Differential invariants and exact solutions of the Einstein equation and the Einstein-Maxwell equation

V. A. Yumaguzhin

Program Systems Institute of RAS, Pereslavl'-Zalesskiy, Russia
yuma@diffiety.botik.ru

- Valentin Lychagin, Valeriy Yumaguzhin, *Differential invariants and exact solutions of the Einstein-Maxwell equation*,
  Analysis and Mathematical Physics, 2016.
  DOI: 10.1007/s13324-016-0127-7.
- Valentin Lychagin, Valeriy Yumaguzhin, *Differential invariants and exact solutions of the Einstein equation*,
  Analysis and Mathematical Physics, 2016.
  DOI: 10.1007/s13324-016-0130-z.

1. The Einstein equation

Let M be a 4-dimensional oriented manifold, x^0, x^1, x^2, x^3 its local coordinate system, and g a Lorentzian metric on M.

The Levi-Civita connection of g:

$$g = g_{ij} dx^i dx^j, \quad \Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right).$$

The curvature tensor $\mathrm{C}_g = (\mathrm{C}^l_{ijk})$ of g:

$$\mathrm{C}^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{jk}}{\partial x^i} + \Gamma^l_{jm} \Gamma^m_{ik} - \Gamma^l_{im} \Gamma^m_{jk}.$$

The Ricci tensor$\mathrm{Ric}_g = (\mathrm{R}_{ij})$ and the scalar curvature R of g:

$$\mathrm{R}_{ij} = \mathrm{C}^m_{mij}, \quad \mathrm{R} = g^{mn} \mathrm{R}_{mn} .$$

The *vacuum Einstein equation*:

$$\mathrm{Ric}_g - \Lambda\, g = 0, \tag{1}$$

where Λ is a cosmological constant.

2. Weyl tensor

The covariant curvature tensor: $C_{ijkl} = g_{im}C^m_{jkl}$

$$C_{ijkl} = \frac{1}{2}(\mathrm{R}_{ik}\, g_{jl} + \mathrm{R}_{jl}\, g_{ik} - \mathrm{R}_{il}\, g_{jk} - \mathrm{R}_{jk}\, g_{il}) + \frac{\mathrm{R}}{6}(g_{il}g_{jk} - g_{ik}g_{jl}) + \mathrm{W}_{ijkl}$$

The Weyl tensor $\mathrm{W}_g = (\mathrm{W}_{ijkl})$ has the following properties:

$$\mathrm{W}_{ijkl} = -\mathrm{W}_{jikl}, \quad \mathrm{W}_{ijkl} = -\mathrm{W}_{ijlk}, \quad \mathrm{W}_{ijkl} = \mathrm{W}_{klij},$$
$$\mathrm{W}_{ijkl} + \mathrm{W}_{iljk} + \mathrm{W}_{iklj} = 0,$$
$$g^{ij}\,\mathrm{W}_{ijkl} = 0.$$

The Weyl operator:

$$\widehat{\mathrm{W}}_g : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M, \quad \widehat{\mathrm{W}}^{ij}{}_{kl} = g^{im}g^{jn}\,\mathrm{W}_{mnkl}$$

3. Complex structure on $\Lambda^2 T^* M$

The metric g induces the metric on $\Lambda^2 T^* M$:

$$g(\alpha, \beta) = g^{j_1 k_1} g^{j_2 k_2} \alpha_{j_1 j_2} \beta_{k_1, k2}, \quad \forall \alpha, \beta \in \Lambda^2 T^* M.$$

The Hodge operator

$$* : \Lambda^2 T^* M \longrightarrow \Lambda^{4-2} T^* M$$

satisfies to the condition $*^2 = -1$ and defines the complex vector bundle structure on $\Lambda^2 T^* M$:

$$i \cdot \omega \overset{def}{=} * \, \omega, \quad \forall \, \omega \in \Lambda^2 T^* M.$$

The $\mathbb{C}$-valued and $\mathbb{C}$-bilinear non degenerate 2-form h is defined on $\Lambda^2 T^* M$:

$$h(\alpha, \beta) = g(\alpha, \beta) - i \cdot g(* \alpha, \beta), \quad \forall \alpha, \beta \in \Lambda^2 T^* M.$$

It can be proved that

$$h(\alpha, \beta) = - * (* \alpha \wedge \beta) + i * (\alpha \wedge \beta), \quad \forall \alpha, \beta \in \Lambda^2 T^* M.$$

4. Normed eigenvectors of the Weyl operator

Operators $\widehat{\mathrm{W}}$ and $*$ commute, i. e.,

$$*\,\widehat{\mathrm{W}}_g = \widehat{\mathrm{W}}_g\, *\,.$$

The $\mathbb{C}$-linear operator $\widehat{\mathrm{W}}$ is symmetric w.r.t. the 2-form h, i. e.

$$h(\widehat{\mathrm{W}}_g(\alpha), \beta) = h(\alpha, \widehat{\mathrm{W}}_g(\beta)), \quad \forall \alpha, \beta \in \Lambda^2 T^* M.$$

A 2-dimensional subspace $V \subset T_a M$ is elliptic or hyperbolic if the restriction $g|_V$ has signature $(-,-)$ or $(+,-)$ respectively.

Proposition. Let ω be a normed eigenvector of the Weyl operator, i. e. $h(\omega, \omega) = 1$. Then:
(1) ω is decomposable 2-form,
(2) the plane E corresponding to ω is elliptic,
(3) the plane H corresponding to $*\omega$ is hyperbolic and orthogonal to E.

5. Integrability conditions

Let E and H be elliptic and hyperbolic distributions on M generated by a normed eigenvector ω of the Weyl operator $\widehat{\mathrm{W}}_g$.

$$T(M) = E \oplus H, \quad E \perp H.$$

E and H are completely integrable iff their differential invariants, curvature tensors, $\mathcal{R}_E$ and $\mathcal{R}_H$ are trivial.

$$\mathcal{R}_E(X,Y) \stackrel{def}{=} P_H([P_E(X), P_E(Y)]),$$
$$\mathcal{R}_H(X,Y) \stackrel{def}{=} P_E([P_H(X), P_H(Y)]),$$

where X, Y are vector fields in M and $P_E : T(M) \to E$ $P_H : T(M) \to H$ are projectors.

6. Completely integrability of E and H and form of g

Let E and H be completely integrable,
x_2, x_3 1st integrals of E and
x_0, x_1 1st integrals of H.
Then x_0, x_1, x_2, x_3 are local coordinates in M and
g has the following form in these coordinates

$$g = g^H + g^E,$$

where

$$g^H = \sum_{i,j=0}^{1} g^H_{ij}(x_0, x_1, x_2, x_3)dx_i dx_j \quad \text{with signature } (1, -1),$$

$$g^E = \sum_{i,j=2}^{3} g^E_{ij}(x_0, x_1, x_2, x_3)dx_i dx_j \quad \text{with signature } (-1, -1).$$

7. Total geodesic distributions

Recall that that a submanifold S of the Lorentzian manifold (M, g) is said to be *totally geodesic* if:
(1) tangent planes to S are not tangent to the light cones and
(2) every geodesic γ of (M, g) such that $\gamma(0) \in S$ and $\dot{\gamma}(0) \in T_{\gamma(0)}(S)$ belongs S.
Condition (2) is equivalent to the following one: the covariant derivative $\nabla_X Y$ is tangent to S for all vector fields X, Y tangent to S.

We say that completely integrable distributions E and H are *total geodesic* if their integral manifolds are total geodesic.

E and H are total geodesic iff their differential invariants

$$\mathcal{A}_E(X,Y) \stackrel{def}{=} P_H\big(\nabla_{P_E(X)} P_E(Y)\big),$$
$$\mathcal{A}_H(X,Y) \stackrel{def}{=} P_E(\nabla_{P_H(X)} P_H(Y)),$$

are trivial.

8. Totally geodesic solutions

We say that g is *totally geodesic solution* of the Einstein equation if there is a normed eigenvector ω of the operator W_g such that the distributions H and E are totally geodesic.

Proposition Let g be total geodesic. Then, in coordinates x_0, x_1, x_2, x_3, given by the above 1-st integrals, the metric g has the form $g = g^H + g^E$, where

$$g^H = \sum_{i,j=0}^{1} g_{ij}^H(x_0, x_1) dx_i dx_j \quad \text{and} \quad g^E = \sum_{i,j=2}^{3} g_{ij}^E(x_2, x_3) dx_i dx_j .$$

The coordinates x_0, x_1, x_2, x_3 are defined up to gauge transformations

$$(x_0, x_1) \to \big(X^0(x_0, x_1), X^1(x_0, x_1)\big),$$
$$(x_2, x_3) \to \big(X^2(x_2, x_3), X^3(x_2, x_3)\big).$$

Therefore, these coordinates can be chosen to be *isothermal* for metrics g^H and g^E, i. e.,

$$g^H = e^{\alpha(x_0,x_1)}\big(dx_0^2 - dx_1^2\big), \quad g^E = e^{\beta(x_2,x_3)}\big(-dx_2^2 - dx_3^2\big).$$

9. Reduction to Liouville equations

Substituting the last expression of g in the Einstein equation, we get the system of hyperbolic and elliptic Liouville equations:

$$\begin{aligned} \frac{\partial^2\alpha}{\partial {x_0}^2} - \frac{\partial^2\alpha}{\partial {x_1}^2} + 2\Lambda e^{\alpha} = 0, \\ \frac{\partial^2\beta}{\partial {x_2}^2} + \frac{\partial^2\beta}{\partial {x_3}^2} - 2\Lambda e^{\beta} = 0. \end{aligned} \qquad (2)$$

For the first time a solution of the hyperbolic equation was obtained by J. Liouville in 1853,

$$\alpha(x_0, x_1) = \ln\Big(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda v^2}\Big),$$

where function $v(x_0, x_1)$ satisfies the wave equation $v_{x_0x_0} - v_{x_1x_1} = 0$. Then L. Bianchi in 1879 got the other solution of the same form

$$\alpha(x_0, x_1) = \ln\Big(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda \cos^2(v)}\Big)$$

for this equation.

10. Solutions of the Liouville equations

We will find solutions of the Liouville equations in the forms:

$$\alpha(x_0, x_1) = \ln\big(h_1(v)(v_{x_0}^2 - v_{x_1}^2)\big), \tag{3}$$

where $v(x_0, x_1)$ satisfies the wave equation $v_{x_0 x_0} - v_{x_1 x_1} = 0$ and h_1 is a smooth function,

$$\beta(x_2, x_3) = \ln\big(h_2(u)(u_{x_2}^2 + u_{x_3}^2)\big), \tag{4}$$

where $u(x_2, x_3)$ satisfies the Laplace equation $u_{x_2 x_2} + u_{x_3 x_3} = 0$ and h_2 is a smooth function.

Functions (3) and (4) satisfy the corresponding Liouville equations iff h_1 and h_2 are solutions of the following ODEs respectively:

$$yy'' - (y')^2 + 2\Lambda y^3 = 0, \quad \text{and} \quad yy'' - (y')^2 - 2\Lambda y^3 = 0,$$

10. Solutions of the ODEs

The ODE

$$yy'' - (y')^2 + ky^3 = 0, \quad k \in \mathbb{R}\backslash\{0\},$$

has two families of general solutions

$$y_1(x) = 2/k\, a^2 \cosh^2\big((x+b)/a\big),$$
$$y_2(x) = -2/k\, a^2 \cos^2\big((x+b)/a\big),$$

and the family of singular solutions

$$y_3(x) = -2/k(x+b)^2,$$

where $a, b \in \mathbb{R}$, $a \neq 0$.

11. List of totally geodesic solutions

Below, $a_1, a_2, b_1, b_2 \in \mathbb{R}$, and $a_1, a_2 \neq 0$ in all formulas.

1. Assume that $v_{x_0}^2 - v_{x_1}^2 > 0$ in a domain. Then we have the following solutions:

1.1. $h_1 = y_1,\ h_2 = y_2$

$$g = \frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda a_1^2 \cosh^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda a_2^2 \cos^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2).$$

1.2. $h_1 = y_1,\ h_2 = y_3$

$$g = \frac{(v_{x_0}^2 - v_{x_1}^2)}{\Lambda a_1^2 \cosh^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) - \frac{(u_{x_2}^2 + u_{x_3}^2)}{\Lambda (u+b_2)^2}(dx_2^2 + dx_3^2).$$

2. Assume that $v_{x_0}^2 - v_{x_1}^2 < 0$ in a domain. Then we have the following solutions:

2.1. $h_1 = y_2,\ h_2 = y_2$

$$g = -\frac{2(v_{x_0}^2 - v_{x_1}^2)}{\Lambda a_1^2 \cos^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) - \frac{(u_{x_2}^2 + u_{x_3}^2)}{\Lambda a_2^2 \cos^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2).$$

2.2. $h_1 = y_2,\ h_2 = y_3$

$$g = -\frac{(v_{x_0}^2 - v_{x_1}^2)}{\Lambda a_1^2 \cos^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda (u+b_2)^2}(dx_2^2 + dx_3^2).$$

2.3. $h_1 = y_3,\ h_2 = y_2$

$$g = -\frac{(v_{x_0}^2 - v_{x_1}^2)}{\Lambda (v+b_1)^2}(dx_0^2 - dx_1^2) - \frac{(u_{x_2}^2 + u_{x_3}^2)}{\Lambda a_2^2 \cos^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2).$$

2.4. $h_1 = y_3,\ h_2 = y_3$

$$g = -\frac{(v_{x_0}^2 - v_{x_1}^2)}{\Lambda(u+b_2)^2}(dx_0^2 - dx_1^2) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda(u+b_2)^2}(dx_2^2 + dx_3^2).$$

- John K. Beem, Paul Ehrlich, Kevin Easley , *Global Lorentzian Geometry, Second Edition*, Marcel Dekker, Inc., New York - Basel - Hong Kong, 1996, pp. 650.
- Arthur A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin - Heidelberg - New York, 1987, pp. 516.
- L. Bianchi, *Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante*// Ann. Scuola Norm. Superiore Pisa, Ser. 1, 2, 26 (1879) p. 285-341.
- J. Liouville, *Sur l'equation aux differences partielles* $d^2 \log \lambda / du\, dv \pm \lambda/(2a^2) = 0$// J. Math. Pure Apple., 18, (1853) 71.