**Noname manuscript No.**
(will be inserted by the editor)

# Cut Locus and Optimal Synthesis in Sub-Riemannian Problem on the Lie Group SH(2)

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**Abstract** Global optimality analysis in sub-Riemannian problem on the Lie group SH(2) is considered. We cutout open dense domains in the preimage and in the image of the exponential mapping based on the description of Maxwell strata. We then prove that the exponential mapping restricted to these domains is a diffeomorphism. Based on the proof of diffeomorphism, the cut time, i.e., time of loss of global optimality is computed on SH(2). We also consider the global structure of the exponential mapping and obtain an explicit description of cut locus and optimal synthesis.

**Keywords** Sub-Riemannian geometry, Special hyperbolic group SH(2), Maxwell points, Cut time, Conjugate time, Optimal synthesis

**Mathematics Subject Classification (2010)** 49J15, 93B27, 93C10, 53C17, 22E30

## 1 Introduction

In this work we complete our study of the sub-Riemannian problem on the Lie group SH(2) which is the group of motions of pseudo Euclidean plane. The work was initiated in [1] where we defined the sub-Riemannian problem. The control system comprises two 3-dimensional left invariant vector fields and a 2-dimensional linear control vector. We applied PMP to the control system and obtained the corresponding Hamiltonian system. In [2] we proved the Liouville integrability of the Hamiltonian system. We calculated the Hamiltonian flow such that the extremal trajectories were parametrized in terms of Jacobi elliptic functions [1]. Since PMP states only the first order optimality conditions, the trajectory resulting from PMP are only potentially optimal called extremal trajectories or geodesics. Further analysis based on second order optimality conditions is then needed to segregate the optimal trajectories or the minimizing geodesics. It is well known that the candidate optimal trajectories lose optimality either at the Maxwell points or at the conjugate points [3],[4],[5]. Based on the optimality analysis one is able to state the time of loss of global optimality known as the cut time. Rigorous techniques for this optimality analysis have evolved over the years from research on related sub-Riemannian problems on various Lie groups, see e.g., [4], [5], [6], [7]. These techniques were employed in [1] and [8] to compute the Maxwell strata and the conjugate locus in the problem under investigation. An effective upper bound on the cut time was also computed.

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In this paper we extend the global optimality analysis similar to [9]. We decompose the image $M = \mathrm{SH}(2)$ and the preimage of the exponential mapping into open dense sets based on the Maxwell strata and conjugate loci and prove that the exponential mapping between these sets is a diffeomorphism. This leads naturally to the proof that the cut time is equal to the first Maxwell time. Finally, we analyze the global structure of the exponential mapping and obtain explicit characterization of the cut locus and the optimal synthesis on the manifold SH(2).

The paper is organized as follows. In Section 2, we review the results from [1] and [8] as ready reference. Sections 3 and 4 contain the main results of this work. In Section 3 we state and prove the conditions for exponential mapping being a diffeomorphism and compute the cut time. Section 4 pertains to explicit characterization of the Maxwell strata and the cut locus in terms of a stratification of SH(2). In Section 5 we conclude this work.

## 2 Previous Work

### 2.1 Problem Statement

The Lie group SH(2) is a 3-dimensional group of roto-translations of the pseudo Euclidean plane [10]. The sub-Riemannian problem on the Lie group SH(2) reads as follows [1]:

$$\dot{x} = u_1 \cosh z, \quad \dot{y} = u_1 \sinh z, \quad \dot{z} = u_2, \tag{2.1}$$

$$q = (x, y, z) \in M = \mathrm{SH}(2) \cong \mathbb{R}^3, \quad x, y, z \in \mathbb{R}, \quad (u_1, u_2) \in \mathbb{R}^2, \tag{2.2}$$

$$q(0) = (0, 0, 0), \qquad q(t_1) = q_1 = (x_1, y_1, z_1), \tag{2.3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{2.4}$$

By Cauchy-Schwarz inequality, the sub-Riemannian length functional $l$ minimization problem (2.4) is equivalent to the problem of minimizing the following action functional with fixed $t_1$ [11]:

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2) dt \to \min. \tag{2.5}$$

### 2.2 Known Results

We now briefly review the results from [1] and [8] as a ready reference in this paper. System (2.1) satisfies the bracket generating condition and is hence globally controllable [12],[13]. Existence of optimal trajectories for the optimal control problem (2.1)–(2.5) follows from Filippov's theorem [3]. We applied PMP [3] to (2.1)–(2.5) to derive the normal Hamiltonian system. It turns out that the vertical part of the normal Hamiltonian system is a double covering of a mathematical pendulum. The normal Hamiltonian system is given as:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \quad \lambda = (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \quad 2S^1_\gamma = \mathbb{R}/(4\pi\mathbb{Z}), \tag{2.6}$$

$$\dot{x} = \cos\frac{\gamma}{2} \cosh z, \quad \dot{y} = \cos\frac{\gamma}{2} \sinh z, \quad \dot{z} = \sin\frac{\gamma}{2}. \tag{2.7}$$

The total energy integral of the pendulum (2.6) is given as:

$$E = \frac{c^2}{2} - \cos\gamma, \quad E \in [-1, +\infty). \tag{2.8}$$

The initial cylinder of the vertical subsystem is decomposed into the following subsets based upon the pendulum energy that correspond to various pendulum trajectories:

$$C = \bigcup_{i=1}^{5} C_i,$$

where,

$$C_1 = \{\lambda \in C \mid E \in (-1, 1)\}, \tag{2.9}$$

$$C_2 = \{\lambda \in C \mid E \in (1, \infty)\}, \tag{2.10}$$

$$C_3 = \{\lambda \in C \mid E = 1, c \neq 0\}, \tag{2.11}$$

$$C_4 = \{\lambda \in C \mid E = -1,\, c = 0\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n,\, c = 0\}, \quad n \in \mathbb{N}, \tag{2.12}$$

$$C_5 = \{\lambda \in C \mid E = 1,\, c = 0\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n + \pi,\, c = 0\}, \quad n \in \mathbb{N}. \tag{2.13}$$

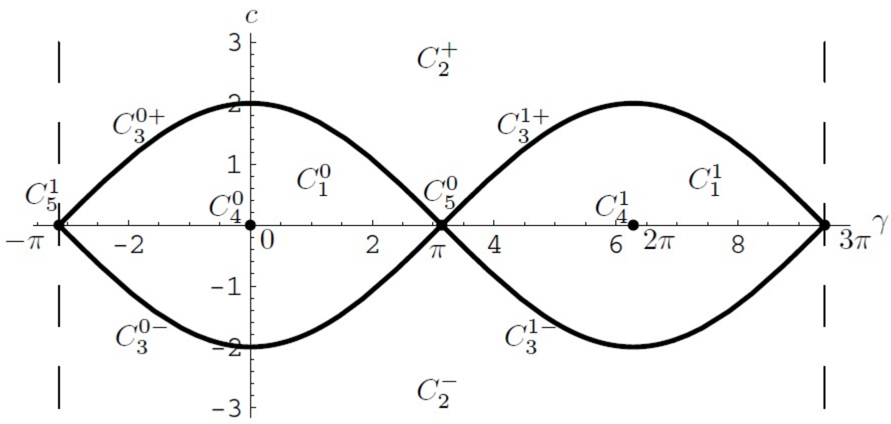


**Fig. 1** Stratification of the Phase Cylinder $C$ of the Pendulum

We defined elliptic coordinates $(\varphi, k)$ for $\lambda \in \cup_{i=1}^{3} C_i \subset C$ and proved that the flow of the pendulum is rectified in these coordinates. Note that $k$ was defined as the reparametrized energy and $\varphi$ was defined as the reparametrized time of motion of the pendulum [1]. Integration of the horizontal subsystem in elliptic coordinates follows from integration of the vertical subsystem and the resulting extremal trajectories are parametrized by the Jacobi elliptic functions $\operatorname{sn}(\varphi,k)$, $\operatorname{cn}(\varphi,k)$, $\operatorname{dn}(\varphi,k)$, $\mathrm{E}(\varphi,k) = \int_0^{\varphi} \operatorname{dn}^2(t,k)dt$ (Theorems 5.1–5.5 [1]). The results of integration for $\lambda \in C_i, \quad i = 1,\ldots,5$, are summarized as:

– Case 1 : $\lambda = (\varphi, k) \in C_1$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left[\left(w + \frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)] + \left(\frac{k}{w(1-k^2)} - kw\right)[\operatorname{sn}\varphi_t - \operatorname{sn}\varphi]\right] \\ \frac{1}{2}\left[\left(w - \frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)] - \left(\frac{k}{w(1-k^2)} + kw\right)[\operatorname{sn}\varphi_t - \operatorname{sn}\varphi]\right] \\ s_1 \ln\left[(\operatorname{dn}\varphi_t - k\operatorname{cn}\varphi_t).w\right] \end{pmatrix}, \tag{2.14}$$

where $w = \frac{1}{\operatorname{dn}\varphi - k\operatorname{cn}\varphi}$, $s_1 = \operatorname{sgn}\left(\cos\frac{\gamma}{2}\right)$ and $\varphi_t = \varphi + t$.

– Case 2 : $\lambda = (\psi, k) \in C_2$

$$\begin{aligned} x_t &= \frac{1}{2}\left(\frac{1}{w(1-k^2)} - w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ &+ \frac{1}{2}\left(kw + \frac{k}{w(1-k^2)}\right)[\operatorname{sn}\psi_t - \operatorname{sn}\psi], \\ y_t &= -\frac{s_2}{2}\left(\frac{1}{w(1-k^2)} + w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ &+ \frac{s_2}{2}\left(kw - \frac{k}{w(1-k^2)}\right)[\operatorname{sn}\psi_t - \operatorname{sn}\psi], \\ z_t &= s_2 \ln\left[(\operatorname{dn}\psi_t - k\operatorname{cn}\psi_t).w\right], \end{aligned} \tag{2.15}$$

where $\psi = \frac{\varphi}{k}, \quad \psi_t = \frac{\varphi_t}{k} = \psi + \frac{t}{k}$ and $w = \frac{1}{\operatorname{dn}\psi - k\operatorname{cn}\psi}$, $s_2 = \operatorname{sgn} c$, $k' = \sqrt{1-k^2}$.

– Case 3 : $\lambda = (\varphi, k) \in C_3$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left[\frac{1}{w}(\varphi_t - \varphi) + w(\tanh\varphi_t - \tanh\varphi)\right] \\ \frac{s_2}{2}\left[\frac{1}{w}(\varphi_t - \varphi) - w(\tanh\varphi_t - \tanh\varphi)\right] \\ -s_1 s_2 \ln[w \operatorname{sech}\varphi_t] \end{pmatrix}, \tag{2.16}$$

where $w = \cosh\varphi$.

– Case 4 : $\lambda = (\varphi, k) \in C_4$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}\left(\cos\frac{\gamma}{2}\right)t \\ 0 \\ 0 \end{pmatrix}. \tag{2.17}$$

– Case 5 : $\lambda = (\varphi, k) \in C_5$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{sgn}\left(\sin\frac{\gamma}{2}\right)t \end{pmatrix}. \tag{2.18}$$

The phase portrait of the pendulum admits a discrete group of symmetries $G = \{Id, \varepsilon^1, \ldots, \varepsilon^7\}$. The symmetries $\varepsilon^i$ are reflections and translations about the coordinates axes $(\gamma, c)$. The reflection symmetries in the phase portrait of a standard pendulum are given as:

$$\begin{aligned}
\varepsilon^1 &: (\gamma, c) \to (\gamma, -c),\\
\varepsilon^2 &: (\gamma, c) \to (-\gamma, c),\\
\varepsilon^3 &: (\gamma, c) \to (-\gamma, -c),\\
\varepsilon^4 &: (\gamma, c) \to (\gamma + 2\pi, c),\\
\varepsilon^5 &: (\gamma, c) \to (\gamma + 2\pi, -c),\\
\varepsilon^6 &: (\gamma, c) \to (-\gamma + 2\pi, c),\\
\varepsilon^7 &: (\gamma, c) \to (-\gamma + 2\pi, -c).
\end{aligned} \tag{2.19}$$

According to Proposition 6.3 [1], the action of reflections on endpoints of extremal trajectories can be defined as $\varepsilon^i : q \mapsto q^i$, where $q = (x, y, z) \in M, \quad q^i = (x^i, y^i, z^i) \in M$ and,

$$\begin{aligned}
(x^1, y^1, z^1) &= (x \cosh z - y \sinh z,\ x \sinh z - y \cosh z,\ z),\\
(x^2, y^2, z^2) &= (x \cosh z - y \sinh z,\ -x \sinh z + y \cosh z,\ -z),\\
(x^3, y^3, z^3) &= (x,\ -y,\ -z),\\
(x^4, y^4, z^4) &= (-x,\ y,\ -z),\\
(x^5, y^5, z^5) &= (-x \cosh z + y \sinh z,\ x \sinh z - y \cosh z,\ -z),\\
(x^6, y^6, z^6) &= (-x \cosh z + y \sinh z,\ -x \sinh z + y \cosh z,\ z),\\
(x^7, y^7, z^7) &= (-x,\ -y,\ z).
\end{aligned} \tag{2.20}$$

These symmetries are exploited to state the general conditions on Maxwell strata in terms of the functions $z_t$ and $R_i(q)$ given as:

$$R_1 = y \cosh \frac{z}{2} - x \sinh \frac{z}{2}, \quad R_2 = x \cosh \frac{z}{2} - y \sinh \frac{z}{2}. \tag{2.21}$$

We define the Maxwell sets $MAX^i, \quad i = 1, \ldots, 7$, resulting from the reflections $\varepsilon^i$ of the extremals in the preimage of the exponential mapping $N$ as:

$$\mathrm{MAX}^i = \left\{ \nu = (\lambda, t) \in N = C \times \mathbb{R}^+ \quad | \quad \lambda \neq \lambda^i, \quad \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(\lambda^i, t) \right\},$$

where $\lambda = \varepsilon^i(\lambda)$. The corresponding Maxwell strata in the image of the exponential mapping are defined as:

$$\mathrm{Max}^i = \mathrm{Exp}(\mathrm{MAX}^i) \subset M.$$

In [8] Proposition 3.7 we proved that the first Maxwell points corresponding to the reflection symmetries of the vertical subsystem lie on the plane $z = 0$ and the corresponding Maxwell time $t_1^{\mathrm{Max}}(\lambda)$ is given as :

$$\lambda \in C_1 \implies t_1^{\mathrm{Max}}(\lambda) = 4K(k), \tag{2.22}$$
$$\lambda \in C_2 \implies t_1^{\mathrm{Max}}(\lambda) = 4kK(k), \tag{2.23}$$
$$\lambda \in C_3 \cup C_4 \cup C_5 \implies t_1^{\mathrm{Max}}(\lambda) = +\infty. \tag{2.24}$$

Similarly we proved that the first conjugate time $t_1^{\mathrm{conj}}(\lambda)$ is bounded as (Theorems 4.1–4.3) [8]:

$$\lambda \in C_1 \implies 4K(k) \le t_1^{\mathrm{conj}}(\lambda) \le 2p_1^1(k), \tag{2.25}$$
$$\lambda \in C_2 \implies 4kK(k) \le t_1^{\mathrm{conj}}(\lambda) \le 2k\, p_1^1(k), \tag{2.26}$$
$$\lambda \in C_4 \implies t_1^{\mathrm{conj}}(\lambda) = 2\pi, \tag{2.27}$$
$$\lambda \in C_3 \cup C_5 \implies t_1^{\mathrm{conj}}(\lambda) = +\infty. \tag{2.28}$$

where $p_1^1(k)$ is the first positive root of the function $f_1(p) = \mathrm{cn}p\,\mathrm{E}(p) - \mathrm{sn}p\,\mathrm{dn}p$, which is bounded as $p_1^1(k) \in (2K(k), 3K(k))$. Note that we defined:

$$\varphi_t = \tau + p, \quad \varphi = \tau - p \implies \tau = \frac{1}{2}(\varphi_t + \varphi), \ p = \frac{t}{2} \text{ when } \nu = (\lambda, t) \in N_1 \cup N_3, \tag{2.29}$$

$$\psi_t = \frac{\varphi_t}{k} = \tau + p, \quad \psi = \frac{\varphi}{k} = \tau - p \implies \tau = \frac{1}{2k}(\varphi_t + \varphi), \ p = \frac{t}{2k} \text{ when } \nu = (\lambda, t) \in N_2. \tag{2.30}$$

Here and below $N_i = C_i \times \mathbb{R}_+$.

## 3 Upper Bound on Cut Time

In this section we describe the basic properties of the upper bound on cut time obtained in [8].

Define the following function $\mathbf{t} : C \to (0, +\infty]$,

$$\mathbf{t}(\lambda) = \min \left( t_1^{\mathrm{Max}}(\lambda), t_1^{\mathrm{conj}}(\lambda) \right), \quad \lambda \in C.$$

Equalities (2.22)–(2.28) yield the explicit representation of this function:

$$\lambda \in C_1 \implies \mathbf{t}(\lambda) = 4K(k), \tag{3.1}$$
$$\lambda \in C_2 \implies \mathbf{t}(\lambda) = 4kK(k), \tag{3.2}$$
$$\lambda \in C_4 \implies \mathbf{t}(\lambda) = 2\pi, \tag{3.3}$$
$$\lambda \in C_3 \cup C_5 \implies \mathbf{t}(\lambda) = +\infty. \tag{3.4}$$

In [8] we proved the upper bound:

$$t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda), \quad \lambda \in C. \tag{3.5}$$

We now prove that inequality (3.5) is in fact an equality (see Theorem 4.2). The general scheme of the proof is as follows [5], [7]:

1. The exponential mapping $\mathrm{Exp} : N = C \times \mathbb{R}_+ \to M$ parametrizes all optimal geodesics, but also all non-optimal ones, since all the geodesics $\mathrm{Exp}(\lambda, t)$ with $t > \mathbf{t}(\lambda)$ are not optimal.
2. We reduce the domain of the exponential mapping so that it does not include these a priori non-optimal geodesics:
$$\widehat{N} = \{(\lambda, t) \in N \quad | \quad t \leq \mathbf{t}(\lambda)\}.$$
We also reduce the range of the exponential mapping so that it does not contain the initial point for which the optimal geodesic is trivial:
$$\widehat{M} = M \backslash \{q_0\}.$$
Then $\mathrm{Exp} : \widehat{N} \to \widehat{M}$ is surjective, but not injective, due to Maxwell points.
3. We exclude Maxwell points in the image of Exp:
$$\widetilde{M} = \left\{ q \in M \quad | \quad \varepsilon^i(q) \neq q \right\},$$
and reduce respectively the preimage of Exp:
$$\widetilde{N} = \mathrm{Exp}^{-1}\left(\widetilde{M}\right).$$
The mapping $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ is injective. Moreover, it is non-degenerate since $t_1^{\mathrm{conj}}(\lambda) \geq \mathbf{t}(\lambda)$.
4. We take connected components in preimage and image of Exp :
$$\widetilde{N} = \cup D_i, \qquad \widetilde{M} = \cup M_i.$$
Each of the mappings $\mathrm{Exp} : D_i \to M_i$ is non-degenerate and proper. Moreover, all $D_i$ and $M_i$ are smooth 3-dimensional manifolds, connected and simply connected. By Hadamard's global diffeomorphism theorem [14], each $\mathrm{Exp} : D_i \to M_i$ is a diffeomorphism. Thus $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ is a diffeomorphism as well.
5. Further, we consider the action of the exponential mapping on the boundary of the 3-dimensional diffeomorphic domains:
$$\mathrm{Exp} : N' \to M', \quad N' = \widehat{N} \backslash \widetilde{N}, \qquad M' = \widehat{M} \backslash \widetilde{M}.$$
We construct a stratification in the preimage and the image of Exp :
$$N' = \cup N_i', \quad M' = \cup M_i',$$
$$\dim N_i', \ \dim M_i' \in \{0, 1, 2\},$$
where all $N_i'$ are disjoint, while some $M_i'$ coincide with others. Further, we prove that all $\mathrm{Exp} : N_i' \to M_i'$ are diffeomorphisms by the same argument.

6. On the basis of the global diffeomorphic structure of the exponential mapping thus described, we get the following results:

$$\begin{aligned} t_{\mathrm{cut}}(\lambda) &= \mathbf{t}(\lambda), \quad \lambda \in C, \\ \mathrm{Max} &= \cup \left\{ M_i' \quad | \quad \exists\, j \neq i \text{ such that } M_j' = M_i' \right\}, \\ \mathrm{Cut} &= \mathrm{cl}(\mathrm{Max}) \setminus \{q_0\}, \\ \mathrm{Cut} \cap \mathrm{Conj} &= \partial(\mathrm{Max}) \setminus \{q_0\}. \end{aligned}$$

We show that the optimal synthesis is double valued on the Maxwell set Max, and is one valued on $\widehat{M}\backslash\mathrm{Max}$. The central notion of our approach is the stratification in the preimage and in the image of Exp :

$$\begin{aligned} \widehat{N} &= (\cup D_i) \cup \left(\cup N_i'\right), \\ \widehat{M} &= (\cup M_i) \cup \left(\cup M_i'\right), \\ \dim(D_i) &= \dim(M_i) = 3, \\ \dim(N_i'), \ \dim(M_i') &\in \{0, 1, 2\}, \end{aligned}$$

such that all the corresponding strata are diffeomorphic via the exponential mapping, i.e., $\mathrm{Exp} : D_i \to M_i$ and $\mathrm{Exp} : N_i' \to M_i'$ are diffeomorphisms.

It is well known [7],[14] that for any smooth manifolds $X$ and $Y$ of equal dimensions, a smooth mapping $f : X \to Y$ is a diffeomorphism if $f$, $X$ and $Y$ satisfy the following conditions **P1** – **P4**:
**P1** - $X$ is connected,
**P2** - $Y$ is connected and simply connected,
**P3** - $f$ is non-degenerate,
**P4** - $f$ is proper, i.e., for any compact set $K \subset Y$ the inverse image $f^{-1}(K) \subset X$ is also compact.
We now consider the invariance properties of the function $\mathbf{t}$ with respect to the reflections $\varepsilon^i \in G$ and the vertical part of the Hamiltonian vector field:

$$\vec{H}_\nu = c\frac{\partial}{\partial \gamma} - \sin\gamma \frac{\partial}{\partial c} \in \mathrm{Vec}(C).$$

**Proposition 3.1**

(1) *The function* $\mathbf{t}$ *is invariant w.r.t. the reflections* $\varepsilon^i \in G$ *and the flow of* $\vec{H}_\nu$*:*

$$\mathbf{t} \circ \varepsilon^i(\lambda) = \mathbf{t} \circ e^{t\vec{H}_\nu}(\lambda) = \mathbf{t}(\lambda), \quad \lambda \in C, \quad \varepsilon^i \in G, \quad t \in \mathbb{R}.$$

(2) *The function* $\mathbf{t} : C \to (0, +\infty]$ *is in fact a function* $\mathbf{t}(E)$ *of the energy* $E = \frac{c^2}{2} - \cos\gamma$ *of pendulum* (2.6).

*Proof* The reflections $\varepsilon^i \in G$ (2.19) and the flow of $\vec{H}_\nu$ preserve the subsets $C_i$ of the cylinder $C$ and on each of these subsets, the function $\mathbf{t}$ is expressed as a function of the energy $E$ of the pendulum since we have equalities (3.1)–(3.4) and,

$$\begin{aligned} \lambda \in C_1 &\implies k = \sqrt{\frac{E+1}{2}}, \\ \lambda \in C_2 &\implies k = \sqrt{\frac{2}{E+1}}, \\ \lambda \in C_4 &\implies E = -1, \\ \lambda \in C_3 \cup C_5 &\implies E = 1. \end{aligned}$$

This proves item (2) of this proposition. Item (1) follows since the energy $E$ is invariant w.r.t. $\varepsilon^i$ and $\vec{H}_\nu$. □

A plot of $\mathbf{t}(E)$ is shown in Figure 2. Regularity properties of the function $\mathbf{t}(E)$ visible in its plot are proved in the following statement.

**Proposition 3.2**

(1) *The function* $\mathbf{t}(\lambda)$ *is smooth on* $C_1 \cup C_2$.
(2) $\lim_{E\to-1} \mathbf{t}(E) = 2\pi, \quad \lim_{E\to 1} \mathbf{t}(E) = +\infty, \quad \lim_{E\to+\infty} \mathbf{t}(E) = 0.$
(3) *The function* $\mathbf{t} : C \to (0, +\infty]$ *is continuous.*

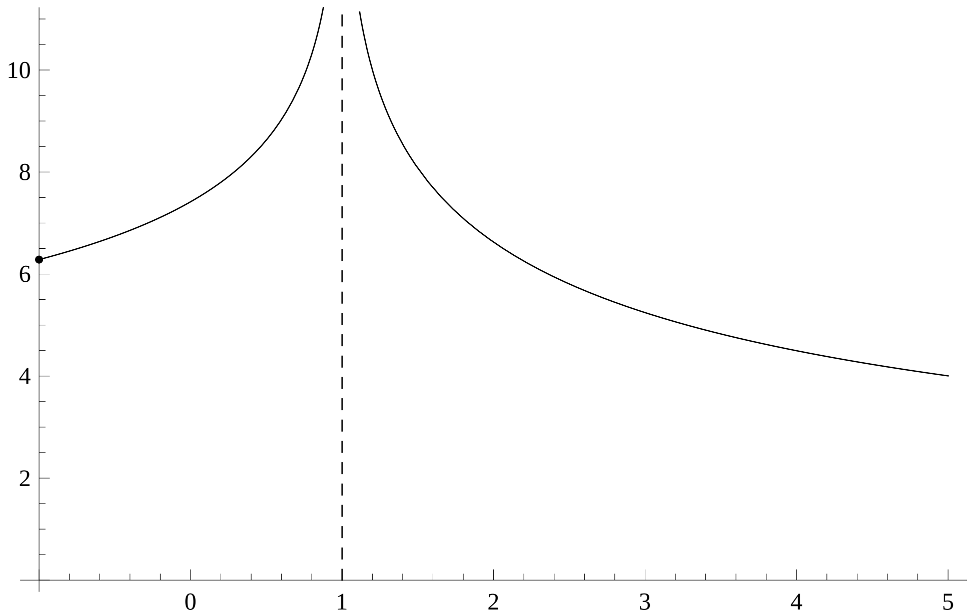


**Fig. 2** Plot of the function $\mathbf{t}(E)$

*Proof* Item (1) follows from (3.1) and (3.2). The limits in item (2) follow from (3.1) and (3.2), and from the limits $\lim_{k\to+0} K(k) = \frac{\pi}{2}, \quad \lim_{k\to 1-} K(k) = +\infty$. Then continuity of $\mathbf{t}(\lambda)$ follows on $C_4$:

$$\lambda \to \bar{\lambda} \in C_4 \implies E(\lambda) \to E(\bar{\lambda}) = -1 \implies \mathbf{t}(\lambda) \to 2\pi = \mathbf{t}(\bar{\lambda}).$$

Continuity on $C_3 \cup C_5$ follows since

$$\lambda \to \bar{\lambda} \in C_3 \cup C_5 \implies E(\lambda) \to E(\bar{\lambda}) = 1 \implies \mathbf{t}(\lambda) \to +\infty = \mathbf{t}(\bar{\lambda}).$$

Thus $\mathbf{t}(\lambda)$ is continuous on $C$ and item (3) is proved. □

## 3.1 Decompositions in the Image of the Exponential Mapping

Consider the set $\widehat{M} = M \backslash \{q_0\}$. From Filippov's theorem and Pontryagin's Maximum Principle [3], we already know that any point $q \in \widehat{M}$ can be joined with $q_0$ by an optimal trajectory $q(s) = \mathrm{Exp}(\lambda, s)$ such that $q(t) = q, \quad (\lambda, t) \in N$. Then $\mathrm{Exp}(N) \supset \widehat{M}$. However the Maxwell points $q \in \widehat{M}$ have non unique preimage under the exponential mapping. Hence the mapping $\mathrm{Exp} : N \to \widehat{M}$ is surjective, but not injective. In order to separate Maxwell points we consider the set that contains all such points:

$$M' = \left\{ q \in M \quad | \quad z = 0, \quad x^2 + y^2 \neq 0 \right\},$$

and its complement $\widetilde{M}$ in $\widehat{M}$:

$$\widetilde{M} = \{ q \in M \quad | \quad z \neq 0 \},$$
$$\widehat{M} = \widetilde{M} \sqcup M',$$

where $\sqcup$ is the union of disjoint sets.

### *3.1.1 Decompositions in $\widetilde{M}$*

The plane $z = 0$ cuts the domain $\widetilde{M}$ into two half spaces as:

$$\widetilde{M} = M_1 \sqcup M_2,$$
$$M_1 = \{ q \in M \quad | \quad z > 0 \}, \tag{3.6}$$
$$M_2 = \{ q \in M \quad | \quad z < 0 \}. \tag{3.7}$$

Note that the decomposition of the manifold $M$ is simpler in description of cut time on SH(2) than similar decomposition of $M$ in related problems on SE(2) [5] and on the Engel group [7].

| Id,$\varepsilon^1$,$\varepsilon^6$,$\varepsilon^7$ | $\varepsilon^2$,$\varepsilon^3$,$\varepsilon^4$,$\varepsilon^5$ |
|---|---|
| $M_1$ | $M_2$ |
| $M_2$ | $M_1$ |

**Table 1** Action of $\varepsilon^i$ on $M_j$

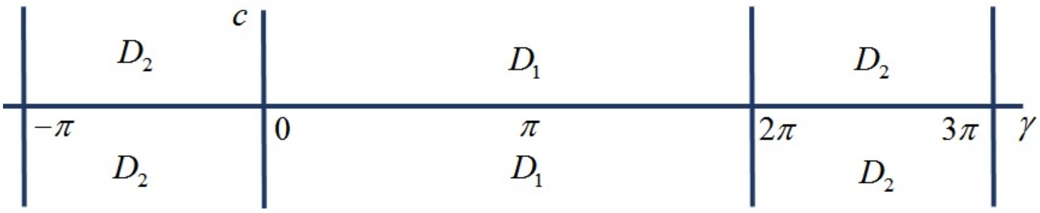


**Fig. 3** Projections of $D_i$ to Phase Cylinder $C$ of the Pendulum at $t = 0$

**Proposition 3.3** *Reflections $\varepsilon^j \in G$ permute the domains $M_1$ and $M_2$ according to Table* 1.

*Proof* Follows immediately from the definitions of the actions of reflections (2.20). □

**Proposition 3.4** *The domains $M_1, M_2$ are open, connected and simply connected.*

*Proof* From the definition of the sets $M_1$, $M_2$ (3.6)–(3.7) it follows that the domains $M_i$ are homeomorphic to $\mathbb{R}^3$ and therefore they are open, connected and simply connected. □

## 3.2 Decomposition in the Preimage of the Exponential Mapping

We now consider the following set $\widehat{N} \subset N$ corresponding to all potentially optimal geodesics:

$$\widehat{N} = \{(\lambda, t) \in N \quad | \quad t \leq \mathbf{t}(\lambda)\}.$$

By existence of the optimal geodesics, $\mathrm{Exp}(\widehat{N}) \supset \widehat{M}$. In order to separate the Maxwell points in the preimage of the exponential mapping, introduce further the sets:

$$\begin{aligned}
\widehat{N} &= \widetilde{N} \sqcup N',\\
N' &= \left\{(\lambda, t) \in \cup_{i=1}^{3} \widehat{N}_i \quad | \quad t = \mathbf{t}(\lambda) \text{ or } \sin \frac{\gamma_{t/2}}{2} = 0\right\} \cup \widehat{N}_4,\\
\widehat{N}_i &= N_i \cap \widehat{N}, \quad i = 1, \dots, 4,\\
\widetilde{N} &= \left\{(\lambda, t) \in \cup_{i=1}^{3} N_i \quad | \quad t < \mathbf{t}(\lambda), \quad \sin \frac{\gamma_{t/2}}{2} \neq 0\right\} \cup N_5.
\end{aligned}$$

### *3.2.1 Decomposition in $\widetilde{N}$*

We now introduce the connected components $D_i$ of the set $\widetilde{N}$:

$$\begin{aligned}
\widetilde{N} &= D_1 \sqcup D_2,\\
D_1 &= \left\{(\lambda, t) \in \cup_{i=1}^{3} N_i \quad | \quad t < \mathbf{t}(\lambda), \quad \sin\left(\frac{\gamma_{t/2}}{2}\right) > 0\right\},\\
D_2 &= \left\{(\lambda, t) \in \cup_{i=1}^{3} N_i \quad | \quad t < \mathbf{t}(\lambda), \quad \sin\left(\frac{\gamma_{t/2}}{2}\right) < 0\right\},
\end{aligned}$$

where $D_i$ are defined explicitly in coordinates in Table 2 (in the sets $N_1, N_2, N_3$). Projections of the sets $D_i$ to the initial phase cylinder are shown in Figure 3. We note that for $t < \mathbf{t}(\lambda) = t_1^{\mathrm{Max}}(\lambda)$ the values of $p$ are given from formulas (2.29)–(2.30), and the values of $t_1^{\mathrm{Max}}(\lambda)$ are given in (2.22)–(2.24). The values of $\tau$ in Table 2 were calculated by using the definition of elliptic coordinates [1], formulas for Jacobi elliptic functions [?] and values of $\gamma$ and $c$ from Figure 1. Note that enumeration of the sets $D_i$ is chosen to correspond to the sets $M_i$ for further analysis.

We now establish an important fact about the domains $D_i$ that is vital in proving that the exponential mapping transforms $D_i$ diffeomorphically.

**Proposition 3.5** *Reflections $\varepsilon^j \in G$ permute the domains $D_1$ and $D_2$ as shown in Table* 3.

| $D_i$ | $D_1$ | | $D_2$ | |
|---|---|---|---|---|
| $\lambda$ | $C_1^0$ | $C_1^1$ | $C_1^0$ | $C_1^1$ |
| $p$ | $(0,2K)$ | $(0,2K)$ | $(0,2K)$ | $(0,2K)$ |
| $\tau$ | $(0,2K)$ | $(2K,4K)$ | $(2K,4K)$ | $(0,2K)$ |
| $\lambda$ | $C_2^+$ | $C_2^-$ | $C_2^+$ | $C_2^-$ |
| $p$ | $(0,2K)$ | $(0,2K)$ | $(0,2K)$ | $(0,2K)$ |
| $\tau$ | $(0,2K)$ | $(-2K,0)$ | $(2K,4K)$ | $(0,2K)$ |
| $\lambda$ | $C_3^{0+}\cup C_3^{1-}$ | $C_3^{0-}\cup C_3^{1+}$ | $C_3^{0+}\cup C_3^{1-}$ | $C_3^{0-}\cup C_3^{1+}$ |
| $p$ | $(0,+\infty)$ | $(0,+\infty)$ | $(0,+\infty)$ | $(0,+\infty)$ |
| $\tau$ | $(0,+\infty)$ | $(-\infty,0)$ | $(-\infty,0)$ | $(0,+\infty)$ |

**Table 2** Decomposition $\widetilde{N}=\cup_{i=1}^2 D_i$

| Id,$\varepsilon^1$,$\varepsilon^6$,$\varepsilon^7$ | $\varepsilon^2$,$\varepsilon^3$,$\varepsilon^4$,$\varepsilon^5$ |
|---|---|
| $D_1$ | $D_2$ |
| $D_2$ | $D_1$ |

**Table 3** Action of $\varepsilon^i$ on $D_j\subset\widetilde{N}$

*Proof* In paper [1] we defined the action of reflections $\varepsilon^j: N\to N$ so that it satisfies the following properties:

$$\varepsilon^j(\lambda,t)=\left(\varepsilon^j\circ e^{t\overrightarrow{H}_\nu}(\lambda),t\right),\quad \text{if}\quad \varepsilon^j_*\overrightarrow{H}_\nu=-\overrightarrow{H}_\nu,$$
$$\varepsilon^j(\lambda,t)=\left(\varepsilon^j(\lambda),t\right),\quad \text{if}\quad \varepsilon^j_*\overrightarrow{H}_\nu=\overrightarrow{H}_\nu,$$

where $\varepsilon^j_*\left(\overrightarrow{H}_\nu\right)$ is the pushforward of $\overrightarrow{H}_\nu$ under the reflection $\varepsilon^j$. Recall that $\varepsilon^j_*\overrightarrow{H}_\nu=-\overrightarrow{H}_\nu$, for $j=1,2,5,6$ because these symmetries reverse the direction of time and $\varepsilon^j_*\overrightarrow{H}_\nu=\overrightarrow{H}_\nu$, for $j=3,4,7$ because these symmetries preserve the direction of time [1]. Hence, it is sufficient to prove the case $\varepsilon^2(D_1)=D_2$ as proof of all other cases $\varepsilon^j(D_i)=D_k$ is similar. In order to prove the inclusion $\varepsilon^j(D_1)\subset D_2$ we take any $(\lambda,t)=(\gamma,c,t)\in D_1$ and prove that

$$\varepsilon^2:(\lambda,t)\mapsto(\lambda^2,t)=(\gamma^2,c^2,t)\in D_2.$$

By Proposition 3.1,

$$\mathbf{t}(\lambda^2)=\mathbf{t}\circ\varepsilon^2\circ e^{t\overrightarrow{H}_\nu}(\lambda)=\mathbf{t}(\lambda).$$

Thus $t<\mathbf{t}(\lambda)$. Moreover, at instant $t/2$ the trajectories of the vertical subsystem are given as:

$$\lambda_{t/2}=(\gamma_{t/2},c_{t/2})=e^{\overrightarrow{H}_\nu t/2}(\lambda),$$
$$\lambda^2_{t/2}=\left(\gamma^2_{t/2},c^2_{t/2}\right)=e^{\overrightarrow{H}_\nu t/2}\left(\lambda^2\right),$$

Since $\lambda^2=\varepsilon^2\circ e^{\overrightarrow{H}_\nu t}(\lambda)$, we have

$$\lambda^2_{t/2}=e^{\overrightarrow{H}_\nu t/2}\circ\varepsilon^2\circ e^{\overrightarrow{H}_\nu t}(\lambda)=\varepsilon^2\circ e^{-\overrightarrow{H}_\nu t/2}\circ e^{\overrightarrow{H}_\nu t}(\lambda)=\varepsilon^2\circ e^{\overrightarrow{H}_\nu t/2}(\lambda)=\varepsilon^2(\lambda_{t/2}).\tag{3.8}$$

In proof of (3.8) we used the fact that for any diffeomorphism $F: M\to M$ and a vector field $\overrightarrow{V}$ on a manifold $M$, $F_*\overrightarrow{V}=-\overrightarrow{V}\iff F\circ e^{t\overrightarrow{V}}=e^{-t\overrightarrow{V}}\circ F$. Clearly, $\varepsilon^2(\lambda_{t/2})=\left(\gamma^2_{t/2},c^2_{t/2}\right)$ and from (6.3) [1] we have:

$$\left(\gamma^2_{t/2},c^2_{t/2}\right)=\left(-\gamma_{t/2},c_{t/2}\right).$$

Thus $\sin\frac{\gamma^2_{t/2}}{2}=\sin\frac{-\gamma_{t/2}}{2}<0$. We proved that $(\lambda^2,t)\in D_2$, thus $\varepsilon^2(D_1)\subset D_2$. Similarly it follows that $\varepsilon^2(D_2)\subset D_1$. Since $\varepsilon^2\circ\varepsilon^2=\mathrm{Id}$, then $\varepsilon^2\circ\varepsilon^2(D_1)=D_1\implies\varepsilon^2(D_1)=D_2$. □

**Proposition 3.6** *The domains $D_1,D_2\subset\widetilde{N}$ are open and connected.*

*Proof* Since $\varepsilon^2: N\to N$ is a diffeomorphism and $\varepsilon^2(D_1)=D_2$ it suffices to prove that $D_1$ is open and connected. Consider a vector field

$$P=\frac{t}{2}\left(c\frac{\partial}{\partial\gamma}-\sin\gamma\frac{\partial}{\partial c}\right)\in\mathrm{Vec}(N).$$

The flow of this vector field $e^P$ is given as:

$$e^P(\gamma,c,t)=e^P(\lambda,t)=\left(e^{\frac{t}{2}\overrightarrow{H}_\nu}(\lambda),t\right)=\left(\gamma_{t/2},c_{t/2},t\right).$$

Thus $e^P(D_1) = \widetilde{D}_1$ where
$$\widetilde{D}_1 = \left\{(\lambda, t) \in N \quad | \quad \sin\frac{\gamma}{2} > 0, \quad t < \mathbf{t}(\lambda)\right\}.$$
The set $\widetilde{D}_1$ is a subgraph of a continuous function $\lambda \mapsto \mathbf{t}(\lambda)$ on an open connected 2-dimensional domain $\{(\gamma, c) \in C \quad | \quad \gamma \in (0, 2\pi), \quad c \in \mathbb{R}\}$, thus $\widetilde{D}_1$ is open and connected. Since $D_1 = e^{-P}(\widetilde{D}_1)$ therefore $D_1$ is also open and connected. □

**Proposition 3.7** *There hold the inclusions:*

(1) $\mathrm{Exp}(D_i) \subset M_i, \quad i = 1, 2,$
(2) $\mathrm{Exp}(\widetilde{N}) \subset \widetilde{M},$
(3) $\mathrm{Exp}(N') \subset M'.$

*Proof*

(1) It suffices to prove only that $\mathrm{Exp}(D_1) \subset M_1$, in view of the reflections $\varepsilon^j$. Notice the decomposition:
$$D_1 = (D_1 \cap N_1) \sqcup (D_1 \cap N_2) \sqcup (D_1 \cap N_3) \sqcup (D_1 \cap N_5). \tag{3.9}$$
Let $(\lambda, t) \in D_1 \cap N_1 = \left\{(\lambda, t) \in N_1 \quad | \quad t < \mathbf{t}(\lambda), \quad \sin\frac{\gamma_{t/2}}{2} > 0\right\}$, thus $p = \frac{t}{2} \in (0, 2K(k))$. Further, from formula (5.3) [1] we have $s_1 \mathrm{sn}\tau > 0$. Now recall formula (3.2) [8]:
$$\sinh z_t = s_1 \frac{2k\,\mathrm{sn}p\,\mathrm{sn}\tau}{\Delta}, \quad \Delta = 1 - k^2 \mathrm{sn}^2 p\,\mathrm{sn}^2\tau. \tag{3.10}$$
Then we get $\sinh z_t > 0$, thus $z_t > 0$, i.e., $\mathrm{Exp}(\lambda, t) \in M_1$. We proved that $\mathrm{Exp}(D_1 \cap N_1) \subset M_1$. All other required inclusions $\mathrm{Exp}(D_1 \cap N_j) \subset M_1, \quad j = 2, 3, 5$, are proved similarly, and the inclusion $\mathrm{Exp}(D_1) \subset M_1$ follows.
(2) Since $\widetilde{N} = D_1 \cup D_2$ and $\widetilde{M} = M_1 \cup M_2$, the inclusion $\mathrm{Exp}(\widetilde{N}) \subset \widetilde{M}$ follows from item (1).
(3) We have $N' = \left(N' \cap N_1\right) \sqcup \left(N' \cap N_2\right) \sqcup \left(N' \cap N_3\right) \sqcup N_4$.
Let $(\lambda, t) \in N' \cap N_1 = \left\{(\lambda, t) \in \widehat{N}_1 \quad | \quad t = \mathbf{t}(\lambda) \text{ or } \sin\frac{\gamma_{t/2}}{2} = 0\right\}$, then similarly to the proof of item (1) we get $p = 2K(k)$ or $\mathrm{sn}\tau = 0$, thus $z_t = 0$ by (3.10). From (3.6) [8] we get $R_2(q_t) = \frac{2s_1}{1-k^2}\mathrm{dn}\tau\, f_2(p) \neq 0$, and therefore $x^2 + y^2 \neq 0$. We proved that $\mathrm{Exp}(N' \cap N_1) \subset M'$. It follows similarly that $\mathrm{Exp}(N' \cap N_j) \subset M', \quad j = 2, 3$. Finally, if $(\lambda, t) \in \widehat{N}_4$, then
$$q_t = (x_t, y_t, z_t) = (t, 0, 0) \in M'.$$
Consequently, $\mathrm{Exp}(N') \subset M'$. □

**Theorem 3.1** *For* $\lambda \in \cup_{i=1}^5 C_i$, *we have* $t_1^{\mathrm{conj}}(\lambda) \geq t_1^{\mathrm{Max}}(\lambda)$.

*Proof* Apply equations (2.22)–(2.24) and (2.25)–(2.28). □

**Proposition 3.8** *The restriction* $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ *is non-degenerate.*

*Proof* From Theorem 3.1, $t_1^{\mathrm{conj}}(\lambda) \geq t_1^{\mathrm{Max}}(\lambda)$. Since for any $\nu = (\lambda, t) \in \widetilde{N}$ we have $t < \mathbf{t}(\lambda)$ and therefore exponential mapping is non-degenerate $\forall \nu = (\lambda, t) \in \widetilde{N}$ . □

Hence we proved properties **P1, P2** and **P3** for the exponential mapping $\mathrm{Exp} : D_i \to M_i$. It only remains to prove condition **P4** now to establish that the exponential mapping $\mathrm{Exp} : D_i \to M_i$ is indeed a diffeomorphism.

### 3.3 Diffeomorphic Properties of the Exponential Mapping

In this subsection we prove that the exponential mapping $\mathrm{Exp} : D_i \to M_i, \quad i = 1, 2$, is proper. First we recall an equivalent formulation of the properness property.

**Definition 1** Let $X$ be a topological space and $\{x_n\} \subset X$ a sequence. We write $x_n \to \partial X$ if there is no compact $K \subset X$ such that $x_n \in K$ for any $n \in \mathbb{N}$.

*Remark 1* Let $X, Y$ be topological spaces and $F : X \to Y$ a continuous mapping. The mapping $F$ is proper iff for any sequence $\{x_n\} \subset X$ there holds the implication:
$$x_n \to \partial X \implies F(x_n) \to \partial Y.$$
Below we apply this properness test to the mapping $\mathrm{Exp} : D_1 \to M_1$.

**Lemma 1** *Let $\{q_n\} \subset M_1$. We have $q_n \to \partial M_1$ iff there is a subsequence $\{n_k\}$ on which one of the conditions holds:*

(1) $z \to 0$,
(2) $z \to +\infty$,
(3) $x \to \infty$,
(4) $y \to \infty$.

*Proof* Any compact set in $M_1$ is contained in a compact set $\left\{q \in M_1 \quad | \quad \varepsilon \leq z \leq \frac{1}{\varepsilon}, \quad |x| \leq \frac{1}{\varepsilon}, \quad |y| \leq \frac{1}{\varepsilon}\right\}$ for some $\varepsilon \in (0,1)$. □

**Lemma 2** *Let $\{\nu_n\} \subset D_1$, then $\nu_n \to \partial D_1$ iff there is a subsequence $\{n_k\}$ on which one of the following conditions hold:*

(1) $\gamma_{t/2} \to 0$,
(2) $\gamma_{t/2} \to 2\pi$,
(3) $c_{t/2} \to \infty$,
(4) $t \to 0$,
(5) $t \to +\infty$,
(6) $\mathbf{t}(\lambda) - t \to 0$.

*Proof* Any compact set in $D_1$ is contained in a compact set

$$\left\{\nu \in N \mid \gamma_{t/2} \in [\varepsilon, 2\pi - \varepsilon], \left|c_{t/2}\right| \leq \frac{1}{\varepsilon}, \, t \in [\varepsilon, \frac{1}{\varepsilon}], \, \mathbf{t}(\lambda) - t \geq \varepsilon\right\},$$

for some $\varepsilon \in (0,1)$. □

**Proposition 3.9** *The mapping* $\mathrm{Exp} : D_i \to M_i, \quad i = 1, 2,$ *is proper.*

*Proof* In view of the reflections $\varepsilon^j$, it suffices to consider the case $\mathrm{Exp} : D_1 \to M_1$. Let $\{\nu_n\} \subset D_1$, $\nu_n \to \partial D_1$, we have to show that $q_n = \mathrm{Exp}(\nu_n) \to \partial M_1$. Taking into account decomposition (3.9), we can consider the cases $\{\nu_n\} \subset D_1 \cap N_j, \quad j = 1, 2, 3, 5$.

Let $\{\nu_n\} \subset D_1 \cap N_1$, $\nu_n \to \partial D_1$. We will need the following formulas for the extremals $\lambda_t = e^{t\vec{H}}(\lambda), \quad \lambda \in C_1$, obtained in [1] and [8]:

$$\begin{aligned}
\sin \frac{\gamma_t}{2} &= s_1 k \,\mathrm{sn}(\varphi_t),\\
\frac{c_t}{2} &= k \,\mathrm{cn}(\varphi_t),\\
\sinh z_t &= s_1 \frac{k \,\mathrm{sn} p \,\mathrm{sn}\tau}{\Delta}, \quad \Delta = 1 - k^2 \mathrm{sn}^2 p \,\mathrm{sn}^2 \tau,\\
R_2(q_t) &= f_2(p) \frac{2 s_1}{1 - k^2} \mathrm{dn}\tau, \quad f_2(p) = \mathrm{dn} p \,\mathrm{E}(p) - k^2 \mathrm{sn} p \,\mathrm{cn} p.
\end{aligned}$$

Notice that $p = \frac{t}{2}, \quad \tau = \varphi + \frac{t}{2}$, and consider all the cases (1)–(6) of Lemma 2.

(1) If $\gamma_{t/2} \to 0$, then $\sin \frac{\gamma_{t/2}}{2} = s_1 k \,\mathrm{sn}\tau \to 0$, thus $\sinh z_t \to 0$, so $z_t \to 0$, hence $q_n \to \partial M_1$(Lemma 1, (1)).
(2) If $\gamma_{t/2} \to 2\pi$, then $\sin \frac{\gamma_{t/2}}{2} = s_1 k \,\mathrm{sn}\tau \to 0$, thus $\sinh z_t \to 0$, so $z_t \to 0$, hence $q_n \to \partial M_1$.
(3) The case $c_{t/2} \to \infty$ is impossible.
(4) If $t \to 0$, then $p \to 0$, thus $z_t \to 0$.
(5) Let $t \to +\infty$, then $p \to +\infty$. Since $p \in (0, 2K(k))$ then $k \to 1$. Denote $u = \mathrm{am}(p) \in (0, \pi)$. On a subsequence we have $u \to \bar{u} \in [0, \pi]$ and we will suppose so in the sequel.
  (a) If $\bar{u} \in [0, \pi)$, then $p = F(u, k) \to F(\bar{u}, 1) = \int_0^{\bar{u}} \frac{dt}{\cos(t)} < +\infty$, a contradiction.
  (b) Let $\bar{u} = \frac{\pi}{2}$, thus $\mathrm{sn} p = \sin u \to 1$, $\mathrm{cn} p = \cos(u) \to c$.
    i. If $\mathrm{sn}\tau \to 1$, then $\Delta \to 0$, thus $z_t \to \infty$.
    ii. Let $\mathrm{sn}\tau \to \bar{s} \neq 1$, then $\mathrm{dn}\tau \to \sqrt{1 - \bar{s}^2} \neq 0$. Denote

$$g_2(u) = f_2(F(u,k)) = \sqrt{1 - k^2 \sin^2 u} \mathrm{E}(u, k) - k^2 \sin(u) \cos(u).$$

We prove now that $\frac{g_2(u)}{1-k^2} \to +\infty$, then $\frac{f_2(u)}{1-k^2} \to +\infty$, thus $R_2(q_t) \to \infty$, so $x_t^2 + y_t^2 + z_t^2 \to \infty$, whence $q_t \to \partial M_1$. Denote $k' = \sqrt{1 - k^2} \to 0$. We can suppose that on a subsequence $\frac{\cos u}{k'} \to \alpha \in [0, +\infty]$. We have

$$\begin{aligned}
k^2 \sin(u) \cos(u) &= \sin(u) \cos(u) + o(k'^2),\\
\sqrt{1 - k^2 \sin^2 u} &= \sqrt{\cos^2 u + k'^2 - k'^2 \cos^2 u}.
\end{aligned}$$

Now we estimate $E(u,k)$ from below:

$$\begin{aligned}
E(u,k)-\sin(u) &= \int_0^u \sqrt{1-k^2\sin^2 t}dt - \int_0^u \cos(t)dt = \int_0^u \frac{1-k^2\sin^2 t-\cos^2 t}{\sqrt{1-k^2\sin^2 t}+\cos t}dt \\
&> \frac{1-k^2}{2}\int_0^u \sin^2 t\, dt \\
&= \frac{1-k^2}{4}\left(u-\frac{\sin(2u)}{2}\right) \\
&= \frac{\pi}{8}k'^2(1+o(1)).
\end{aligned}$$

Thus,

$$E(u,k) > \sin(u) + \frac{\pi}{8}k'^2(1+o(1)).$$

A. Let $\alpha \in [0,+\infty)$. Then $\cos(u) = \alpha k' + o(k')$, $\quad \sin(u) = 1 + o(1)$, thus

$$\begin{aligned}
k^2\sin(u)\cos(u) &= \alpha k' + o(k'), \\
\sqrt{1-k^2\sin^2(u)} &= \sqrt{1+\alpha^2}k' + o(k'), \\
E(u,k) &= 1 + o(1), \\
\sqrt{1-k^2\sin^2 u}\, E(u,k) &= \sqrt{1+\alpha^2}k' + o(k'), \\
g_2(u) &= \left(\sqrt{1+\alpha^2}-\alpha\right)k' + o(k'), \\
\frac{g_2(u)}{k'^2} &= \frac{(\sqrt{1+\alpha^2}-\alpha)}{k'}(1+o(1)) \to \infty,
\end{aligned}$$

and the claim follows.

B. Let $\alpha = +\infty$, thus $k' = o(\cos(u))$. Then

$$\begin{aligned}
k^2\sin(u)\cos(u) &= \sin(u)\cos(u) - k'^2\cos(u) + o\left(k'^2\cos(u)\right), \\
\sqrt{1-k^2\sin^2 u} &= \cos(u)\sqrt{1+\frac{k'^2}{\cos^2 u} + o\left(\frac{k'^2}{\cos^2 u}\right)} \\
&= \cos(u) + \frac{1}{2}\frac{k'^2}{\cos(u)} + o\left(\frac{k'^2}{\cos(u)}\right), \\
\sqrt{1-k^2\sin^2 u}\, E(u,k) &> \cos(u)\sin(u) + \frac{1}{2}\frac{k'^2}{\cos(u)} + o\left(\frac{k'^2}{\cos(u)}\right), \\
g_2(u) &> \frac{1}{2}\frac{k'^2}{C}(1+o(1)), \\
\frac{g_2(u)}{k'^2} &> \frac{1}{2C}(1+o(1)) \to +\infty,
\end{aligned}$$

and the claim follows.

iii. Let $u \in (0,\pi)$, then $f_2(p) = g_2(u) \to |\cos\bar{u}|\,(E(\bar{u},1)+\sin\bar{u}) > 0$, thus

$$\frac{f_2(p)}{\sqrt{1-k^2}} \to +\infty.$$

Since $\frac{\mathrm{dn}\tau}{\sqrt{1-k^2}} \geq 1$, then $R_2(q_t) \to \infty$, so $x_t^2 + y_t^2 + z_t^2 \to \infty$, whence $q_t \to \partial M_1$.

iv. If $\bar{u} = \pi$, then $\mathrm{sn} p = \sin(u) \to 0$, thus $z_t \to 0$.

(6) Let $\mathbf{t}(\lambda) - t \to 0$. Recall that $\mathbf{t}(\lambda) = 4K(k)$ for $\lambda \in C_1$, thus $4K(k) - t \to 0$. Since $k \in (0,1)$, then there is a subsequence $\{n_m\}$ on which $k \to \bar{k} \in [0,1]$. If $\bar{k} \in [0,1)$, then $K(k) \to K(\bar{k}) < +\infty$, thus $t \to 4K(\bar{k})$, so $p = 2K(\bar{k})$. Consequently, $\sinh z_t \to 0$, whence $q_n \to \partial M_1$ (Lemma 1, (1)). If $\bar{k} = 1$, then $K(k) \to +\infty$, thus $t \to +\infty$, $q_n \to \partial M_1$ by item (5).

Consequently, in each of the cases (1)–(6) of Lemma 2 we get $q_n \to \partial M_1$ for a sequence $\{\nu_n\} \subset D_1 \cap N_1$, $\quad \nu_n \to \partial D_1$. All the rest cases $\{\nu_n\} \subset D_1 \cap N_j$, $\quad j = 2,3,5$, are considered similarly.

Summing up, for any sequence $\{\nu_n\} \subset D_1$ with $\nu_n \to \partial D_1$ we have $\mathrm{Exp}(\nu_n) \to \partial M_1$. Thus the mapping $\mathrm{Exp} : D_1 \to M_1$ is proper. □

Now we get the main result of this section.

**Theorem 3.2** *The mapping* $\mathrm{Exp} : D_i \to M_i, \quad i = 1, 2,$ *is a diffeomorphism.*

*Proof* All of the conditions **P1**–**P4** are satisfied for the mapping $\mathrm{Exp} : D_1 \to M_1$:

- $D_1 \subset N$ and $M_1 \subset M$ are open subsets thus 3-dimensional manifolds (Proposition 3.6, Proposition 3.4),
- **P1 -** $D_1$ is connected (Proposition 3.6),
- **P2 -** $M_1$ is connected and simply connected (Proposition 3.4),
- **P3 -** $\mathrm{Exp}|_{D_1}$ is non-degenerate (Proposition 3.8),
- **P4 -** $\mathrm{Exp} : D_1 \to M_1$ is proper (Proposition 3.9).

Thus $\mathrm{Exp} : D_1 \to M_1$ is a diffeomorphism. By virtue of the reflections, $\mathrm{Exp} : D_2 \to M_2$ is a diffeomorphism as well.

□

**Corollary 1** *The exponential mapping* $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ *is a diffeomorphism.*

*Proof* Follows from Theorem 3.2. □

### 3.4 Cut Time

Now we can prove that inequality (3.5) is in fact an equality for $\lambda \in C\backslash C_4$.

**Theorem 3.3** *If* $\lambda \in C\backslash C_4$*, then* $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$.

*Proof* Let $\lambda \in C\backslash C_4 = \cup_{i=1}^{3} C_i \cup C_5$. In view of inequality (3.5), it remains to prove that $t_{\mathrm{cut}}(\lambda) \geq \mathbf{t}(\lambda)$. Take any $t_1 \in (0, \mathbf{t}(\lambda))$.We need to prove that the geodesic $\mathrm{Exp}(\lambda, t)$ is optimal on the segment $t \in [0, t_1]$.

Consider first the case $\lambda \in \cup_{i=1}^{3} C_i$. If $\sin \frac{\gamma_{t_1/2}}{2} \neq 0$, then $(\lambda, t_1) \in \widetilde{N}$, and $q_1 = \mathrm{Exp}(\lambda, t_1) \in \widetilde{M}$. By virtue of Proposition 3.7 and Theorem 3.2, the point $q_1$ has a unique preimage under the mapping $\mathrm{Exp} : \widehat{N} \to \widehat{M}$. Thus the geodesic $\mathrm{Exp}(\lambda, t)$ is optimal on the segment $t \in [0, t_1]$.

If $\lambda \in \cup_{i=1}^{3} C_i$ and $\sin \frac{\gamma_{t_1/2}}{2} = 0$, then we can choose $t_2 \in (t_1, \mathbf{t}(\lambda))$ such that $\sin \frac{\gamma_{t_2/2}}{2} \neq 0$. By the argument of the preceding paragraph, the geodesic $\mathrm{Exp}(\lambda, t)$ is optimal at the segment $[0, t_2]$, thus at the segment $[0, t_1] \subset [0, t_2]$ as well.

Finally, if $\lambda \in C_5$, then $(\lambda, t_1) \in \widetilde{N}$, and the geodesic $\mathrm{Exp}(\lambda, t), \quad t \in [0, t_1]$, is optimal as above.

We proved that $t_{\mathrm{cut}}(\lambda) \geq \mathbf{t}(\lambda)$, thus $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$ for any $\lambda \in C\backslash C_4$. □

We will be able to prove the equality $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$ for $\lambda \in C_4$ below after the description of the structure of the exponential mapping $\mathrm{Exp} : N' \to M'$. The geodesic $\mathrm{Exp}(\lambda, t), \quad \lambda \in C_4$, requires a separate study since it belongs to the set $M'$ for all $t > 0$.

Intuitively, Theorem 3.3 establishes the fact that since $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ is a diffeomorphism, hence upto time $t < \mathbf{t}(\lambda)$ there is a unique point $\nu = (\lambda, s) \in \widetilde{N}$ that is mapped to a unique extremal trajectory $q_s = \mathrm{Exp}(\lambda, s) \in \widetilde{M}$ that joins $q_0 \in M$ to $q_1 \in \widetilde{M} \subset M$. Hence, the trajectory $q_s = \mathrm{Exp}(\lambda, s) \in \widetilde{M}$ is optimal and therefore $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$. It therefore follows that optimal synthesis in the domain $\widetilde{M}$ is given by:

$$u_i(q) = h_i(\lambda), \quad i = 1, 2, \quad (\lambda, t) = \mathrm{Exp}^{-1}(q) \in \widetilde{N}, \quad q \in \widetilde{M},$$

where $u_i$ are the control variables (i.e., translational and rotational velocities) and $h_i$ are the optimal controls defined in (4.8) [1].

## 4 Exponential Mapping on the Boundary of Diffeomorphic Domains

Until now we have studied the mapping $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ and proved that it is a diffeomorphism. This allowed us to prove that the cut time $t_{\mathrm{cut}}(\lambda) = t_1^{\mathrm{Max}}(\lambda), \quad \lambda \in C\backslash C_4$. In this section we obtain the global structure of the exponential mapping in order to characterize the cut locus and the Maxwell strata and to construct the optimal synthesis. Specifically we study the mapping $\mathrm{Exp} : N' \to M'$ where:

$$N' = \left\{(\lambda, t) \in \cup_{i=1}^{3} N_i \quad | \quad t = t_1^{\mathrm{Max}}(\lambda) \quad \text{or} \quad \sin\left(\frac{\gamma_{t/2}}{2}\right) = 0\right\} \cup \left\{(\lambda, t) \in N_4 \quad | \quad t \leq 2\pi = t_1^{\mathrm{conj}}(\lambda)\right\},$$
$$M' = \left\{q \in M \quad | \quad x^2 + y^2 \neq 0, \quad z = 0\right\}.$$

| $j$ | $\lambda$ | $p$ | $\tau$ | $k$ |
|---|---|---|---|---|
| 1 | $C_1^0$ | $2K$ | $(0, K)$ | $(0, 1)$ |
| 9 | $C_2^+$ | $2K$ | $(0, K)$ | $(0, 1)$ |
| 17 | $C_1^0$ | $2K$ | $K$ | $(0, 1)$ |
| 21 | $C_1^0$ | $2K$ | $0$ | $(0, 1)$ |
| 25 | $C_2^+$ | $2K$ | $0$ | $(0, 1)$ |
| 29 | $C_2^+$ | $2K$ | $K$ | $(0, 1)$ |

**Table 4** Decomposition $N'_j, \quad j \in \{1, 9, 17, 21, 25, 29\}$

| $\lambda$ | $p$ | $\tau$ | $k$ |
|---|---|---|---|
| $C_1^0$ | $(0, 2K)$ | $0$ | $(0, 1)$ |
| $C_2^+$ | $(0, 2K)$ | $0$ | $(0, 1)$ |
| $C_3^{0+}$ | $(0, +\infty)$ | $0$ | $1$ |

**Table 5** Decomposition $N'_j, \quad j = 35$

| $j$ | $\lambda$ | $t$ |
|---|---|---|
| 33 | $C_4^0$ | $2\pi$ |
| 39 | $C_4^0$ | $(0, 2\pi)$ |

**Table 6** Decomposition $N'_j, \quad j \in \{33, 39\}$

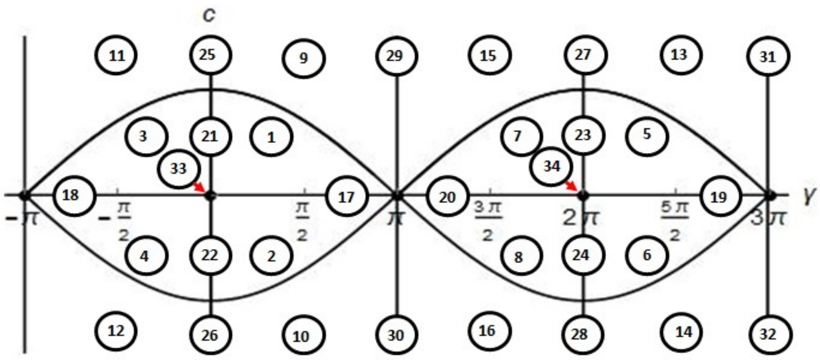


**Fig. 4** The sets $N'_j$ with $t = t_1^{\mathrm{Max}}(\lambda)$ or $\sin\left(\frac{\gamma_{t/2}}{2}\right) = 0$

## 4.1 Stratification of $N'$

We define subsets $N'_j \subset N', \quad j = 1, \ldots, 40$, as follows:

- for $j \in \{1, 9, 17, 21, 25, 29\}$ the sets $N'_j$ are given by Table 4, for $j = 35$ by Table 5 and for $j \in \{33, 39\}$ by Table 6,
- for all the rest $j$ the set $N'_j$ are defined by the action of reflections $\varepsilon^i$ as in (4.1)–(4.4):

$$\varepsilon^i \left(N'_j\right) = N'_{j+i}, \quad i = 1, \ldots, 7, \quad j = 1, 9, \tag{4.1}$$

$$\varepsilon^{2i} \left(N'_{17}\right) = N'_{17+i}, \quad i = 1, 2, 3, \tag{4.2}$$

$$\varepsilon^{2+i} \left(N'_j\right) = N'_{j+i}, \quad i = 1, 2, 3, \quad j = 21, 25, 29, 35, \tag{4.3}$$

$$\varepsilon^4 \left(N'_j\right) = N'_{j+1}, \quad j = 33, 39. \tag{4.4}$$

The following stratification of the set $N'$ follows from the definition of the sets $N'_j$.

**Lemma 3** *The stratification of $N'$ shown in Figures 4,5 is given as:*

$$N' = \sqcup_{j=1}^{40} N'_j. \tag{4.5}$$

From Figures 4, 5 we see the sets $N'_j$ given in Tables 4, 5, 6 pertain to the quadrant of the phase portrait of vertical subsystem for which $\lambda = (\gamma, c) \in C$ such that $\gamma \in [0, \pi]$ and $c \in [0, \infty)$. For $\lambda = (\gamma, c)$ in other parts of phase portrait, the sets $N'_j$ are obtained by the reflection symmetries (4.1)–(4.4) of the vertical subsystem.

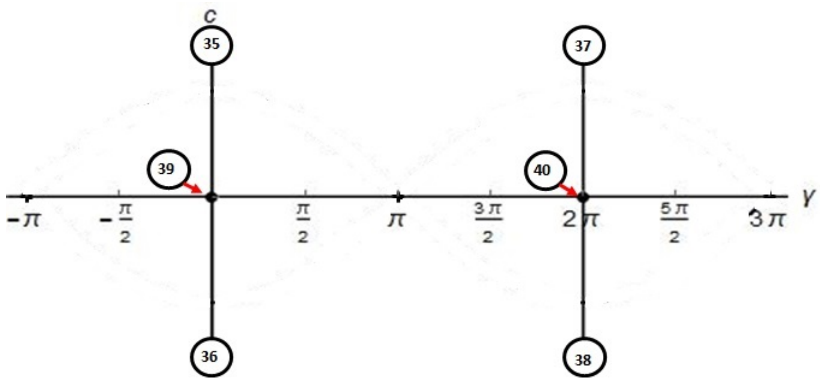


**Fig. 5** The sets $N'_j$ with $t < t_1^{\mathrm{Max}}(\lambda)$, $\sin\frac{\gamma_{t/2}}{2} = 0$

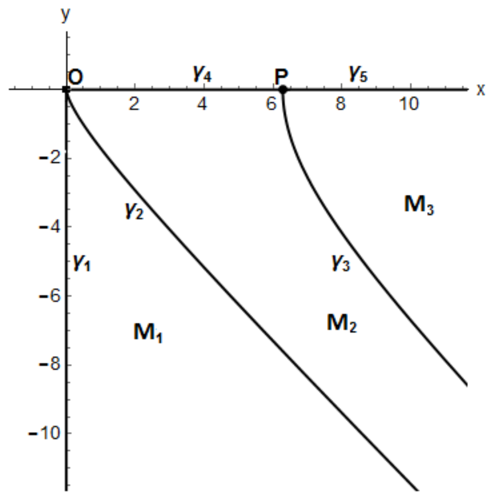


**Fig. 6** Stratification of the quadrant $Q$

4.2 Stratification of a Quadrant of the Plane $z = 0$

Define the following curves and points in the quadrant $Q = \left\{(x, y) \in \mathbb{R}^2 \quad | \quad x \geq 0,\, y \leq 0\right\}$ (see Figure 6):

$$
\begin{aligned}
\gamma_1 :&\quad x = 0, \quad y = y_1(k) = -\frac{4a(k)}{\sqrt{1-k^2}}, \quad k \in (0,1),\\
\gamma_2 :&\quad x = x_2(k) = \frac{4k\,a(k)}{1-k^2}, \quad y = y_2(k) = -\frac{4a(k)}{1-k^2}, \quad k \in (0,1),\\
\gamma_3 :&\quad x = x_3(k) = \frac{4}{1-k^2}E(k), \quad y = y_3(k) = -\frac{4k}{1-k^2}E(k), \quad k \in (0,1),\\
\gamma_4 :&\quad x = x_4(t) = t, \quad y = 0, \quad t \in (0, 2\pi),\\
\gamma_5 :&\quad x = x_5(k) = \frac{4}{\sqrt{1-k^2}}E(k), \quad y = 0, \quad k \in (0,1),\\
P :&\quad x = 2\pi, \quad y = 0,\\
O :&\quad x = 0, \quad y = 0,
\end{aligned}
$$

where $a(k) = E(k) - (1-k^2)K(k)$, $\quad k \in (0,1)$. The curves $\gamma_1, \ldots, \gamma_5$ result from substitution of $t = t_1^{\mathrm{Max}}(\lambda)$, and $\varphi = \tau - p$ from Table 4 in the equations of extremal trajectories for $\lambda \in \cup_{i=1}^{5} C_i$. The curves $\gamma_1, \ldots, \gamma_5$ and the point $P$ are the images of certain sets $\mathrm{Exp}\left(N'_j\right)$ under the projection

$$p : \{q \in M \quad | \quad z = 0\} \to \mathbb{R}^2_{x,y}, \quad (x, y, 0) \mapsto (x, y). \tag{4.6}$$

$$\begin{aligned}
\gamma_1 &= p \circ \operatorname{Exp}\left(N'_{29}\right), \\
\gamma_2 &= p \circ \operatorname{Exp}\left(N'_{25}\right), \\
\gamma_3 &= p \circ \operatorname{Exp}\left(N'_{21}\right), \\
\gamma_4 &= p \circ \operatorname{Exp}\left(N'_{39}\right), \\
\gamma_5 &= p \circ \operatorname{Exp}\left(N'_{17}\right), \\
P &= p \circ \operatorname{Exp}\left(N'_{33}\right).
\end{aligned}$$

These equalities can be verified easily. From [8] we know that the first Maxwell points with $t = t_1^{\mathrm{Max}}(\lambda)$ and conjugate points with $t = t_1^{\mathrm{Max}}(\lambda)$ and $\operatorname{sn}\tau \operatorname{cn}\tau = 0$ lie in the plane $z = 0$. Hence, the curves $\gamma_1, \dots, \gamma_5$ decompose the fourth quadrant of the plane $z = 0$ into various regions (see Figure 6). The regularity and mutual disposition of the curves $\gamma_1, \dots, \gamma_5$ are described in the following lemmas.

**Lemma 4** *The function $a(k)$ satisfies the following properties:*

$$a : (0,1) \to (0,1) \text{ is a diffeomorphism,} \tag{4.7}$$

$$k \to 0 \implies a(k) = \frac{\pi}{4}k^2 + o(k^2), \tag{4.8}$$

$$k \to 1-0 \implies a(k) = 1 - \frac{1}{2}k'^2 \ln\left(\frac{1}{k'}\right) + O(k'^2) \tag{4.9}$$

*where $k' = \sqrt{1-k^2}$. Moreover, the function $a(k)$ is convex.*

*Proof* If $k \to 0$, then

$$\begin{aligned}
K(k) &= \frac{\pi}{2}\left(1 + \frac{k^2}{4}\right) + o(k^2), \\
E(k) &= \frac{\pi}{4}\left(1 - \frac{k^2}{4}\right) + o(k^2),
\end{aligned}$$

which gives asymptotics (4.8). If $k \to 1-0$, then

$$\begin{aligned}
K(k) &= \ln\left(\frac{1}{k'}\right) + o(k'), \\
E(k) &= 1 + \frac{1}{2}k'^2 \ln\left(\frac{1}{k'}\right) + O(k'^2),
\end{aligned}$$

which gives asymptotics (4.9). Finally, property (4.7) follows since

$$\begin{aligned}
\frac{da}{dk} &= k\,K(k) > 0, \\
\lim_{k\to 0} a(k) &= 0, \\
\lim_{k\to 1-0} a(k) &= 1.
\end{aligned}$$

The function $a(k)$ is convex since $\frac{da}{dk} = k\,K(k)$ increases $\forall k \in (0,1)$. □

**Lemma 5** *The function $y = y_1(k)$ defines a diffeomorphism $y_1 : (0,1) \to (-\infty, 0)$. Moreover,*

$$\lim_{k\to 0^+} y_1(k) = 0, \tag{4.10}$$

$$\lim_{k\to 1^-} y_1(k) = -\infty. \tag{4.11}$$

*Proof* The function $y = y_1(k)$ is a strictly decreasing function with:

$$\frac{dy_1}{dk} = \frac{-4kE(k)}{(1-k^2)^{\frac{3}{2}}} < 0, \quad k \in (0,1).$$

Further, Lemma 4 yields the asymptotics:

$$\begin{aligned}
k \to 0 &\implies y_1(k) = \frac{-4a(k)}{\sqrt{1-k^2}} \to 0, \\
k \to 1-0 &\implies y_1(k) \sim -\frac{4}{k'} \to -\infty,
\end{aligned}$$

and the statement of this lemma follows. □

**Lemma 6** *The function $x = x_4(t)$ defines a diffeomorphism $x_4 : (0, 2\pi) \to (0, 2\pi)$. Moreover,*

$$\lim_{t\to 0^+} x_4(t) = 0,$$
$$\lim_{k\to 2\pi^-} x_4(t) = 2\pi.$$

*Proof* Clearly $x_4(t)$ is a smooth bijection with a smooth inverse. Hence it is a diffeomorphsim. The limits can be calculated by direct substitution in $x_4(t)$. □

**Lemma 7** *The function $x = x_5(k)$ defines a diffeomorphism $x_5 : (0, 1) \to (2\pi, +\infty)$. Moreover,*

$$\lim_{k\to 0^+} x_5(k) = 2\pi,$$
$$\lim_{k\to 1^-} x_5(k) = +\infty.$$

*Proof* The function $x = x_5(k)$ is a strictly decreasing function with:

$$\frac{dx_5}{dk} = \frac{4a(k)}{k(1-k^2)^{\frac{3}{2}}} > 0,$$

and

$$k \to 0 \implies E(k) \to \frac{\pi}{2} \implies x_5(k) \to 2\pi,$$
$$k \to 1-0 \implies E(k) \to 1 \implies x_5(k) \to +\infty,$$

and the statement of the lemma follows. □

**Lemma 8** *The functions $x = x_2(k)$, $\quad y = y_2(k) \quad k \in (0, 1)$, define parametrically a function $x = x_2(y)$ which is a diffeomorphism $x_2 : (-\infty, 0) \to (0, +\infty)$ with $\lim_{y\to-\infty} x_2(y) = +\infty$, $\quad \lim_{y\to 0^-} x_2(y) = 0$. Moreover,*

$$-y - 2 < x_2(y) < -y, \quad y \in (-\infty, 0). \tag{4.12}$$

*The curve $\gamma_2$ is convex, has near the origin the asymptotics*

$$y = -\pi^{\frac{1}{3}} x^{\frac{2}{3}} + o\left(x^{\frac{2}{3}}\right), \quad x \to 0, \tag{4.13}$$

*and has an asymptote $y + x + 2 = 0$ as $x \to \infty$.*

*Proof* Notice that

$$k \to 0 \implies x_2(k) \to 0, \quad y_2(k) \to 0,$$
$$k \to 1 \implies x_2(k) \to +\infty, \quad y_2(k) \to -\infty.$$

Also,

$$\frac{dx_2}{dk} = \frac{4\left(\left(1+k^2\right)E(k) - (1-k^2)K(k)\right)}{\left(1-k^2\right)^2} = \frac{4\left(a(k) + k^2E(k)\right)}{k\left(1-k^2\right)^2} > 0,$$
$$\frac{dy_2}{dk} = -\frac{4k\left(2E(k) - (1-k^2)K(k)\right)}{\left(1-k^2\right)^2} = -\frac{4k\left(a(k) + E(k)\right)}{\left(1-k^2\right)^2} < 0,$$

thus the functions $x_2(k)$ and $y_2(k)$ define diffeomorphisms $x_2 : (0, 1) \to (0, +\infty)$ and $y_2 : (0, 1) \to (-\infty, 0)$. So these functions define parametrically the diffeomorphism

$$x = x_2(y), \quad y \in (-\infty, 0), \quad x \in (0, +\infty),$$
$$y = y_2(x), \quad x \in (0, +\infty), \quad y \in (-\infty, 0).$$

Notice that

$$\lim_{y\to-\infty} x_2(y) = \lim_{k\to 1} x_2(k) = +\infty,$$
$$\lim_{y\to 0-} x_2(y) = \lim_{k\to 0-} x_2(k) = 0.$$

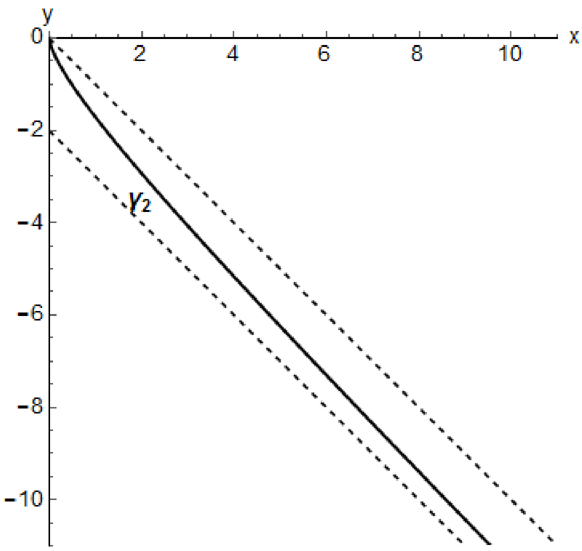


**Fig. 7** The curve $\gamma_2$ and its bounds $y + x = -2, \quad y + x = 0$.

Now we show that the curve $\gamma_2$ is convex. We have

$$\frac{dy_2}{dx} = \frac{dy_2/dk}{dx_2/dk} = \alpha(k),$$

$$\alpha(k) = -k\frac{2E(k) - (1-k^2)K(k)}{(1+k^2)E(k) - (1-k^2)K(k)}, \tag{4.14}$$

$$\frac{d\alpha}{dk} = -\left(1-k^2\right)\frac{3E^2(k) - (5-k^2)E(k)\,K(k) + 2(1-k^2)K^2(k)}{\left((1+k^2)\,E(k) - (1-k^2)\,K(k)\right)^2}. \tag{4.15}$$

Since $a(k) = E(k) - \left(1-k^2\right)K(k) \in (0,1)$, then $\frac{E(k)}{K(k)} \in \left(\left(1-k^2\right),1\right)$. But the numerator of the function $t = \frac{E(k)}{K(k)} \mapsto 3t^2 - \left(5-k^2\right)t + 2\left(1-k^2\right)$ is negative for $t \in \left(\left(1-k^2\right),1\right)$ thus the numerator of fraction (4.15) is positive. Therefore, $\frac{d\alpha}{dk} > 0$, i.e., $\frac{dy_2}{dx}$ is increasing for $k \in (0,1)$ and also increasing for $x \in (0,+\infty)$. Thus the function $y_2(x)$ and its graph, i.e., the curve $\gamma_2$, are convex. The second inequality in (4.12) follows since

$$\frac{x_2(k)}{y_2(k)} = -k > -1, \quad k \in (0,1).$$

The first inequality in (4.12) and existence of the asymptote $y + x + 2 = 0$ follows from equalities:

$$\lim_{k\to 1-}\frac{y_2(k)}{x_2(k)} = -1,$$

$$\lim_{k\to 1-}(y_2(x) + x_2(y)) = -2,$$

$$(y_2(x) + x_2(y)) + 2 = \frac{2}{1+k}(1 + k - 2a(k)) > 0,$$

since $a(k) < k < \frac{1+k}{2}$ for $k \in (0,1)$. Finally asymptotics (4.13) follows since

$$x_2(k) = \pi k^3 + o(k^3), \quad y_2(k) = -\pi k^2 + o(k^2), \quad k \to 0.$$

□

A plot of the curve $\gamma_2$ with its bounds given by (4.12) is shown in Figure 7.

**Lemma 9** *The functions $x = x_3(k), \quad y = y_3(k)$, define parametrically a function $x = x_3(y)$ which is a diffeomorphism $x_3 : (-\infty,0) \to (2\pi,+\infty)$ with $\lim_{y\to-\infty} x_3(y) = +\infty, \quad \lim_{y\to 0^+} x_3(y) = 2\pi$. Moreover,*

$$x_3(y) > 2\pi, \quad x_3(y) > 2 - y, \quad y \in (-\infty,0). \tag{4.16}$$

*The curve $\gamma_3$ is convex and has an asymptote $y + x = 2$ as $x \to \infty$.*

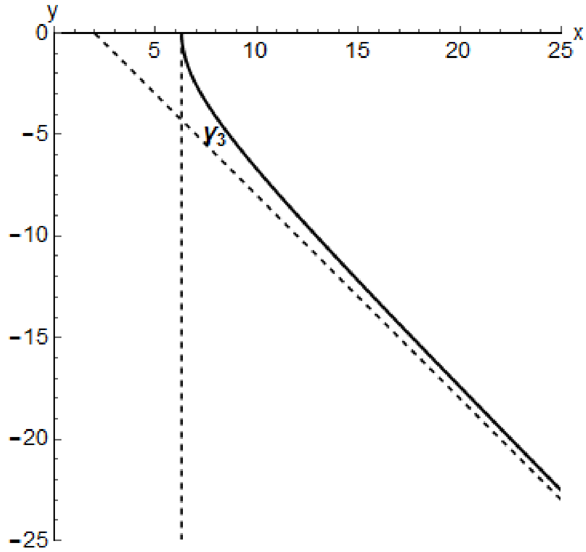


**Fig. 8** The curve $\gamma_3$ and its bounds $y + x = 2, \quad x = 2\pi$.

*Proof* Notice that

$$k \to 0 \implies x_3(k) \to 2\pi, \quad y_3(k) \to 0,$$
$$k \to 1 \implies x_3(k) \to +\infty, \quad y_3(k) \to -\infty.$$

Furthermore,

$$\frac{dx_3}{dk} = \frac{4\left(\left(1+k^2\right)E(k) - (1-k^2)K(k)\right)}{k\,(1-k^2)^2} = \frac{4\left(a(k) + k^2E(k)\right)}{k\,(1-k^2)^2} > 0,$$
$$\frac{dy_3}{dk} = -\frac{4\left(2E(k) - (1-k^2)K(k)\right)}{k\,(1-k^2)^2} = -\frac{4\,(a(k) + E(k))}{k\,(1-k^2)^2} < 0,$$

thus the functions $x_3(k)$ and $y_3(k)$ define diffeomorphisms $x_3 : (0,1) \to (2\pi, +\infty)$ and $y_3 : (0,1) \to (-\infty, 0)$. So these functions define parametrically a diffeomorphism

$$x = x_3(y), \quad y \in (-\infty, 0), \quad x \in (2\pi, +\infty).$$

Notice that

$$\lim_{y \to -\infty} x_3(y) = \lim_{k \to 1} x_3(k) = +\infty,$$
$$\lim_{y \to 0+} x_3(y) = \lim_{k \to 0+} x_3(k) = 2\pi.$$

Since $\frac{dx_3}{dk} > 0$, therefore $x_3(k) > 2\pi$ for $k \in (0,1)$, which gives the first inequality in (4.16). The second inequality in (4.16) and existence of the asymptote $y + x = 2$ follow from the equalities:

$$\lim_{k \to 1} \frac{y_3(k)}{x_3(k)} = -1,$$
$$\lim_{k \to 1} (y_3(x) + x_3(y)) = 2,$$
$$(y_3(x) + x_3(y)) - 2 = \frac{4}{1+k}\left(E(k) - \frac{1+k}{2}\right) > 0.$$

Finally, convexity of the curve $\gamma_3$ follows since

$$\frac{dy_3}{dx} = \frac{dy_3/dk}{dx_3/dk} = \alpha(k),$$

where $\alpha(k)$ is given by (4.14), which is increasing by the proof of Lemma 8. □

A plot of the curve $\gamma_3$ with its bounds given by (4.16) is shown in Fig 8.

**Lemma 10** *For any $y \in (-\infty, 0)$, we have $x_2(y) < x_3(y)$.*

| $j$ | $y$ | $x$ | $z$ |
|---|---|---|---|
| 1 | $(-\infty,0)$ | $(x_3(y),+\infty)$ | 0 |
| 9 | $(-\infty,0)$ | $(0,x_2(y))$ | 0 |
| 17 | 0 | $(2\pi,+\infty)$ | 0 |
| 21 | $(-\infty,0)$ | $x_3(y)$ | 0 |
| 25 | $(-\infty,0)$ | $x_2(y)$ | 0 |
| 29 | $(-\infty,0)$ | 0 | 0 |
| 33 | 0 | $2\pi$ | 0 |
| 35 | $(-\infty,0)$ | $(x_2(y),x_3(y))$ | 0 |
| 39 | 0 | $(0,2\pi)$ | 0 |

**Table 7** Definition of $M'_j \subset p^{-1}(Q)$.

*Proof* It follows from Lemmas 8 and 9 that $x_2(y) < -y < 2 - y < x_3(y), \quad y \in (-\infty, 0)$. □

Lemmas 5–10 allow us to define the following domains in the plane $Q \subset \mathbb{R}^2_{x,y}$:

$$
\begin{aligned}
m_1 &= \left\{(x,y) \in \mathbb{R}^2 \quad | \quad y < 0, \quad 0 < x < x_2(y)\right\},\\
m_2 &= \left\{(x,y) \in \mathbb{R}^2 \quad | \quad y < 0, \quad x_2(y) < x < x_3(y)\right\},\\
m_3 &= \left\{(x,y) \in \mathbb{R}^2 \quad | \quad y < 0, \quad x_3(y) < x\right\},
\end{aligned}
$$

see Figure 6.

**Lemma 11** *The domains $m_1, m_2, m_3 \subset \mathbb{R}^2_{x,y}$ are open, connected and simply connected, with the following boundaries:*

$$
\begin{aligned}
\partial m_1 &= \gamma_1 \cup \gamma_2 \cup \{O\},\\
\partial m_2 &= \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \{O, P\},\\
\partial m_3 &= \gamma_3 \cup \gamma_5 \cup \{P\}.
\end{aligned}
$$

*Moreover, the quadrant $Q$ has the following decomposition into disjoint subsets:*

$$
Q = \left(\cup_{i=1}^{3} m_i\right) \cup \left(\cup_{i=1}^{5} \gamma_i\right) \cup \{O, P\}.
$$

*Proof* Follows from the definition of the domains $m_i$ and from Lemmas 5–10. □

Define the inverse images of the sets $m_i, \gamma_i$, and $P$ via the projection $p$ (4.6):

$$
\begin{aligned}
M'_9 &= p^{-1}(m_1), \quad & M'_{35} &= p^{-1}(m_2), \quad & M'_1 &= p^{-1}(m_3),\\
M'_{29} &= p^{-1}(\gamma_1), \quad & M'_{25} &= p^{-1}(\gamma_2), \quad & M'_{21} &= p^{-1}(\gamma_3),\\
M'_{39} &= p^{-1}(\gamma_4), \quad & M'_{17} &= p^{-1}(\gamma_5), \quad & M'_{33} &= p^{-1}(P).
\end{aligned}
$$

Explicitly, these sets are defined in Table 7.

Now we aim to prove that all the mappings $\mathrm{Exp} : N'_j \to M'_j$ are diffeomorphisms for the sets $N'_j$ and $M'_j$ defined by Tables 4, 5, 6, 7.

**Lemma 12** *For any $j \in \{17, 21, 25, 29, 33, 39\}$ the mapping $\mathrm{Exp} : N'_j \to M'_j$ is a diffeomorphism.*

*Proof* Follows immediately from above lemmas:

- Lemma 7 for $j = 17$,
- Lemma 9 for $j = 21$,
- Lemma 8 for $j = 25$,
- Lemma 5 for $j = 29$,
- Lemma 6 for $j = 39$,
- and it is obvious for $j = 33$. □

Now we consider the mappings of 2-dimensional domains.

**Lemma 13** *The mapping $\mathrm{Exp} : N'_9 \to M'_9$ is a diffeomorphism.*

*Proof* In the coordinates $p = \frac{t}{2k}$ and $\tau = \left(\varphi + \frac{t}{2}\right)/k$, the domain $N_9'$ is given as follows:

$$N_9' : \lambda \in C_2^+, \quad s_2 = 0, \quad p = 2K(k), \quad \tau \in (0, K(k)), \quad k \in (0,1).$$

Introduce further the coordinate $u = \operatorname{am}(\tau)$, then,

$$N_9' : s_2 = 0, \quad p = 2K(k), \quad u \in \left(0, \frac{\pi}{2}\right), \quad k \in (0,1).$$

In these coordinates the exponential mapping $\operatorname{Exp}(\lambda, t) = (x, y, z)$ is given as follows:

$$\begin{aligned} x &= x_9(u,k) = \frac{4ka(k)\cos(u)}{1-k^2}, \\ y &= y_9(u,k) = -\frac{4a(k)\sqrt{1-k^2\sin^2(u)}}{1-k^2}, \\ z &= 0. \end{aligned}$$

Consider the mapping:

$$\begin{aligned} f_9 : D_{u,k} \to \mathbb{R}^2_{x,y}, &\quad (u,k) \mapsto (x_9, y_9), \\ D_{u,k} = \left(0, \frac{\pi}{2}\right)_u &\times (0,1)_k. \end{aligned}$$

We have to show that the mapping $f_9 : D \to m_1$ is a diffeomorphism.

(1) First we show that $f_9(D) \subset m_1$.
We fix any $k \in (0,1)$ and show that the curve $\mathit{\Gamma} : u \to (x_9, y_9), \quad u \in \left(0, \frac{\pi}{2}\right)$, is contained in $m_1$. Compute first the boundary points of $\mathit{\Gamma}$:

$$\begin{aligned} u \to 0 &\implies \mathit{\Gamma}(u) \to (x_2(k), y_2(k)) \in \gamma_2, \\ u \to \frac{\pi}{2} &\implies \mathit{\Gamma}(u) \to (0, y_2(k)) \in \gamma_1. \end{aligned}$$

Further, since

$$\begin{aligned} \frac{\partial x_9}{\partial u} &= -\frac{4ka(k)}{1-k^2}\sin(u) < 0, \\ \frac{\partial y_9}{\partial u} &= \frac{4k^2a(k)}{1-k^2}\frac{\sin(u)\cos(u)}{\sqrt{1-k^2\sin^2(u)}} > 0, \end{aligned}$$

then the curve $\mathit{\Gamma}$ is a graph of the smooth function $x \mapsto y_9(x)$. Since

$$\frac{dy_9}{dx} = \frac{\partial y_9/\partial u}{\partial x_9/\partial u} = -\frac{k\cos(u)}{\sqrt{1-k^2\sin^2(u)}}, \quad \text{for } u \in \left(0, \frac{\pi}{2}\right),$$

then the curve $\mathit{\Gamma}$ is concave. Moreover,

$$\left.\frac{dy_9}{dx}\right|_{u=0} = -k > \alpha(k) = \frac{dy_2}{dx},$$

where $\alpha(k)$ is given by (4.14). Since the curve $\gamma_2$ is convex, it follows that the curve $\mathit{\Gamma}$ lies below the curve $\gamma_2$. Thus $\mathit{\Gamma} \subset m_1$. Consequently, $f_9(D) \subset m_1$.

(2) Since

$$\frac{\partial(x_9, y_9)}{\partial(u,k)} = \frac{16k^2E(k)a(k)\sin(u)}{(1-k^2)^2\sqrt{1-k^2\sin^2(u)}} > 0, \tag{4.17}$$

then the mapping $f_9 : D \to m_1$ is non-degenerate.

(3) Finally we show that the mapping $f_9 : D \to m_1$ is proper.
It is obvious that a sequence $(u_n, k_n) \to \partial D$ iff it has a subsequence on which at least one of the conditions hold:

$$u \to 0, \quad u \to \frac{\pi}{2}, \quad k \to 0, \quad k \to 1. \tag{4.18}$$

On the other hand, a sequence $(x_n, y_n) \to \partial m_1$ iff it has a subsequence on which at least one of the conditions hold:

$$x \to 0, \quad x \to +\infty, \quad y \to 0, \quad y \to -\infty, \quad x_2(y) - x \to 0. \tag{4.19}$$

We show that in each of the cases (4.18) we have one of the cases (4.19). If $k \to 0$, then $x_9 \to 0$ and $y_9 \to 0$. We can assume below that $k \to \bar{k} \in (0,1]$.
Let $\bar{k} \in (0,1)$. If $u \to 0$, then $(x_9, y_9) \to (x_2(k), y_2(k)) \in \gamma_2$ thus $x_2(y) - x \to 0$. If $u \to \frac{\pi}{2}$, then $x_9 \to 0$. Let $\bar{k} = 1$. If $u \to 0$, then $x_9 \to \infty$. If $u \to \frac{\pi}{2}$, then $y_9 \to \infty$.
We proved that the mapping $f_9 : D \to m_1$ is proper.

(4) The sets $D$, $m_1 \subset \mathbb{R}^2$ are open, connected and simply connected.
Thus $f_9 : D \to m_1$ is a diffeomorphism, as well as $\mathrm{Exp} : N_9' \to M_9'$. □

**Lemma 14** *The mapping* $\mathrm{Exp} : N_1' \to M_1'$ *is a diffeomorphism.*

*Proof* In the coordinates $p = \frac{t}{2}$ and $\tau = \varphi + \frac{t}{2}$, the domain $N_1'$ is given as follows:

$$N_1' : \lambda \in C_1^0, \quad s_1 = 0, \quad p = 2K(k), \quad \tau \in (0, K(k)), \quad k \in (0,1).$$

Introduce further the coordinate $u = \mathrm{am}(\tau)$, then

$$N_1' : s_1 = 0, \quad p = 2K(k), \quad u \in \left(0, \frac{\pi}{2}\right), \quad k \in (0,1).$$

In these coordinates the exponential mapping $\mathrm{Exp}(\lambda, t) = (x, y, z)$ is given as follows:

$$\begin{aligned} x &= x_1(u,k) = \frac{4E(k)\sqrt{1 - k^2 \sin^2(u)}}{1 - k^2}, \\ y &= y_1(u,k) = -\frac{4k\, E(k)\cos(u)}{1 - k^2}, \\ z &= 0. \end{aligned}$$

Consider the mapping:

$$\begin{aligned} f_1 : D_{u,k} &\to \mathbb{R}^2_{x,y}, \quad (u,k) \mapsto (x_1, y_1), \\ D_{u,k} &= \left(0, \frac{\pi}{2}\right)_u \times (0,1)_k. \end{aligned}$$

We have to show that the mapping $f_1 : D \to m_3$ is a diffeomorphism.

(1) First we show that $f_1(D) \subset m_3$.
If $(u,k) \in D$, then $x_1(u,k) > 0, \quad y_1(u,k) < 0$, thus $f_1(D) \subset \mathbb{R}^2_{+-} = \{(x,y) \in \mathbb{R}^2 \quad | \quad x > 0, \quad y < 0\}$.
The boundary of the domain $m_3$ in $\mathbb{R}^2_{+-}$ is the curve $\gamma_3$ and along this curve we have $\frac{y_4(k)}{x_4(k)} = -k$. Thus

$$\gamma_3 = \left\{ (x,y) \in \mathbb{R}^2_{+-} \quad | \quad x = \frac{4E\left(-\frac{y}{x}\right)}{1 - \frac{y^2}{x^2}} \right\},$$

so

$$m_3 = \left\{ (x,y) \in \mathbb{R}^2_{+-} \quad | \quad x > \frac{4E\left(-\frac{y}{x}\right)}{1 - \frac{y^2}{x^2}} \right\}.$$

Consider the function

$$\varphi_1(u,k) = x - \left.\frac{4E\left(-\frac{y}{x}\right)}{1 - \frac{y^2}{x^2}}\right|_{x = x_1(u,k),\, y = y_1(u,k)}.$$

We have to show that $\varphi_1(u,k) > 0$ for $(u,k) \in D$. Since

$$\begin{aligned} \varphi_1(u,k) &= \frac{4E(k)\sqrt{1 - k^2 \sin^2(u)}}{1 - k^2} - \frac{4E(\bar{k})}{1 - \frac{k^2 \cos^2 u}{1 - k^2 \sin^2 u}} \\ &= \frac{4\sqrt{1 - k^2 \sin^2(u)}}{1 - k^2} \left( E(k) - E(\bar{k})\sqrt{1 - k^2 \sin^2(u)} \right), \end{aligned}$$

where $\bar{k} = \frac{k\cos(u)}{\sqrt{1 - k^2 \sin^2 u}}$, we have to show that

$$\varphi_2(u,k) = E(k) - E(\bar{k})\sqrt{1 - k^2 \sin^2(u)} > 0, \quad (u,k) \in D.$$

Since $\varphi_2(0,k) = 0$ and

$$\frac{\partial \varphi_2}{\partial u} = \frac{\tan(u)}{\sqrt{1 - k^2 \sin^2(u)}} \varphi_3(u,k),$$

where $\varphi_3(u,k) = \left(1 - k^2 \sin^2(u)\right) E(\bar{k}) - \left(1 - k^2\right) K(\bar{k})$, it is sufficient to show that $\varphi_3(u,k) > 0$ for all $(u,k) \in D$. By Lemma 4, we have

$$a(k) = E(k) - \left(1 - k^2\right) K(k) > 0, \quad k \in (0,1),$$

thus

$$a(\bar{k}) = E(\bar{k}) - \left(1 - \bar{k}^2\right) K(\bar{k})$$
$$= \frac{\left(1 - k^2 \sin^2(u)\right) E(\bar{k}) - \left(1 - k^2\right) K(\bar{k})}{1 - k^2 \sin^2(u)} > 0.$$

That is, $\varphi_3(u,k) > 0, \quad \forall (u,k) \in D$. Thus it follows that $f_1(D) \subset m_3$, i.e., $\mathrm{Exp}(N_1') \subset M_1'$.

(2) Since

$$\frac{\partial(x_1, y_1)}{\partial(u,k)} = -\frac{16E(k)\, a(k) \sin(u)}{(1-k^2)^2 \sqrt{1 - k^2 \sin^2(u)}} < 0,$$

then the mapping $f_1 : D \to m_3$ is non-degenerate.

(3) Finally we show that the mapping $f_1 : D \to m_3$ is proper.
In order to show that the mapping $f_1 : D \to m_3$ is proper, we show that if a sequence $(u_n, k_n) \in D$ satisfies one of the conditions:

$$u \to 0, \quad u \to \frac{\pi}{2}, \quad k \to 0, \quad k \to 1,$$

then its image $(x_n, y_n) = f_1(u_n, k_n)$ satisfies one of the conditions:

$$x \to 0, \quad x \to +\infty, \quad y \to 0, \quad y \to \infty, \quad x_3(y) - x \to 0.$$

We can assume that $k \to \bar{k} \in (0,1], \quad u \in \bar{u} \in [0, \frac{\pi}{2}]$. If $\bar{k} = 0$, then $y_1 \to 0$.
Let $\bar{k} \in (0,1)$. If $\bar{u} \to 0$, then $(x_1, y_1) \to (x_3(k), y_3(k)) \in \gamma_3$, thus $x_3(y) - x \to 0$. If $\bar{u} = \frac{\pi}{2}$, then $y_1 \to 0$. Let $\bar{k} = 1$. If $\bar{u} \in [0, \frac{\pi}{2})$, then $x_1 \to \infty$, $y_1 \to \infty$. Let $\bar{u} = \frac{\pi}{2}$, then

$$y_1 \sim -\frac{4\cos(u)}{1-k^2},$$
$$x_1 \sim 4\sqrt{\frac{1}{1-k^2} + k^2 \left(\frac{\cos(u)}{1-k^2}\right)^2}.$$

We can assume that $\frac{\cos(u)}{1-k^2} \to d \in [0, +\infty)$. If $d \in [0, +\infty)$, then $x_1 \to +\infty$, and if $d = +\infty$, then $y_1 \to \infty$.
We proved that the mapping $f_1 : D \to m_3$ is proper.

(4) The sets $D$, $m_3 \subset \mathbb{R}^2$ are open, connected and simply connected.

Thus $f_1 : D \to m_3$ is a diffeomorphism, as well as the mapping $\mathrm{Exp} : N_1' \to M_1'$. □

**Lemma 15** *The mapping* $\mathrm{Exp} : N_{35}' \to M_{35}'$ *is a diffeomorphism.*

*Proof* It follows from Tables 5, 7 that

$$N_{35}' = \left\{ (\lambda, t) \in N \quad | \quad \gamma_{\frac{t}{2}} = 0, \quad c_{\frac{t}{2}} > 0, \quad t \in (0, \mathbf{t}(\lambda)) \right\},$$
$$M_{35}' = \{ q \in M \quad | \quad z = 0, \quad y < 0, \quad x_2(y) < x < x_3(y) \}.$$

Further we have an obvious decomposition

$$N_{35}' = N_{35,1}' \sqcup N_{35,2}' \sqcup N_{35,3}',$$
$$N_{35,j}' = N_{35}' \cap N_j, \quad j = 1,2,3.$$

(1) We show first that $\mathrm{Exp}(N_{35}') \subset M_{35}'$.
Consider the set $N_{35,2}'$. In the coordinates $p = \frac{t}{2k}$ and $\tau = \left(\varphi + \frac{t}{2}\right)/k$, the domain $N_{35,2}'$ is given as follows:

$$N_{35,2}' : \lambda \in C_2^+, \quad s_2 = 1, \quad p = (0, 2K(k)), \quad \tau = 0, \quad k \in (0,1).$$

Introduce further the coordinate $u = \mathrm{am}(p)$, then

$$N_{35,2}' : \lambda \in C_2^+, \quad s_2 = 1, \quad u = (0, 2\pi), \quad \tau = 0, \quad k \in (0,1).$$

In these coordinates the exponential mapping $\mathrm{Exp}(\lambda, t) = (x, y, z), \quad (\lambda, t) \in N_{35,2}'$ is given as follows:

$$x = x_{35}(u,k) = \frac{2k}{1-k^2} \left[ \sin(u) \sqrt{1 - k^2 \sin^2(u)} - \cos(u)\, \alpha(u,k) \right],$$
$$y = y_{35}(u,k) = -\frac{2}{1-k^2} \left[ \sqrt{1 - k^2 \sin^2(u)}\, \alpha(u,k) - k^2 \sin(u) \cos(u) \right],$$
$$z = 0,$$

where $\alpha(u,k) = E(u,k) - \left(1-k^2\right)F(u,k)$. Thus $\operatorname{Exp}(N'_{35,2}) \subset \{q \in M \quad | \quad z=0\}$. Now we show that $x_{35}(u,k) > 0, \quad y_{35}(u,k) < 0$ for $(u,k) \in (0,\frac{\pi}{2}) \times (0,1)$. We have to prove the double inequality

$$\begin{aligned}
&\alpha_1(u,k) < \alpha(u,k) < \alpha_2(u,k), \quad (u,k) \in (0,\frac{\pi}{2}) \times (0,1),\\
&\alpha_1(u,k) = \frac{k^2\sin(u)\cos(u)}{\sqrt{1-k^2\sin^2(u)}},\\
&\alpha_2(u,k) = \frac{\sin(u)\sqrt{1-k^2\sin^2(u)}}{\cos(u)}.
\end{aligned}$$

This double inequality follows since

$$\begin{aligned}
\alpha_1(0,k) = \alpha(0,k) = \alpha_2(0,k) = 0,\\
\frac{\partial}{\partial u}\left(\alpha(u,k) - \alpha_1(u,k)\right) = \left(1-k^2\right)\sin^2(u) > 0,\\
\frac{\partial}{\partial u}\left(\alpha_2(u,k) - \alpha(u,k)\right) = 1-k^2 > 0.
\end{aligned}$$

Thus $x_{35}(u,k) > 0, \quad y_{35}(u,k) < 0$ for $(u,k) \in \left(0,\frac{\pi}{2}\right) \times (0,1)$. If $u \in [\frac{\pi}{2},\pi), \quad k \in (0,1)$, then $\sin(u) > 0, \quad \cos(u) \leq 0, \quad \alpha(u,k) > 0$, thus $x_{35}(u,k) > 0, \quad y_{35}(u,k) < 0$. We proved that $\operatorname{Exp}(N'_{35,2}) \subset \{q \in M \quad | \quad z=0, \quad x>0, \quad y<0\}$. The sets $N'_{35,1}$ and $N'_{35,3}$ are considered similarly. Thus it follows that

$$\operatorname{Exp}(N'_{35}) \subset \mathbb{R}^2_{+-} := \{q \in M \quad | \quad z=0, \quad x>0, \quad y<0\}.$$

We now show that $\operatorname{Exp}(N'_{35}) \subset M'_{35}$. Notice the decomposition

$$\mathbb{R}^2_{+-} = M'_1 \sqcup M'_9 \sqcup M'_{21} \sqcup M'_{25} \sqcup M'_{35}.$$

By contradiction, let $\operatorname{Exp}(N'_{35}) \not\subset M'_{35}$, then $\operatorname{Exp}(N'_{35}) \cap \left(M'_1 \sqcup M'_9 \sqcup M'_{21} \sqcup M'_{25}\right) \ni q$. Let $q \in \operatorname{Exp}(N'_{35}) \cap M'_1$ (the cases of intersection with $M'_9, M'_{21}, M'_{25}$ are considered similarly). Then there exist $(\lambda_{35}, t_{35}) \in N'_{35}$, $(\lambda_1, t_1) \in N'_1$ such that $q = \operatorname{Exp}(\lambda_{35}, t_{35}) = \operatorname{Exp}(\lambda_1, t_1)$. Notice that

$$(\lambda_{35}, t_{35}) \in N'_{35} \implies t_{35} < t_{\text{cut}}(\lambda_{35}), \tag{4.20}$$

$$(\lambda_1, t_1) \in N'_1 \implies t_1 < t_{\text{cut}}(\lambda_1). \tag{4.21}$$

If $t_{35} < t_1$, then the trajectory $\operatorname{Exp}(\lambda_1, t), \quad t \in [0,t_1]$, is not optimal which contradicts to (4.21). If $t_{35} \geq t_1$, then the trajectory $\operatorname{Exp}(\lambda_{35}, t), \quad t \in [0, t_{35}+\varepsilon]$ is not optimal for small $\varepsilon > 0$ which contradicts to (4.20). Thus $\operatorname{Exp}(N'_{35}) \cap M'_1 = \emptyset$. Then it follows that $\operatorname{Exp}(N'_{35}) \subset M'_{35}$.

(2) We now prove that $\operatorname{Exp} : N'_{35} \to M'_{35}$ is non-degenerate.
Let $\nu = (\lambda, t) \in N'_{35,2}$. In the coordinates $(p,\tau,k)$ on $N'_{35,2}$, we have $p \in (0, 2K(k)), \quad \tau = 0, \quad k \in (0,1)$. Since $t < 4K(k) = t_{\text{cut}}(\lambda) \leq t_1^{\text{conj}}(\lambda)$, therefore the Jacobian $\frac{\partial q}{\partial \nu}(\nu) \neq 0$. We have

$$\frac{\partial q}{\partial \nu} = \frac{\partial(x,y,z)}{\partial(p,\tau,k)} = \begin{vmatrix} x_p & x_\tau & x_k \\ y_p & y_\tau & y_k \\ z_p & z_\tau & z_k \end{vmatrix}.$$

Since $\operatorname{Exp}\left(N'_{i,2}\right) \subset \{q \in M \quad | \quad z=0\}$, then $z_p(\nu) = z_k(\nu) = 0$, thus

$$\frac{\partial q}{\partial \nu}(\nu) = \frac{\partial(x,y)}{\partial(p,k)}(\nu)\, z_\tau(\nu) \neq 0,$$

so $\frac{\partial(x,y)}{\partial(p,k)}(\nu) \neq 0$. Since $\nu \in N'_{35,2}$ is arbitrary, then $\operatorname{Exp}|_{N'_{35,2}}$ is non-degenerate. Similarly it follows that Exp is non-degenerate at any point $\nu \in N'_{35,1} \cup N'_{35,3}$.

(3) The mapping $\operatorname{Exp} : N'_{35} \to M'_{35}$ is proper. This follows similarly to the proof of properness of $\operatorname{Exp} : D_1 \to M_1$.

(4) It is obvious that $M'_{35}$ is a connected, simply connected 2-dimensional manifold. In order to prove the same property for $N'_{35}$, consider the vector field

$$\overrightarrow{P} = c\frac{\partial}{\partial \gamma} - \sin\gamma\frac{\partial}{\partial c} \in \operatorname{Vec}(N).$$

Since

$$e^{t/2\overrightarrow{P}}\left(N'_{35}\right) = \{(\lambda,t) \in N \quad | \quad \gamma = 0, \quad c > 0, \quad t < \mathbf{t}(\lambda)\}$$

is a connected, simply connected 2-dimensional manifold, the same properties hold for the set $N'_{35}$.
Then it follows that $\operatorname{Exp} : N'_{35} \to M'_{35}$ is a diffeomorphism. □

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x$ | $x$ | $x$ | $x$ | $-x$ | $-x$ | $-x$ | $-x$ |
| $y$ | $-y$ | $y$ | $-y$ | $y$ | $-y$ | $y$ | $-y$ |

**Table 8** Action of $\varepsilon^i$ in the plane $\{z = 0\}$

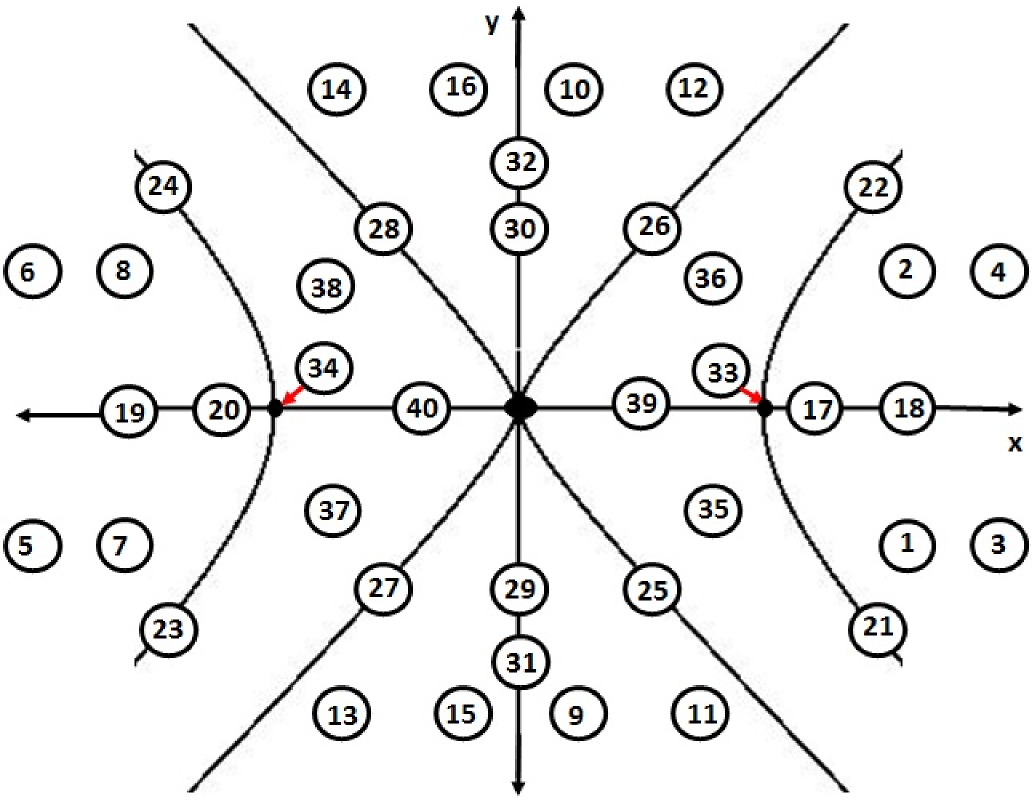


**Fig. 9** Stratification of $M'$

### 4.3 Stratification of the set $M'$

Define subsets $M'_j \subset M'$, $\quad j = 1, \dots, 40$, as follows:

- For $j \in \{1, 9, 17, 21, 25, 29, 33, 35, 39\}$, the sets $M_j$ are given by Table 7,
- For the rest $j$ the sets $M'_j$ are given by equalities (4.22)–(4.25):

$$\varepsilon^i \left(M'_j\right) = M'_{j+i}, \quad i = 1, \dots, 7, \quad j = 1, 9, \tag{4.22}$$

$$\varepsilon^{2i} \left(M'_{17}\right) = M'_{17+i}, \quad i = 1, 2, 3, \tag{4.23}$$

$$\varepsilon^{2+i} \left(M'_j\right) = M'_{j+i}, \quad i = 1, 2, 3, \quad j = 21, 25, 29, 35, \tag{4.24}$$

$$\varepsilon^4 \left(M'_j\right) = M'_{j+1}, \quad j = 33, 39. \tag{4.25}$$

**Lemma 16** *A stratification of $M'$ is given as:*

$$M' = \sqcup_{j=1}^{40} M'_j. \tag{4.26}$$

*Proof* Follows from Lemma 11 and the description of the action of reflections $\varepsilon^i$ in the plane $\{z = 0\}$, see Table 8.

□

Stratification (4.26) is shown in Figure 9.

**Theorem 4.1** *For any $i = 1, \dots, 40$, the mapping* Exp : $N'_i \to M'_i$ *is a diffeomorphism.*

*Proof* Follows from Lemmas 12–15 via the symmetries $\varepsilon^i$ of the exponential mapping. □

Define the following important sets:

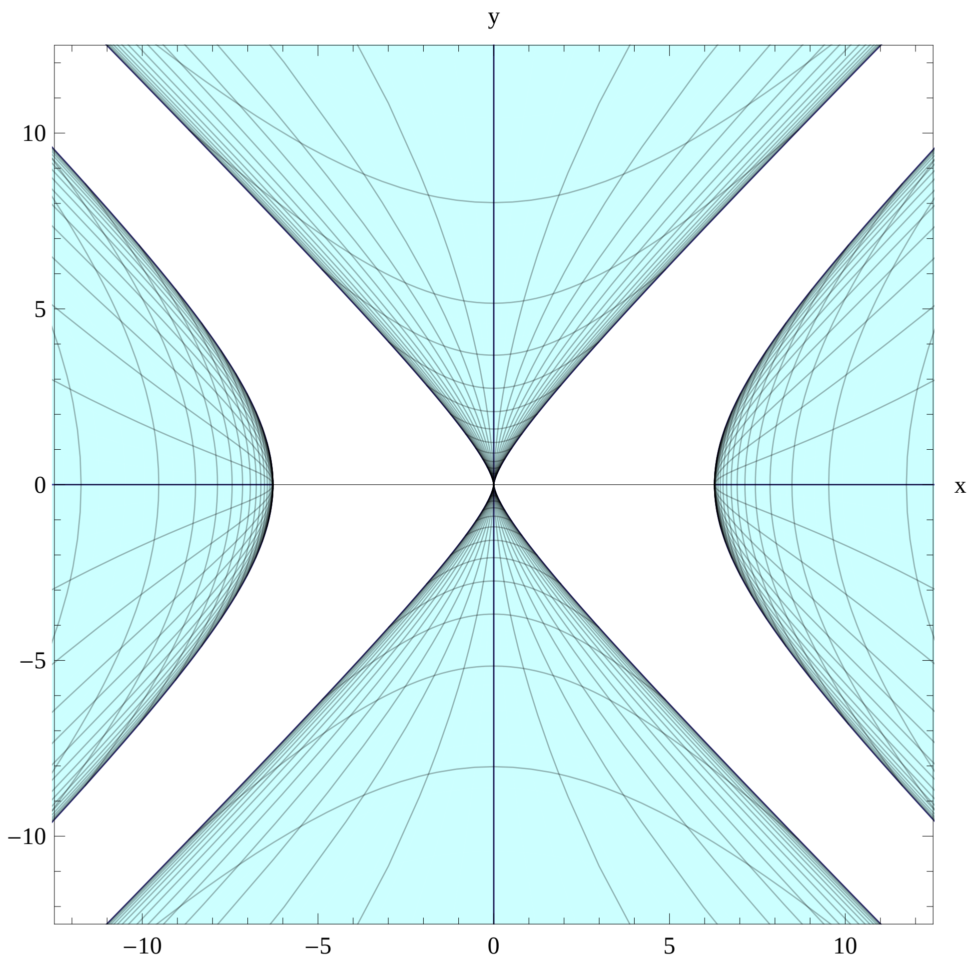


**Fig. 10** Cut Locus

- the cut locus $\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t_{\mathrm{cut}}(\lambda)) \quad | \quad \lambda \in C\}$,
- the first Maxwell set
  $\mathrm{Max} = \left\{ q_1 \in M \quad | \quad \exists \text{ minimizers } q'(t) \not\equiv q''(t), \quad t \in [0, t_1], \text{ such that } q'(t_1) = q''(t_1) = q_1 \right\}$.
- the first conjugate locus $\mathrm{Conj} = \left\{ \mathrm{Exp}(\lambda, t_1^{\mathrm{conj}}(\lambda)) \quad | \quad \lambda \in C \right\}$,
- the rest of the points in $M'$ compared with Cut, i.e., $\mathrm{Rest} = M' \backslash \mathrm{Cut}$.

We have the following explicit description of these sets:

$$\begin{aligned} \mathrm{Cut} &= \cup \left\{ M'_i \quad | \quad i = 1, \dots, 34 \right\}, \\ \mathrm{Max} &= \cup \left\{ M'_i \quad | \quad i = 1, \dots, 20, 29, \dots, 32 \right\}, \\ \mathrm{Conj} \cap \mathrm{Cut} &= \cup \left\{ M'_i \quad | \quad i = 21, \dots, 28, 33, 34 \right\}, \\ \mathrm{Rest} &= \cup \left\{ M'_i \quad | \quad i = 35, \dots, 40 \right\}, \end{aligned}$$

Thus we get the following decomposition of the sets $M'$:

$$\begin{aligned} M' &= \mathrm{Cut} \sqcup \mathrm{Rest}, \\ \mathrm{Cut} &= \mathrm{Max} \sqcup (\mathrm{Conj} \cap \mathrm{Cut}). \end{aligned}$$

The global structure of the cut locus is shown in Figure 10. From our analysis of the exponential mapping, we get the following description of the cut time and the optimal synthesis on SH(2).

**Theorem 4.2** *We have the following explicit description of the cut time, $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$ for any $\lambda \in C$. In detail:*

$$\begin{aligned} \lambda \in C_1 &\implies t_{\mathrm{cut}}(\lambda) = t_1^{\mathrm{Max}}(\lambda) = 4K(k), \\ \lambda \in C_2 &\implies t_{\mathrm{cut}}(\lambda) = t_1^{\mathrm{Max}}(\lambda) = 4kK(k), \\ \lambda \in C_4 &\implies t_{\mathrm{cut}}(\lambda) = t_1^{\mathrm{conj}}(\lambda) = 2\pi, \\ \lambda \in C_3 \cup C_5 &\implies t_{\mathrm{cut}}(\lambda) = +\infty. \end{aligned}$$

*Proof* If $\lambda \in C \setminus C_4$, then we know from Theorem 3.3 that $t_{\text{cut}}(\lambda) = \mathbf{t}(\lambda) = t_1^{\text{Max}}(\lambda)$. It remains to consider the case $\lambda \in C_4^0 \cup C_4^1$. Let $\lambda \in C_4^0$, then $q_t = \text{Exp}(\lambda, t) = (t, 0, 0)$. For any $t \in [0, t_1]$, $\quad t_1 = \mathbf{t}(\lambda) = 2\pi$, the point $q_t$ is connected with $q_0$ by a unique geodesic $\text{Exp}(\lambda^1, s)$, $\quad s \in (0, s_1]$, with $(\lambda^1, s_1) \in \widehat{N}$, namely $(\lambda^1, s_1) = (\lambda, t) \in N'_{39}$ for $t \in (0, 2\pi)$, and $(\lambda^1, s_1) = (\lambda, t) \in N'_{33}$ for $t = 2\pi$. Thus the geodesic $q_t$, $\quad t \in [0, t_1]$ is a minimizer.

It follows that $t_{\text{cut}}(\lambda) = \mathbf{t}(\lambda) = t_1^{\text{conj}}(\lambda) = 2\pi$ for $\lambda \in C_4^0$. By applying a reflection $\varepsilon^i$, we get a similar equality for $\lambda \in C_4^1$. □

From the above description of the structure of the exponential mapping, we get the following statement.

**Theorem 4.3**

1. For every point $q_1 \in \widetilde{M} \cup \text{Rest}$, there exists a unique minimizer $q(t)$, $\quad t \in [0, t_1]$, for which the endpoint $q(t_1) = q_1$ is neither a cut point nor a conjugate point.
2. For any point $q_1 \in \text{Max}$, there exist exactly two minimizers that connect $q_0$ to $q_1$ for which $q_1$ is a cut point but not a conjugate point.
3. For any point $q_1 \in \text{Conj} \cap \text{Cut}$, there exists a unique minimizer that connects $q_0$ to $q_1$ for which $q_1$ is both a cut and a conjugate point, but not a Maxwell point.

## 5 Sub-Riemannian Caustics and Sphere

In [8] we presented plots of sub-Riemannian sphere and sub-Riemannian wavefront in the rectifying coordinates $(R_1, R_2, z)$. Here we perform another graphic study of the essential sub-Riemannian objects, i.e., sub-Riemannian caustic and sub-Riemannian sphere. Recall that the sub-Riemannian caustic which is the first conjugate locus is given as:

$$\text{Conj} = \left\{ \text{Exp}\left(\lambda, t_1^{\text{conj}}(\lambda)\right) \quad | \quad \lambda \in C \right\}.$$

The caustic is presented in Figure 11. The component starting at $(0, 0, 0)$ is the local component of the caustic whereas other two parts on right and left side are the parts of the global component of the first caustic. The red colored surface inside the local and global components of the caustic is the cut locus whereas we see that the boundary of cut locus forms the boundary of the caustic. A zoomed version of the local component of the caustic is separately shown in Figure 12. It is evident that it is a four cusp surface as predicted in [**?**]. A combined plot of first and second caustic is also shown in Figure 13. Note that in the local component of the caustic, the first caustic is solid and the second caustic is transparent whereas in the global component of the caustic, the second caustic is solid and the first caustic is transparent. The sub-Riemannian sphere $S_R(q_0; R)$ at $q_0$ is the set of end-points of minimizing geodesics of sub-Riemannian length $R$ and starting from $q_0$:

$$S_R = \{\text{Exp}(\lambda, R) \in M \quad | \quad \lambda \in C, \quad t_{\text{cut}}(\lambda) \geq R\} = \{q \in M \quad | \quad d(q_0, q) = R\}.$$

The following plots are presented:

1. Sphere of radius $R = \pi$ (Figure 14),
2. Sphere of radius $R = 2\pi$ (Figure 15),
3. Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = \pi$ (Figure 16),
4. Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = 2\pi$ (Figure 17),
5. Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = 3\pi$ (Figure 18),
6. Matryoshka of hemispheres $z < 0$ of radii $R = \pi$ and $R = 2\pi$ (Figure 19).

## 6 Conclusion

The global optimality analysis and structure of exponential mapping for the sub-Riemannian problem on the Lie group SH(2) was considered. We cutout open dense domains by Maxwell strata in the preimage and in the image of exponential mapping and prove that restriction of the exponential mapping to these domains is a diffeomorphism. This fact leads to the proof that the cut time in the sub-Riemannian problem on the Lie group SH(2) is equal to the first Maxwell time. We then describe the global structure of the exponential mapping and obtain a stratification of the cut locus in the plane $z = 0$. Consequently, the problem of finding optimal trajectories from any initial point $q_0 \in M$ to another point $q_1 \in M$, $\quad z \neq 0$ is reduced to solving a set of algebraic equations. Summing up, a complete optimal synthesis for the sub-Riemannian problem on the Lie group SH(2) was constructed.

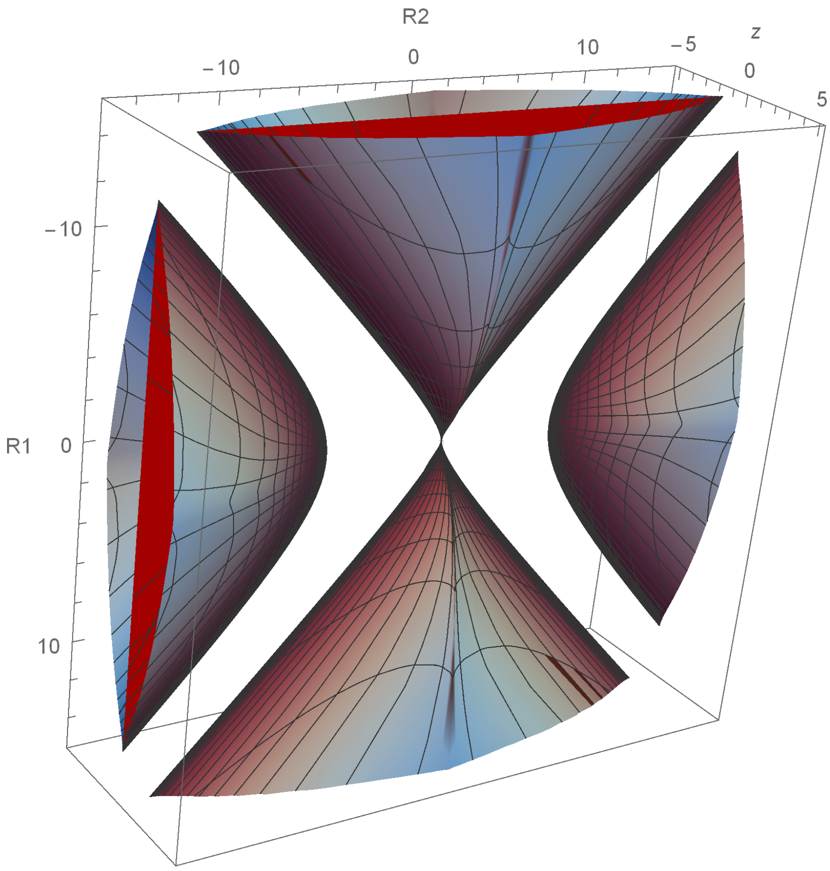


**Fig. 11** Sub-Riemannian caustic and cut locus

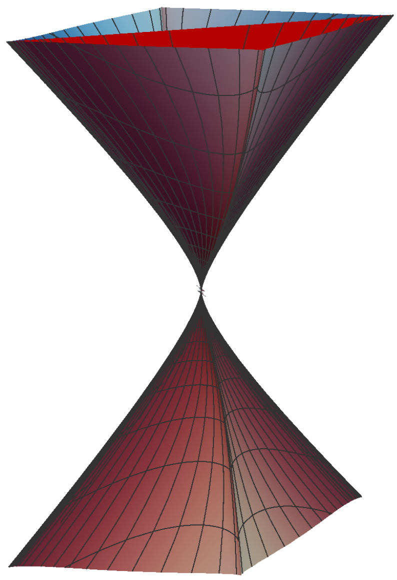

**Fig. 12** Local component of sub-Riemannian caustic and cut locus

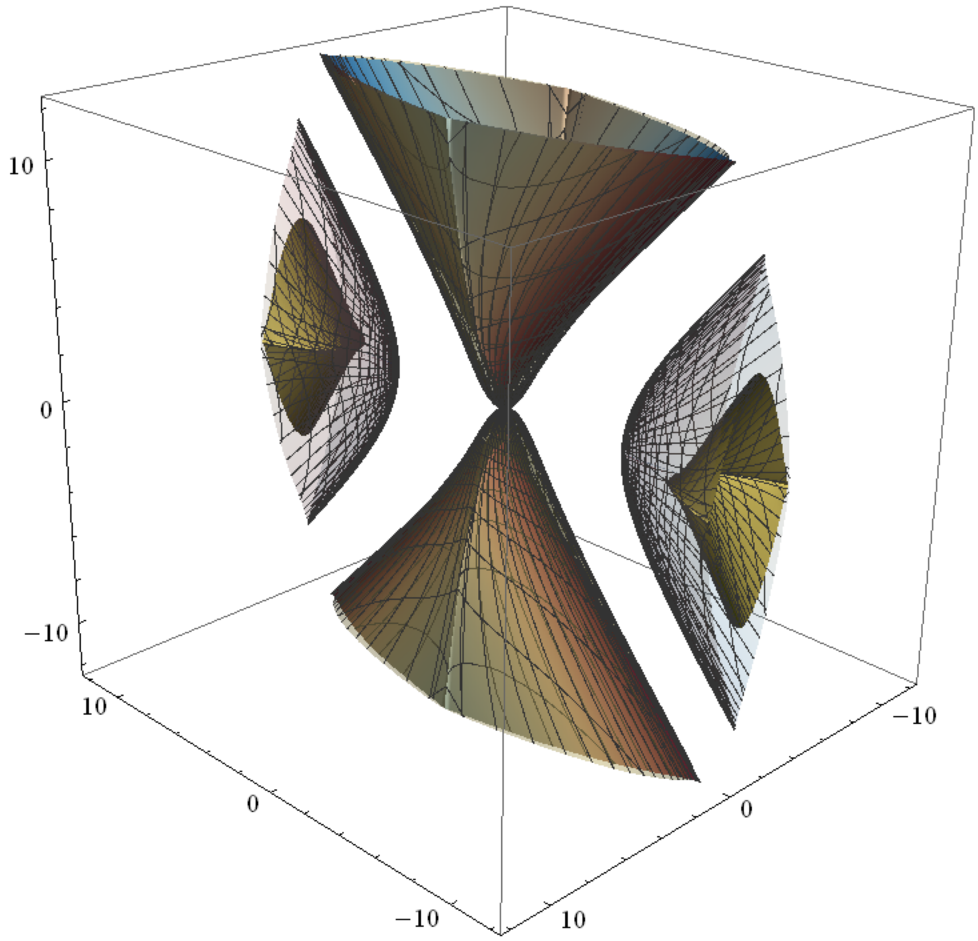


**Fig. 13** Sub-Riemannian first and second caustic

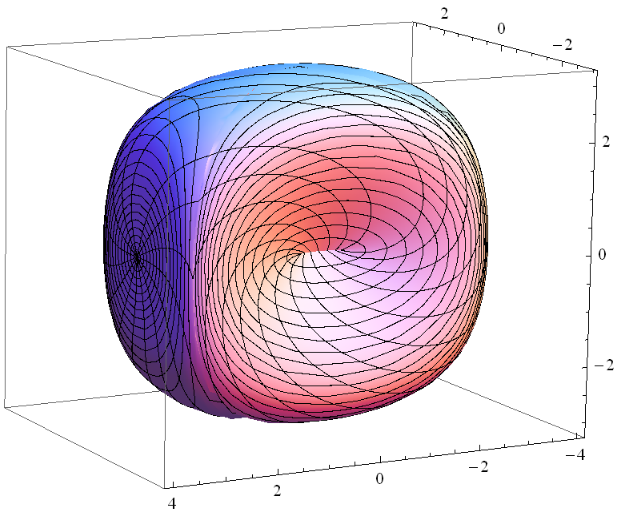


**Fig. 14** Sub-Riemannian sphere of radius $R = \pi$

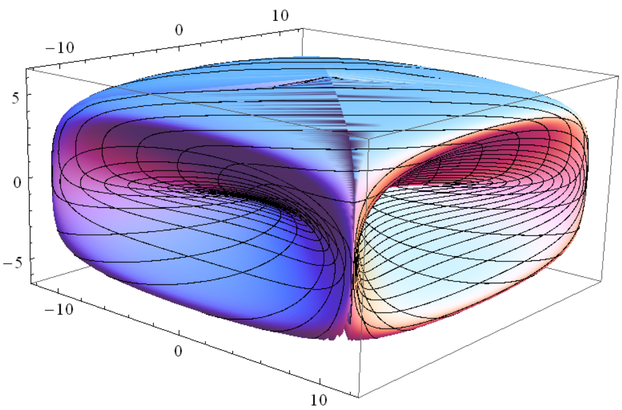


**Fig. 15** Sub-Riemannian sphere of radius $R = 2\pi$

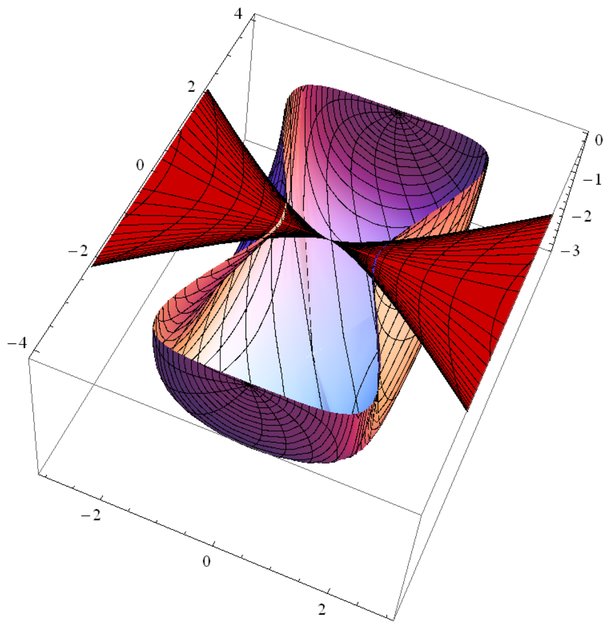


**Fig. 16** Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = \pi$

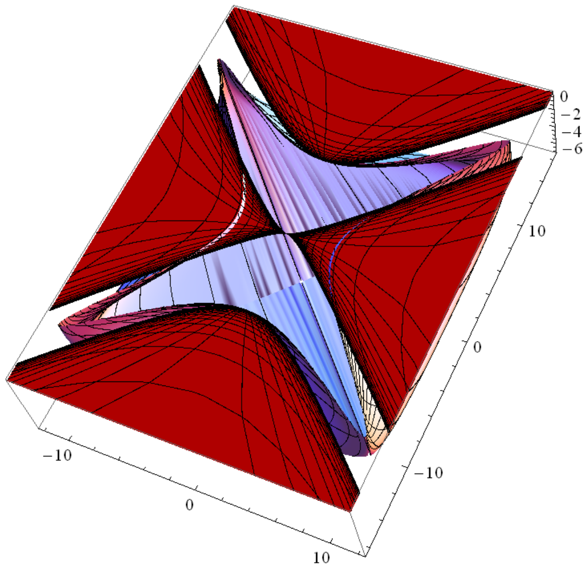


**Fig. 17** Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = 2\pi$

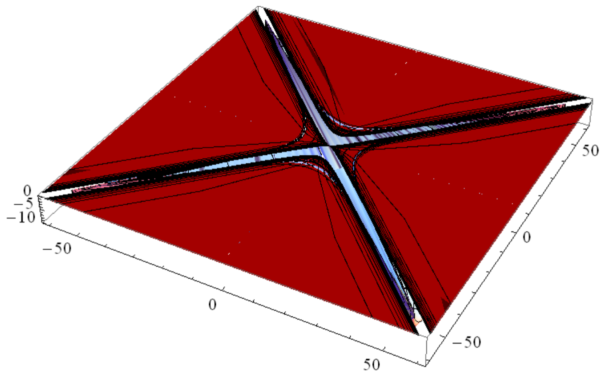


**Fig. 18** Intersection of the cut locus with the hemisphere $z < 0$ of radius $R = 2\pi$

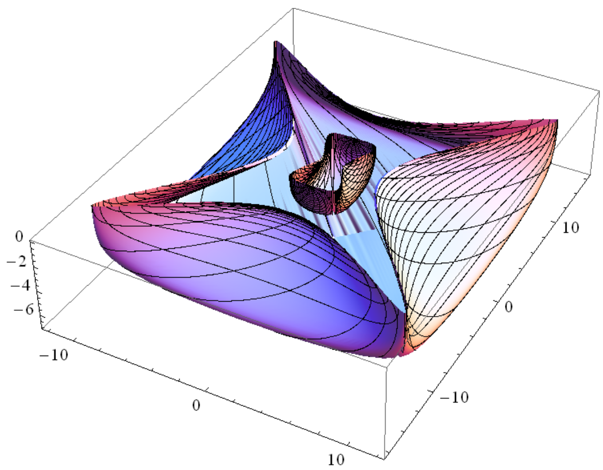


**Fig. 19** Matryoshka of hemispheres $z < 0$ of radii $R = \pi$ and $R = 2\pi$

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