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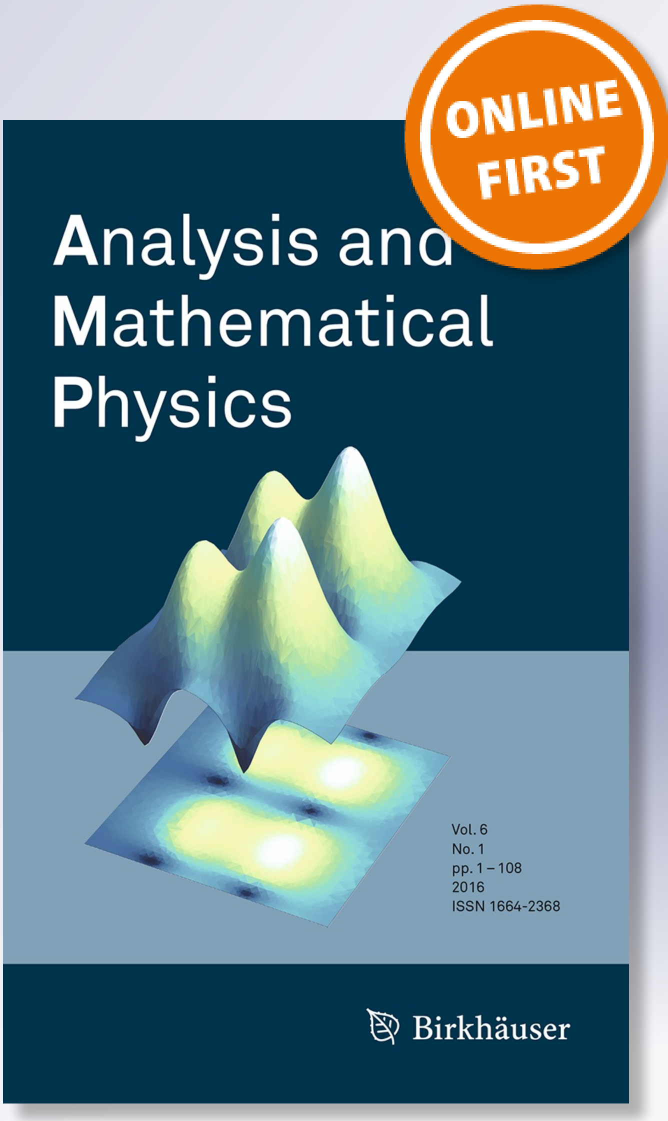

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Differential invariants and exact solutions of the Einstein–Maxwell equation

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Abstract We construct explicit solutions of the Einstein–Maxwell equations in the case when distributions defined by the Faraday tensor are completely integrable and totally geodesic.

Keywords Einstein–Maxwell equation · Jet bundle · Differential invariant · Explicit solution

Mathematics Subject Classification 83C22 · 35A30 · 58J70 · 58D27 · 53A55

1 Introduction

Let M be an oriented 4-dimensional manifold.

The *Einstein-Maxwell equation* (see, for example, [6]) is the following system of PDEs on Lorentzian metric g and differential 2-form F (Faraday tensor) on M :

$$\begin{aligned}&\mathrm{Ric}\,(g) - \Lambda\, g - \frac{8\pi k}{c^4}\,\mathfrak{T}\,(F) = 0,\\ &dF = 0,\ \ d * F = 0.\end{aligned}$$

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Here Λ and k are cosmological and gravitational constants respectively, c is the velocity of light, and $\mathcal{T}(F)$ is the *energy-momentum tensor of electromagnetic field* F.

In this paper we use relativistic differential invariants (see [8]) of solutions of the Einstein–Maxwell equation to get explicit solutions of this equation.

The first invariants one gets from the linear operator $\widehat{F}: TM \to TM$, defining by the Faraday tensor. This operator has characteristic polynomial of the form

$$\lambda^4 + I_1\lambda^2 + I_2,$$

where I_1 and I_2 are invariants of solutions.

We say that a point of manifold M is *Rainich singular* if

$$I_1^2 + I_2^2 = 0$$

at this point.

Let $R_s \subset M$ be the set of all Rainich singular points. Then on the submanifold $M \backslash R_s$ of Rainich regular points tangent spaces TM split in the direct sum of two planes which are $\widehat{F}$—invariant : hyperbolic H and elliptic E. We show that finding solutions of Einstein–Maxwell equation for which hyperbolic and elliptic distributions are completely integrable and totally geodesic is reduced to solution of the Liouville equations and gives us the explicit formulae for solutions of Einstein–Maxwell equation depending on two functions in one variable and one harmonic function.

We would like to thank Valeriya Yumaguzhina for valuable and helpful discussions.

2 Faraday tensor

In this section we briefly recall some facts on Faraday tensor (see, for example, [8,9]).

Let's fix a point $a \in M$ and let $\widehat{F} : T \to T$ be the linear operator acting in the tangent space T $(= T_a M)$ as follows

$$g\left(\widehat{F}X, Y\right) = F(X, Y)$$

for all tangent vectors $X, Y \in T$.

Then $\widehat{F}$ is skew symmetric operator (with respect to metric g) and therefore the characteristic polynomial of $\widehat{F}$ contains only even degree terms:

$$\lambda^4 + I_1\lambda^2 + I_2.$$

It was proved (see for example, [8,9]) that the condition $I_1^2 + I_2^2 \neq 0$ is necessary and sufficient for existence of invariant hyperbolic $H \subset T$ (i.e. g on H is hyperbolic or indefinite) and invariant elliptic $E \subset T$ (i.e. g on E is negative) planes such that $H \perp E$ and $T = H \oplus E$.

Then the restriction of operator $\widehat{F}$ on the hyperbolic plane has eigenvalues $\pm l$ and the restriction on the elliptic plane has eigenvalues $\pm i\, m$, where $l, m \in \mathbb{R}$, and

$$I_1 = m^2 - l^2, \quad I_2 = -l^2 m^2.$$

The projectors on invariant planes are

$$P_H = \frac{1}{J}\left(\widehat{F}^2 + \frac{I_1 + J}{2}\right) \quad \text{and} \quad P_E = -\frac{1}{J}\left(\widehat{F}^2 + \frac{I_1 - J}{2}\right),$$

where

$$J^2 = I_1^2 - 4I_2 = (l^2 + m^2)^2 \neq 0.$$

Moreover, there is an orthonormal basis (e_0, e_1, e_2, e_3), where $g(e_0, e_0) = 1$ and $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -1$ such that

$$\widehat{F} = \begin{bmatrix} 0 & l & 0 & 0 \\ l & 0 & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & -m & 0 \end{bmatrix}.$$

3 Totally geodesic distributions

Recall (see for example, [1] or [3] and [3]) that a submanifold $S \subset M$ is said to be *totally geodesic* (or auto parallel) if

- Tangent planes to S does not tangent to the light cones, and
- Every geodesic of the restriction of g on S is a geodesic of g in M, or the covariant derivative $\nabla_X Y$ is tangent to S every time when vector fields X and Y are tangent to S, here ∇ is the Levi-Civita connection on M.

Let now D be a completely integrable distribution on M and $\dim D = 2$. We say that D *is totally geodesic* if its integral submanifolds are totally geodesic.

If D is a completely integrable distribution such that its planes do not tangent to the light cones and $Q\colon T \to D^{\perp}$ is the projector on the orthogonal complement to D, then tensor

$$\mathcal{A}_D\,(X, Y) = Q\,(\nabla_X Y)\,,$$

where X, Y are vector fields tangent to D, vanishes if and only if D is totally geodesic.

Remark also that the distribution D is completely integrable if and only if its curvature form (see, [5])

$$\mathcal{R}_D\,(X, Y) = Q\,([X, Y])\,,$$

where X, Y are vector fields tangent to D, is equal to zero.

Summarizing we get the following result.

Proposition 1 *Let D be a 2-dimensional distribution on M such that its planes do not tangent to the light cones. Then D is totally geodesic if and only if the tensors $\mathcal{A}_D$ and $\mathcal{R}_D$ are trivial.*

4 Totally geodesic solutions

Let $s = (g, F)$ be a solution of the Einstein- Maxwell equation. We say that s is *totally geodesic* in some domain $\mathcal{O} \subset M$, if the distributions H of hyperbolic and E of elliptic planes are totally geodesic, i.e. due to the above Proposition:

$$\mathcal{R}_H = 0, \quad \mathcal{A}_H = 0, \quad \mathcal{R}_E = 0, \quad \mathcal{A}_E = 0.$$

Assume that a solution s is totally geodesic and let (x_0, x_1) be independent 1st integrals for the hyperbolic distribution H and let (x_2, x_3) be independent 1st integrals for the elliptic distribution E.

Theorem 2 *Let $s = (g, F)$ be a totally geodesic solution of the Einstein-Maxwell equation. Then in coordinates (x_0, x_1, x_2, x_3) given by the 1st integrals metric g has the form*

$$g = g^H + g^E,$$

where

$$g^H = \sum_{i,j=0}^{1} g_{ij}^H (x_0, x_1)\, dx_i dx_j, \quad g^E = \sum_{i,j=2}^{3} g_{ij}^E (x_2, x_3)\, dx_i dx_j.$$

Proof Let us show, for example, that

$$\partial_k g_{ij}^H = 0,$$

for $k = 2, 3$.

One has

$$\partial_k g_{ij}^H = \partial_k g \left(\partial_i, \partial_j\right) = g \left(\nabla_{\partial_k} \partial_i, \partial_j\right) + g \left(\partial_i, \nabla_{\partial_k} \partial_j\right).$$

But $\nabla_{\partial_k} \partial_i = \nabla_{\partial_i} \partial_k$ and $\nabla_{\partial_i} \partial_k \perp \langle \partial_0, \partial_1 \rangle$. Hence $g(\nabla_{\partial_k} \partial_i, \partial_j) = 0$.

Coordinates (x_0, x_1, x_2, x_3) defined up to gauge transformations

$$(x_0, x_1) \to \left(X_0 (x_0, x_1), X_1 (x_0, x_1)\right),$$
$$(x_2, x_3) \to \left(X_2 (x_2, x_3), X_3 (x_2, x_3)\right).$$

Therefore, we can choose these coordinates to be “isothermal” for metrics g^H and g^E, i.e.

$$g^H = e^{\alpha(x_0,x_1)}(dx_0^2 - dx_1^2) \tag{1}$$

and

$$g^E = e^{\beta(x_2,x_3)}(-dx_2^2 - dx_3^2). \tag{2}$$

In these coordinates the Faraday tensor takes the form

$$F = -2l\, e^{\alpha(x_0,x_1)}\, dx_0 \wedge dx_1 + 2m\, e^{\beta(x_2,x_3)}\, dx_2 \wedge dx_3 \tag{3}$$

where

$$l^2 = \frac{J - I_1}{2}, \quad m^2 = \frac{J + I_1}{2}.$$

5 Explicit solutions

Take a totally geodesic solution of the Einstein–Maxwell equation and write down it in form (1), (2), (3).

Substituting these expressions in the Maxwell equations we get the following system of differential equations:

$$\partial_2(le^\alpha) = 0, \quad \partial_3(le^\alpha) = 0, \quad \partial_0(me^\beta) = 0, \quad \partial_1(me^\beta) = 0,$$
$$\partial_2(me^\alpha) = 0, \quad \partial_3(me^\alpha) = 0, \quad \partial_0(le^\beta) = 0, \quad \partial_1(le^\beta) = 0.$$

Taking in account that α and β are functions of (x_0, x_1) and (x_2, x_3) respectively we get that functions l and m are constants.

Therefore, invariants I_1, I_2, J are constants also.

Substituting now (3,2,1) in the Einstein equation we get the following system differential equations on functions α and β:

$$\begin{aligned} &\frac{\partial^2\alpha}{\partial {x_0}^2} - \frac{\partial^2\alpha}{\partial {x_1}^2} + k_1 e^\alpha = 0, \\ &\frac{\partial^2\beta}{\partial {x_2}^2} + \frac{\partial^2\beta}{\partial {x_3}^2} + k_2 e^\beta = 0, \end{aligned} \tag{4}$$

where

$$k_1 = 2\Big(\frac{kJ}{c^4} + \Lambda\Big) \quad \text{and} \quad k_2 = 2\Big(\frac{kJ}{c^4} - \Lambda\Big).$$

System (4) consist of two well known Liouville equations.

First of all, let us note that both these equations have infinite dimensional group of symmetries.

Namely, the hyperbolic Liouville equation has conformal (with respect to metric $d{x_0}^2 - d{x_1}^2$) group of symmetries generated by point transformations of the form

$$a(x_0, x_1)\partial_{x_0} + b(x_0, x_1)\partial_{x_1} - (a_{x_0} + b_{x_1})\partial_\alpha,$$

where functions a, b satisfy the wave equations:

$$a_{x_0} = b_{x_1}, \quad a_{x_1} = b_{x_0}.$$

The elliptic Liouville equation has also conformal (with respect to metric $-dx_2^2 - dx_3^2$) group of symmetries generated by point transformations

$$a(x_2, x_3)\partial_{x_2} + b(x_2, x_3)\partial_{x_3} - (a_{x_2} + b_{x_3})\partial_\beta,$$

where coefficients a, b satisfy the Cauchy-Riemann equations:

$$a_{x_2} = b_{x_3}, \quad a_{x_3} = -b_{x_2}.$$

Secondly, we will find solutions of system (4) in the following form:

- For hyperbolic case

$$\alpha(x_0, x_1) = \ln\big(h_1(v)(v_{x_0}^2 - v_{x_1}^2)\big), \tag{5}$$

where function $v(x_0, x_1)$ satisfies the wave equation

$$v_{x_0x_0} - v_{x_1x_1} = 0$$

and h_1 is a smooth function in one variable;

- For elliptic case with $k_2 \neq 0$

$$\beta(x_2, x_3) = \ln\big(h_2(u)(u_{x_2}^2 + u_{x_3}^2)\big), \tag{6}$$

where function $u(x_2, x_3)$ satisfies the Laplace equation

$$u_{x_2x_2} + u_{x_3x_3} = 0$$

and h_2 is a smooth function in one variable;

For elliptic case with $k_2 = 0$

$$\beta(x_2, x_3) \ \text{ is a harmonic function.} \tag{7}$$

Note that the solution of form (5)

$$\alpha(x_0, x_1) = \ln\left(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda v^2}\right)$$

for the hyperbolic equation first was obtained by Liouville in [7] and after Bianchi [2] got the other solution of the same form

$$\alpha(x_0, x_1) = \ln\left(\frac{v_{x_0}^2 - v_{x_1}^2}{\Lambda \cos^2(v)}\right)$$

for this equation.

Proposition 3 *Functions* (5) *and* (6) *satisfy the corresponding Liouville equations:*

$$\alpha_{x_0 x_0} - \alpha_{x_1 x_1} + k_1 e^{\alpha} = 0 \quad and \quad \beta_{x_2 x_3} + \beta_{x_2 x_3} + k_2 e^{\beta} = 0, \quad k_2 \neq 0,$$

if and only if the functions h_1 *and* h_2 *are solutions of the ordinary differential equations:*

$$yy'' - (y')^2 + k_1 y^3 = 0 \quad and \quad yy'' - (y')^2 + k_2 y^3 = 0$$

respectively.

Remark 4 The ordinary differential equation

$$yy'' - (y')^2 + ky^3 = 0, \quad k \in \mathbb{R}\backslash\{0\}, \tag{8}$$

has two families of general solutions

$$y_1(x) = 2/k\, a^2 \cosh^2\big((x + b)/a\big),$$
$$y_2(x) = -2/k\, a^2 \cos^2\big((x + b)/a\big),$$

and family of singular solutions

$$y_3(x) = -2/k(x + b)^2,$$

where $a, b \in \mathbb{R}$ and $a \neq 0$.

Thus, we get the following result.

Theorem 5 *Any totally geodesic solution of the Einstein-Maxwell equation has the form*

$$g = e^{\alpha(x_0, x_1)}(dx_0^2 - dx_1^2) - e^{\beta(x_2, x_3)}(dx_2^2 + dx_3^2),$$
$$F = -2l\, e^{\alpha(x_0, x_1)} dx_0 \wedge dx_1 + 2m\, e^{\beta(x_2, x_3)} dx_2 \wedge dx_3,$$

where (α, β) *is a solution of PDEs system* (4)*,* x_0, x_1 *and* x_2, x_3 *are first integrals of the hyperbolic and elliptic distributions respectively, and eigenvalues* $\pm l$ *and* $\pm im$ *are constants related with invariants* I_1, I_2, J *in the following way*

$$l^2 = (J - I_1)/2, \quad m^2 = (J + I_1)/2, \quad J^2 = I_1^2 - 4I_2^2.$$

6 Appendix

Taking into account Proposition 3 and Remark 4, we represent in this section a complete list of totally geodesic solutions

$$
\begin{aligned}
g &= e^{\alpha(x_0,x_1)}(dx_0^2 - dx_1^2) - e^{\beta(x_2,x_3)}(dx_2^2 + dx_3^2),\\
F &= -2l\, e^{\alpha(x_0,x_1)} dx_0 \wedge dx_1 + 2m\, e^{\beta(x_2,x_3)} dx_2 \wedge dx_3,
\end{aligned}
$$

of the Einstein–Maxwell equation such that the functions α and β have form (5) and (6) or (7) respectively:

$$
\begin{aligned}
&\alpha(x_0,x_1) = \ln\bigl(h_1(v)(v_{x_0}^2 - v_{x_1}^2)\bigr), \quad v_{x_0x_0} - v_{x_1x_1} = 0,\\
&\beta(x_2,x_3) = \ln\bigl(h_2(u)(u_{x_2}^2 + u_{x_3}^2)\bigr), \quad u_{x_2x_2} + u_{x_3x_3} = 0, \quad k_2 \neq 0,\\
&\beta(x_2,x_3) \ \text{ is a harmonic function if } \ k_2 = 0.
\end{aligned}
$$

Below, $a_1, a_2, b_1, b_2 \in \mathbb{R}$, $a_1, a_2 \neq 0$ for all formulas.

6.1. Let the point (x_0, x_1, x_2, x_3) be such that $v_{x_0}^2 - v_{x_1}^2 < 0$ and let $k_2 < 0$. Then we have the following solutions in a small neighborhood of this point:

6.1.1. $h_1 = y_3, \; h_2 = y_3$

$$
\begin{aligned}
g &= -2\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}(dx_0^2 - dx_1^2) + 2\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}(dx_2^2 + dx_3^2),\\
F &= 4l\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}\, dx_0 \wedge dx_1 - 4m\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}\, dx_2 \wedge dx_3,
\end{aligned}
$$

6.1.2. $h_1 = y_2, \; h_2 = y_2$

$$
\begin{aligned}
g = &-\frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cos^2\bigl((v+b_1)/a_1\bigr)}(dx_0^2 - dx_1^2)\\
&+ \frac{2(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\bigl((u+b_2)/a_2\bigr)}(dx_2^2 + dx_3^2),\\
F = &\ \frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cos^2\bigl((v+b_1)/a_1\bigr)}\, dx_0 \wedge dx_1\\
&- \frac{4m(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\bigl((u+b_2)/a_2\bigr)}\, dx_2 \wedge dx_3,
\end{aligned}
$$

6.1.3. $h_1 = y_3,\ \ h_2 = y_2$

$$g = -2\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}(dx_0^2 - dx_1^2) + \frac{2(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2),$$
$$F = 4l\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}\, dx_0 \wedge dx_1 - \frac{4m(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\big((u+b_2)/a_2\big)}\, dx_2 \wedge dx_3,$$

6.1.4. $h_1 = y_2,\ \ h_2 = y_3$

$$g = -\frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cos^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) + 2\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}(dx_2^2 + dx_3^2),$$
$$F = \frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cos^2\big((v+b_1)/a_1\big)}\, dx_0 \wedge dx_1 - 4m\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}\, dx_2 \wedge dx_3.$$

6.2. Let $v_{x_0}^2 - v_{x_1}^2 > 0$ and $k_2 < 0$. Then:

6.2.1. $h_1 = y_1,\ \ h_2 = y_3$

$$g = \frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) + 2\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}(dx_2^2 + dx_3^2),$$
$$F = -\frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)}\, dx_0 \wedge dx_1 - 4m\frac{u_{x_2}^2 + u_{x_3}^2}{k_2(u+b_2)^2}\, dx_2 \wedge dx_3,$$

6.2.2. $h_1 = y_1,\ \ h_2 = y_2$

$$\begin{aligned} g = {} & \frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) \\ & + \frac{2(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2), \\ F = {} & -\frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)}\, dx_0 \wedge dx_1 \\ & - \frac{4m(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cos^2\big((u+b_2)/a_2\big)}\, dx_2 \wedge dx_3. \end{aligned}$$

6.3. Let $v_{x_0}^2 - v_{x_1}^2 < 0$ and $k_2 > 0$. Then:

6.3.1. $h_1 = y_3,\ \ h_2 = y_1$

$$g = -2\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}(dx_0^2 - dx_1^2) - \frac{2(u_{x_2}^2 + u_{x_3}^2)}{k_2 a_2^2 \cosh^2\big((u+b_2)/a_2\big)}(dx_2^2 + dx_3^2),$$

$$F = 4l\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}\,dx_0\wedge dx_1 - \frac{4m(u_{x_2}^2+u_{x_3}^2)}{k_2a_2^2\cosh^2\big((u+b_2)/a_2\big)}\,dx_2\wedge dx_3,$$

6.3.2. $h_1 = y_2,\ \ h_2 = y_1$

$$\begin{aligned} g &= -\frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1a_1^2\cos^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2)\\ &\quad - \frac{2(u_{x_2}^2+u_{x_3}^2)}{k_2a_2^2\cosh^2\big((u+b_2)/a_2\big)}(dx_2^2+dx_3^2),\\ F &= \frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1a_1^2\cos^2\big((v+b_1)/a_1\big)}\,dx_0\wedge dx_1\\ &\quad - \frac{4m(u_{x_2}^2+u_{x_3}^2)}{k_2a_2^2\cosh^2\big((u+b_2)/a_2\big)}\,dx_2\wedge dx_3. \end{aligned}$$

6.4. Let $v_{x_0}^2 - v_{x_1}^2 > 0$ and $k_2 > 0$. Then

6.4.1. $h_1 = y_1,\ \ h_2 = y_1$

$$\begin{aligned} g &= \frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1a_1^2\cosh^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2)\\ &\quad - \frac{2(u_{x_2}^2+u_{x_3}^2)}{k_2a_2^2\cosh^2\big((u+b_2)/a_2\big)}(dx_2^2+dx_3^2),\\ F &= -\frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1a_1^2\cosh^2\big((v+b_1)/a_1\big)}\,dx_0\wedge dx_1\\ &\quad + \frac{4m(u_{x_2}^2+u_{x_3}^2)}{k_2a_2^2\cosh^2\big((u+b_2)/a_2\big)}\,dx_2\wedge dx_3. \end{aligned}$$

6.5. Let $k_2 = 0$, then $\beta(x_2, x_3)$ is a harmonic function and we have the following solutions:

6.5.1. $v_{x_0}^2 - v_{x_1}^2 < 0$ and $h_1 = y_3$

$$\begin{aligned} g &= -2\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}(dx_0^2 - dx_1^2) - e^{\beta}(dx_2^2+dx_3^2),\\ F &= 4l\frac{v_{x_0}^2 - v_{x_1}^2}{k_1(v+b_1)^2}\,dx_0\wedge dx_1 + 2m\,e^{\beta}dx_2\wedge dx_3, \end{aligned}$$

6.5.2. $v_{x_0}^2 - v_{x_1}^2 < 0$ and $h_1 = y_2$

$$g = -\frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1a_1^2\cos^2\big((v+b_1)/a_1\big)}(dx_0^2 - dx_1^2) - e^{\beta}(dx_2^2+dx_3^2),$$

$$F = \frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cos^2\big((v+b_1)/a_1\big)}\, dx_0 \wedge dx_1 + 2m\, e^{\beta} dx_2 \wedge dx_3,$$

6.5.3. $v_{x_0}^2 - v_{x_1}^2 > 0$ and $h_1 = y_1$

$$g = \frac{2(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)} (dx_0^2 - dx_1^2) - e^{\beta}(dx_2^2 + dx_3^2),$$
$$F = -\frac{4l(v_{x_0}^2 - v_{x_1}^2)}{k_1 a_1^2 \cosh^2\big((v+b_1)/a_1\big)}\, dx_0 \wedge dx_1 + 2m\, e^{\beta} dx_2 \wedge dx_3,$$

Remark 6 Applying symmetries of the hyperbolic and elliptic Liouville equations to the obtained solutions α and β respectively of these equations, one can get new solutions of system (4) and hence new totally geodesic solutions of the Einstein-Maxwell equation.

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