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Differential invariants and exact solutions of the Einstein equations

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Abstract In this paper (cf. Lychagin and Yumaguzhin, in Anal Math Phys, 2016) a class of totally geodesics solutions for the vacuum Einstein equations is introduced. It consists of Einstein metrics of signature (1,3) such that 2-dimensional distributions, defined by the Weyl tensor, are completely integrable and totally geodesic. The complete and explicit description of metrics from these class is given. It is shown that these metrics depend on two functions in one variable and one harmonic function.

Keywords Vacuum Einstein equation · Jet bundle · Differential invariant · Weyl tensor · Explicit solution · Totally geodesic solution

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1 Introduction

In this paper we follow to the scheme of paper [9] and try to get explicit solutions of the Einstein vacuum equations by putting some extra conditions on their differential invariants [10].

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Let (M, g) be a Lorentzian 4-dimensional oriented manifold and let Ric_g be the Ricci tensor of the metric g.

The *vacuum Einstein equation* is the following system of PDEs on metrics

$$\mathrm{Ric}_g - \Lambda g = 0, \tag{1}$$

where Λ is a cosmological constant.

Denote by W_g the Weyl tensor of metric g and by $\widehat{W}_g$ be the corresponding $\mathbb{C}$-linear endomorphism of the bundle of exterior 2-forms on M.

Then any normalized eigenvector of this operator generates elliptic and hyperbolic 2-dimensional distributions on M. We say that g is totally geodesic if these distributions are completely integrable and totally geodesic.

We show that finding such metrics is reduced to solving of the Liouville equations and therefore gives us the explicit formulae for solutions of the Einstein equation, depending on two functions in one variable and one harmonic function.

2 Preliminaries

2.1 The Weyl tensor

In this section we collect the main properties of the curvature tensor (see, for example, [2] for more details).

Let $C_g \in S^2(\Lambda^2 T^*)$ be the curvature tensor of metric g and let

$$\widehat{C}_g : \Lambda^2 T^* \to \Lambda^2 T^*$$

be the corresponding curvature operator.

The curvature tensor has the decomposition $C_g = s_g + R_g + W_g$ in scalar, Ricci and Weyl parts.

The corresponding decomposition of the curvature operator in terms of the complex vector bundle structure

$$* : \Lambda^2 T^* \to \Lambda^2 T^*,$$

given by the Hodge operator, has very transparent meaning:

the Ricci term corresponds to $\mathbb{C}$-anti linear part of operator $\widehat{C}_g$, and

the Weyl part corresponds to traceless $\mathbb{C}$- linear part of the operator.

In particular, the Weyl operator commutes with the Hodge one:

$$\widehat{W}_g \circ * = * \circ \widehat{W}_g.$$

In addition to the metric on $\Lambda^2 T^*$, induced by g, we consider the $\mathbb{C}$- valued and $\mathbb{C}$-bilinear nondegenerate 2-form

$$h(\alpha, \beta) = g(\alpha, \beta) - i\, g(*\alpha, \beta),$$

where $\alpha, \beta \in \Lambda^2 T^*$.

Then, the Weyl operator is symmetric with respect to this form:

$$h(\widehat{W}_g\alpha, \beta) = h(\alpha, \widehat{W}_g\beta).$$

We say that 2-dimensional subspace V of a tangent space T is elliptic or hyperbolic if the restriction $g|_V$ has signature (0, 2) or (1, 1) respectively.

The following statement holds (see, [9], for example).

Proposition 1 *Let ω be a normed eigenvector of the Weyl operator,* $h(\omega, \omega) = 1$. *Then:*

1. *ω is decomposable 2-form.*
2. *the plane E corresponding to ω is elliptic.*
3. *the plane H corresponding to $*\omega$ is hyperbolic and orthogonal to E.*

2.2 Totally geodesic distributions

Recall (see for example, [1] or [4,5]) that a submanifold $S \subset M$ is said to be *totally geodesic* if

1. tangent planes to S are not tangent to the light cones, and
2. every geodesic of the restriction of g on S is a geodesic of g in M, or the covariant derivative $\nabla_X Y$ is tangent to S when vector fields X and Y are tangent to S. Here ∇ is the Levi-Civita connection on M.

Let now D be a completely integrable 2-dimensional distribution on M. We say that D is *totally geodesic* if its integral submanifolds are totally geodesic.

Let D be a 2-dimensional distribution such that its planes do not tangent to the light cones and $Q : T \to D^{\perp}$ is the projector on the orthogonal complement to D.

Then (see, [6]),

$$\mathcal{R}_D\,(X, Y) = Q\,([X, Y])\,,$$

where X, Y are vector fields tangent to D, is a tensor (called a curvature of the distribution) and this tensor vanishes if and only if the distribution is completely integrable.

For the same reasons,

$$\mathcal{A}_D\,(X, Y) = Q\,(\nabla_X Y)\,,$$

is a tensor too and this tensor vanishes if and only if D is totally geodesic.

Summarizing we get the following result.

Proposition 2 *Let D be a 2-dimensional distribution on M such that its planes do not tangent to the light cones. Then D is totally geodesic if and only if the tensors $\mathcal{A}_D$ and $\mathcal{R}_D$ are trivial.*

3 Totally geodesic solutions

Let g be a solution of the vacuum Einstein equation and ω be a differential 2-form that represents a normed eigenvector of the Weyl operator $\widehat{W}_g$ at each point.

Let H, E be the 2-dimensional hyperbolic and elliptic distributions generated by ω.

We say that metric g is *totally geodesic* if there exists an ω such that the distributions H of hyperbolic and E of elliptic planes are totally geodesic, i.e. due to the Proposition 2:

$$\mathcal{R}_H = 0, \quad \mathcal{A}_H = 0, \quad \mathcal{R}_E = 0, \quad \mathcal{A}_E = 0.$$

Assume now that a solution g is totally geodesic and let (x_0, x_1) be independent 1st integrals for the hyperbolic distribution H and (x_2, x_3) be independent 1st integrals for the elliptic distribution E.

Theorem 3 *Let g be a totally geodesic solution of the Einstein equation. Then, in coordinates (x_0, x_1, x_2, x_3), given by the above 1st integrals, the metric g has the following form:*

$$g = g^H + g^E,$$

where

$$g^H = \sum_{i,j=0}^{1} g_{ij}^H (x_0, x_1)\, dx_i dx_j \quad \text{and} \quad g^E = \sum_{i,j=2}^{3} g_{ij}^E (x_2, x_3)\, dx_i dx_j .$$

Proof Let us show, for example, that

$$\partial_k g_{ij}^H = 0,$$

for $k = 2, 3$.

One has

$$\partial_k g_{ij}^H = \partial_k g\left(\partial_i, \partial_j\right) = g\left(\nabla_{\partial_k} \partial_i, \partial_j\right) + g\left(\partial_i, \nabla_{\partial_k} \partial_j\right).$$

But $\nabla_{\partial_k} \partial_i = \nabla_{\partial_i} \partial_k$ and $\nabla_{\partial_i} \partial_k \perp \langle \partial_0, \partial_1 \rangle$. Hence $g(\nabla_{\partial_k} \partial_i, \partial_j) = 0$.

Remark, that the coordinates (x_0, x_1, x_2, x_3) are defined up to gauge transformations:

$$(x_0, x_1) \to \left(X_0 (x_0, x_1), X_1 (x_0, x_1)\right),$$
$$(x_2, x_3) \to \left(X_2 (x_2, x_3), X_3 (x_2, x_3)\right).$$

Therefore, these coordinates can be chosen to be "isothermal" for metrics g^H and g^E, i.e.

$$g^H = e^{\alpha(x_0, x_1)} \left(dx_0^2 - dx_1^2\right) \tag{2}$$

and

$$g^E = e^{\beta(x_2, x_3)} \left(-dx_2^2 - dx_3^2\right). \tag{3}$$

4 Explicit solutions

Substituting now expressions (3), (2) in the Einstein equation we get the following system differential equations on functions α and β:

$$\begin{aligned} &\frac{\partial^2\alpha}{\partial {x_0}^2} - \frac{\partial^2\alpha}{\partial {x_1}^2} + 2\Lambda e^{\alpha} = 0, \\ &\frac{\partial^2\beta}{\partial {x_2}^2} + \frac{\partial^2\beta}{\partial {x_3}^2} - 2\Lambda e^{\beta} = 0. \end{aligned} \tag{4}$$

Thus system (4) consist of two well known Liouville equations.

First of all, remark that both equations have infinite-dimensional groups of symmetries.

Namely, the hyperbolic Liouville equation has conformal (with respect to metric $d{x_0}^2 - d{x_1}^2$) group of symmetries generated by point transformations of the form

$$a(x_0, x_1)\partial_{x_0} + b(x_0, x_1)\partial_{x_1} - (a_{x_0} + b_{x_1})\partial_{\alpha},$$

where functions a, b satisfy the wave equations:

$$a_{x_0} = b_{x_1}, \quad a_{x_1} = b_{x_0}.$$

The elliptic Liouville equation has also conformal (with respect to metric $-d{x_2}^2 - d{x_3}^2$) group of symmetries generated by point transformations

$$a(x_2, x_3)\partial_{x_2} + b(x_2, x_3)\partial_{x_3} - (a_{x_2} + b_{x_3})\partial_{\beta},$$

where coefficients a, b satisfy the Cauchy-Riemann equations:

$$a_{x_2} = b_{x_3}, \quad a_{x_3} = -b_{x_2}.$$

Secondly, to find solutions of system (4) we represent them in the following forms:

- for hyperbolic case

$$\alpha(x_0, x_1) = \ln\left(h_1(v)\left(v_{x_0}^2 - v_{x_1}^2\right)\right), \tag{5}$$

where function $v(x_0, x_1)$ satisfies the wave equation

$$v_{x_0x_0} - v_{x_1x_1} = 0$$

and h_1 is a smooth function,

- for elliptic case

$$\beta(x_2, x_3) = \ln\left(h_2(u)\left(u_{x_2}^2 + u_{x_3}^2\right)\right), \tag{6}$$

where function $u(x_2, x_3)$ satisfies the Laplace equation

$$u_{x_2 x_2} + u_{x_3 x_3} = 0$$

and h_2 is a smooth function.

Note that the solution of form (5)

$$\alpha(x_0, x_1) = \ln\left(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda v^2}\right)$$

for the hyperbolic equation first was obtained by J. Liouville in [8] and after L. Bianchi [3] got the other solution of the same form

$$\alpha(x_0, x_1) = \ln\left(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda \cos^2(v)}\right)$$

for this equation.

Proposition 4 *Functions* (5) *and* (6) *satisfy the corresponding Liouville equations:*

$$\alpha_{x_0 x_0} - \alpha_{x_1 x_1} + 2\Lambda e^{\alpha} = 0,$$

and

$$\beta_{x_2 x_3} + \beta_{x_2 x_3} - 2\Lambda e^{\beta} = 0,$$

if and only if the functions h_1 and h_2 are solutions of the following ordinary differential equations:

$$yy'' - (y')^2 + 2\Lambda y^3 = 0,$$

and

$$yy'' - (y')^2 - 2\Lambda y^3 = 0,$$

respectively.

Remark 5 The ordinary differential equation

$$yy'' - (y')^2 + ky^3 = 0, \quad k \in \mathbb{R}\backslash\{0\}, \tag{7}$$

has two families of general solutions

$$y_1(x) = 2/k\, a^2 \cosh^2\big((x+b)/a\big),$$
$$y_2(x) = -2/k\, a^2 \cos^2\big((x+b)/a\big),$$

and the family of singular solutions

$$y_3(x) = -2/k(x+b)^2,$$

where $a, b \in \mathbb{R}$, $a \neq 0$.

Thus, we get the following result.

Theorem 6 *Any totally geodesic solution of the Einstein equation locally has the form*

$$g = e^{\alpha(x_0,x_1)}(dx_0^2 - dx_1^2) - e^{\beta(x_2,x_3)}(dx_2^2 + dx_3^2),$$

where (α, β) *is a solution of PDEs system* (4)*,* x_0, x_1 *and* x_2, x_3 *are first integrals of the hyperbolic and elliptic distributions respectively.*

Appendix

Taking into account Proposition 4 and Remark 5, we represent in this section a list of totally geodesic solutions

$$g = e^{\alpha(x_0,x_1)}(dx_0^2 - dx_1^2) - e^{\beta(x_2,x_3)}(dx_2^2 + dx_3^2),$$

of the vacuum Einstein equation such that the functions α and β have form (5) and (6) respectively:

$$\alpha(x_0, x_1) = \ln\left(h_1(v)\left(v_{x_0}^2 - v_{x_1}^2\right)\right), \quad v_{x_0x_0} - v_{x_1x_1} = 0,$$
$$\beta(x_2, x_3) = \ln\left(h_2(u)\left(u_{x_2}^2 + u_{x_3}^2\right)\right), \quad u_{x_2x_2} + u_{x_3x_3} = 0.$$

Below, $a_1, a_2, b_1, b_2 \in \mathbb{R}$, and $a_1, a_2 \neq 0$ in all formulae.

5.1. Assume that $v_{x_0}^2 - v_{x_1}^2 > 0$ in a domain. Then we have the following solutions:
5.1.1. $h_1 = y_1,\ h_2 = y_2$

$$g = \frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda a_1^2 \cosh^2\big((v+b_1)/a_1\big)}\left(dx_0^2 - dx_1^2\right) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda a_2^2 \cos^2\big((u+b_2)/a_2\big)}\left(dx_2^2 + dx_3^2\right).$$

5.1.2. $h_1 = y_1,\ h_2 = y_3$

$$g = \frac{\left(v_{x_0}^2 - v_{x_1}^2\right)}{\Lambda a_1^2 \cosh^2\big((v+b_1)/a_1\big)}\left(dx_0^2 - dx_1^2\right) - \frac{\left(u_{x_2}^2 + u_{x_3}^2\right)}{\Lambda (u+b_2)^2}\left(dx_2^2 + dx_3^2\right).$$

5.2. Assume that $v_{x_0}^2 - v_{x_1}^2 < 0$ in a domain. Then we have the following solutions:

5.2.1. $h_1 = y_2,\ \ h_2 = y_2$

$$g = -\frac{2\left(v_{x_0}^2 - v_{x_1}^2\right)}{\Lambda a_1^2 \cos^2\left((v+b_1)/a_1\right)}\left(dx_0^2 - dx_1^2\right) - \frac{\left(u_{x_2}^2 + u_{x_3}^2\right)}{\Lambda a_2^2 \cos^2\left((u+b_2)/a_2\right)}\left(dx_2^2 + dx_3^2\right).$$

5.2.2. $h_1 = y_2,\ \ h_2 = y_3$

$$g = -\frac{(v_{x_0}^2 - v_{x_1}^2)}{\Lambda a_1^2 \cos^2\left((v+b_1)/a_1\right)}(dx_0^2 - dx_1^2) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda (u+b_2)^2}(dx_2^2 + dx_3^2).$$

5.2.3. $h_1 = y_3,\ \ h_2 = y_2$

$$g = -\frac{\left(v_{x_0}^2 - v_{x_1}^2\right)}{\Lambda (v+b_1)^2}\left(dx_0^2 - dx_1^2\right) - \frac{\left(u_{x_2}^2 + u_{x_3}^2\right)}{\Lambda a_2^2 \cos^2\left((u+b_2)/a_2\right)}\left(dx_2^2 + dx_3^2\right).$$

5.2.4. $h_1 = y_3,\ \ h_2 = y_3$

$$g = -\frac{\left(v_{x_0}^2 - v_{x_1}^2\right)}{\Lambda (u+b_2)^2}\left(dx_0^2 - dx_1^2\right) - \frac{u_{x_2}^2 + u_{x_3}^2}{\Lambda (u+b_2)^2}\left(dx_2^2 + dx_3^2\right).$$

Remark 7

1. The Liouville equations have singular solutions with singularities located at points for the elliptic case and on the characteristics for the hyperbolic case. They give us singular solutions of the Einstein equations with singularities located on curves tangent to light cones.
2. Applying symmetries of the hyperbolic and elliptic Liouville equations to the obtained solutions α and β respectively of these equations, one can get new solutions of system (4) and hence new totally geodesic solutions of the Einstein equation.

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