben-Artzi '98** where the **waiting time phenomenon** (briefly WTP) was proven. For nonlinear diffusion case $p > 2$ infinite WTP was proved by **Laurençot, Vazquez '07** for $1 < q < p - 1$. WTP means that from some (finite or infinite) time the solution's support after a moment T^* stops it's growth. In addition, in this situation the **total mass decay** (TMD) arises when the parameter q is less or equal to the critical exponent q^* (for the linear diffusion case $q^* = \frac{N+2}{N+1}$). Both WTP and TMD appear thanks to the influence of the Hamilton-Jacobi term $|\nabla u|^q$ because when $1 < q < p - 1$ then the nonlinear absorption term becomes more and more dominant and the diffusion plays a secondary role for the large times.

Let us recall separately the article of **Tedeev, Andreucci, Ughi '04**. They have been studied the viscous Hamilton-Jacobi equation

$$u_t - \varepsilon div(|\nabla u|^{p-2}\nabla u) + \mu|\nabla u^{\nu}|^q = 0. \tag{3}$$

. Their approach gives not only the possibility to prove WTP, TMD and find the critical exponent for the nonlinear case but to estimate precisely the rate of the decay in terms of critical exponent. Namely, they have proved that the following estimate is true: $\int u(z,t)dz \leq Ct^{-A}$, where $A = \frac{(q^*-q)(\nu N+1)}{H}$, and $q^* = \frac{K+N}{\nu N+1}$ is a generalized critical exponent. Here $H = p(\nu q - 1) - q(p-2) > 0,\ K = N(p-2) + p$. This estimate can be reduced to the case of the linear diffusion if $p = 2, \nu = 1, H = 2, K = 2.$
As a PhD student of professor Tedeev, I have inherited this technique to use it to more complicate viscous Hamilton-Jacobi equation in metric settings.

Let us consider the following Cauchy problem

$$\frac{\partial u}{\partial t} = div_G(|D_G u|^{p-2} D_G u) - a(\rho(z)) f(t) |D_G u^{\nu}|^q, \qquad (4)$$

$(z, t) \in S_T = \mathbb{R}^{N+M} \times (0, T),$

$$u(z, 0) = u_0(z) \in L_1(\mathbb{R}^{M+N}), u_0(z) \geq 0, u_0(z) \not\equiv 0 \text{ a.e.}, \qquad (5)$$

$z = (x, y) \in \mathbb{R}^{N+M}$. Here $p > 2,\ 1 < q < p,\ \nu q > p - 1.$

Let $f(t), a(\rho(z))$ be non-negative and measurable. Moreover,

$$(\mathbf{H_1})\ \begin{array}{l} a(s) \text{ is a continuous nondecrising function,} \\ \text{such that for all } s>0: s^q/a(s) \text{ also non decreases.} \end{array}$$

Let a function $\rho(z)$ hereinafter is either a CC-distance or a homogeneous distance.

$$(\mathbf{H_2})\ \begin{array}{l} f(t) \text{ is a continuous nondecreasing function,} \\ \text{such that for all } t>0 \text{ there exists a number } \mu: \\ 0<\mu<\frac{\nu q-p+1}{p-1},\ t^\mu/f(t) \text{ also non decreases.} \end{array}$$

Definition

Nonnegative function $u(z,t) \in L_{\infty,loc}(S_T)$ we will call a **weak solution of the equation** (4) in $S_T = \mathbb{R}^{N+M} \times (0,T)$ if for any $t: \ T > t > 0, \ R > 0$,

$$u \in C((0,T), L_{2,loc}(\mathbb{R}^{N+M})),$$

$$|D_G u|^p, a(\rho(z)) f(t) |D_G u^\nu|^q \in L_{1,loc}(S_T),$$

and satisfies:

$$\iint\limits_{B_R \times (t,T)} -u\eta_\tau + (|D_G u|^{p-2} D_G u) D_G \eta + \eta a(\rho(z)) f(\tau) |D_G u^\nu|^q dxdyd\tau = 0,$$

where $\eta(x,y,t)$ is any smooth function with the support from $B_R \times (t,T)$.

Definition

A weak solution of the equation (4) which is a function $u(z,t) \in L_{\infty,loc}(S_T)$ we will call a **weak solution of the problem** (4), (5) if

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^{N+M}} u\eta(z)dz = \int\limits_{\mathbb{R}^{N+M}} u_0\eta(z)dz, \forall \eta(z) \in C_0^\infty(\mathbb{R}^{N+M}).$$

Let $\varphi(s)$ be a function inverse to $a(s)^{p-2}s^H$, where $H = p(\nu q - 1) - q(p-2) > 0, K = Q(p-2) + p$. An important role here will be played by the critical function

$$\omega(t): \ \omega(t) \equiv \frac{\varphi\left(\frac{t^{\nu q - p + 1}}{f(t)^{p-2}}\right)}{t^{\frac{1}{K}}}. \tag{6}$$

Let us introduce some notations $\Delta_{p,G}u =: div_G(|D_G u|^{p-2} D_G u)$ and some auxiliary results.

Tedeev, Markasheva-main problem

$$u_t = \Delta_{p,G}u - a(x)f(t)|D_G u^{\nu}|^q$$

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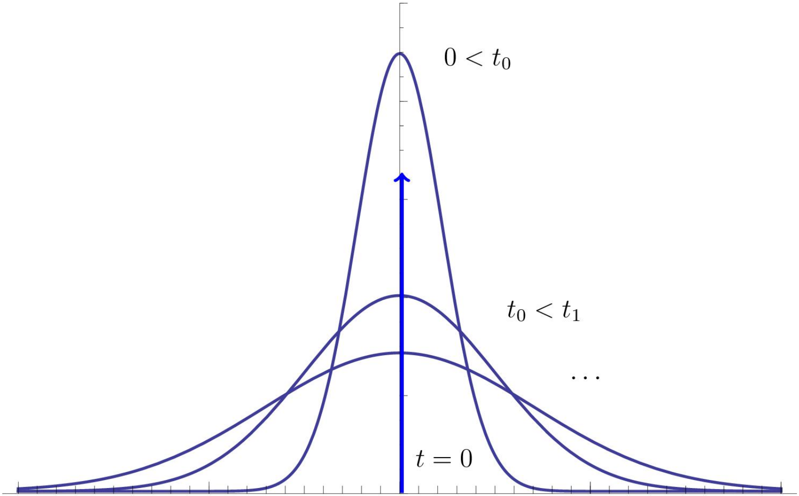
Tedeev, Markasheva-asymptotic estimates for p-Laplacian

$$||u||_{\infty} \le c_2 t^{-\frac{Q}{K}} ||u_o||_1^{\frac{p}{K}}$$

Remark



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$$c_1 t^{-\frac{Q}{K}} ||u_o||_1^{\frac{p}{K}} \leq ||u||_\infty \leq c_2 t^{-\frac{Q}{K}} ||u_o||_1^{\frac{p}{K}}$$

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$$c_3 t^{\frac{1}{K}} ||u_o||_1^{\frac{p-2}{K}} \leq Z(t) \leq 4R_0 + c_4 t^{\frac{1}{K}} ||u_o||_1^{\frac{p-2}{K}}$$

Theorem 1

Let conditions $(\mathbf{H_1})$, $(\mathbf{H_2})$ hold. Then there exists u a weak solution of (4). Moreover, if $\omega(t)$ is a function which was defined in (6), *supp* $u_0 \subset B_{R_0}$, $R_0 < \infty$, then for big enough t we have

$$Z(t) \leq C_7 \omega(t) t^{\frac{1}{K}}. \tag{7}$$

Remark

Solutions of the problem (4), (5) are subsolutions for the corresponding auxiliary Cauchy problem to the model equation with parabolic p-Laplace and classical estimates are true in all cases. Thus, whenever $\omega(t) \to 0$ as $t \to \infty$ the estimate (7) is more precise.

Theorem 2

Let u be a weak solution of (4) and conditions $(\mathbf{H_1})$, $(\mathbf{H_2})$ hold, $supp\ \ u_0 \subset B_{R_0}$, $R_0 < \infty$. And let $0 < \alpha < M(\nu q - 1)/q$. Then for big enough t we have

$$\int\limits_{\mathbb{R}^{M+N}} udz \leq C_{10}\omega^{\frac{K}{p-2}}(t), \tag{8}$$

$$||u||_{\infty,\mathbb{R}^{N+M}} \leq C_{11}\omega^{\frac{p}{p-2}}(t)t^{-\frac{Q}{K}}. \tag{9}$$

Thus, one can see immediately that the condition of the total mass decay is

$$\lim_{t\to\infty} \omega(t) = 0. \tag{10}$$

There are two border cases in which we can neglect the condition of the bounded support of the initial data saving total mass decay phenomenon.

Theorem 3

Let u be a weak solution of (4) and conditions $(\mathbf{H_1})$, $(\mathbf{H_2})$ hold. Then for big enough t, provided that $\nu = 1, \lambda > 1$ and $0 < \alpha < M(q-1)/q$, the following estimate is true

$$\int\limits_{\mathbb{R}^{N+M}} u(t)dz \leq \int\limits_{\rho > \omega(t) t^{\frac{1}{K}}} u_0 dz + C_{12}\omega^{\frac{K}{p-2}}(t). \tag{11}$$

In case $p = 2$ and $0 < \alpha < M(\nu q - 1)/q$, the following holds

$$\int\limits_{\mathbb{R}^{N+M}} u(t)dz \leq \int\limits_{\rho > \sqrt{t}} u_0 dz + C'_{12} \frac{t^{\frac{Q(\nu q - 1) + q - 2}{2(\nu q - 1)}}}{a(\sqrt{t})^{\frac{1}{\nu q - 1}} f(t)^{\frac{1}{\nu q - 1}}}. \tag{12}$$

Next theorem gives a more precise estimate of the total mass of the solution.

Theorem 4

Let u be a weak solution of (4) and conditions ($\mathbf{H_1}$), ($\mathbf{H_2}$) hold, *supp* $u_0 \subset B_{R_0}, R_0 < \infty$. If the following condition holds:

$$\exists C_8 > 0 : \ C_8 \leq \lim_{t\to\infty} \omega(t), \tag{13}$$

then for big enough $t : \ t \geq t_0 = ||u_0||^{p-2}_{1,\mathbb{R}^{N+M}}/R_0^K$, we have

$$\int\limits_{\mathbb{R}^{N+M}} u dz \leq C_{13} \left(\int\limits_{t_0}^{t} \frac{a(\tau^{\frac{1}{K}})f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}} d\tau \right)^{-\frac{1}{\nu q-1}}. \tag{14}$$

Moreover, the corollary of Theorem 4 shows that total mass decay is preserving in the case of the bounded criterion function $\omega(t)$ as well.

Corollary 1

In particular, (14) implies that if $\exists C > 0, \varrho \in (0,1) : C\varrho \leq \omega(t) \leq C$, then for big enough $t : \ t \geq t_0 = ||u_0||^{p-2}_{1,\mathbb{R}^{N+M}} / R_0^K$, we have

$$\int\limits_{\mathbb{R}^{N+M}} udz \leq C_{14} \left(\ln \left(\frac{t}{t_0} \right) \right)^{-\frac{1}{\nu q - 1}}. \tag{15}$$

Theorem 5

Let u be a weak solution of (4) and conditions ($\mathbf{H_1}$), ($\mathbf{H_2}$) hold, *supp* $u_0 \subset B_{R_0}, R_0 < \infty$. If the following condition holds

$$\exists C_9, \epsilon > 0 : \; C_9 t^{\epsilon} \leq \omega(t), \tag{16}$$

then for big enough t we have

$$\int\limits_{\mathbb{R}^{N+M}} u(t)dz \geq C_{15} > 0, \tag{17}$$

where C_{15} is a positive constant depending only from parameters of the problem and $||u_0||_1$.

Summarizing all previous results, we obtain

Total mass decay criterion.

If $\omega(t)$ is bounded from above then total mass of the solutions decays to zero. If, by the contrary, $\exists C, \epsilon > 0: \; Ct^{\epsilon} \leq \omega(t)$ then total mass decreases to some positive constant.

Remark

We would like to notice that because of the restrictions $(\mathbf{H_1})$, $(\mathbf{H_2})$ the behavior of the criterion function which differs from described above is impossible.

Example. Let u be a weak solution of the problem

$$\frac{\partial u}{\partial t} = div_G(|D_G u|^{p-2} D_G u) - \rho(z)^\gamma t^\beta |D_G u^\nu|^q, \quad (z,t) \in S_T,$$

$u(z,0) = u_0(z) \in L_1(\mathbb{R}^{M+N}), u_0(z) \geq 0, u_0(z) \not\equiv 0$ a.e., $supp \ \ u_0 \subset B_{R_0}, R_0 < \infty$. Let $\gamma < q, \beta < \frac{\nu q - p + 1}{p-1}$. Then for the exponents

$$q^* = \frac{Q + \gamma + K(1+\beta)}{\nu Q + 1}, A = \frac{(q^* - q)(\nu Q + 1)}{H + \gamma(p-2)} \qquad (18)$$

the critical function $\omega(t) = t^{-\frac{A(p-2)}{K}}$.

$$Z(t) \leq C_7 t^{\frac{1}{K} - \frac{A(p-2)}{K}}, \qquad \int\limits_{\mathbb{R}^{N+M}} u(t) dz \leq C_{10} t^{-A},$$

$$||u(t)||_{\infty, \mathbb{R}^{N+M}} \leq C_{11} t^{-\frac{Q}{K} - \frac{Ap}{K}}.$$

Thank you for attention!