# A PDE Approach to Data-Driven Sub-Riemannian Geodesics in $S E(2)^{*}$ 

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#### Abstract

We present a new flexible wavefront propagation algorithm for the boundary value problem for subRiemannian (SR) geodesics in the roto-translation group $S E(2)=\mathbb{R}^{2} \rtimes S^{1}$ with a metric tensor depending on a smooth external cost $\mathcal{C}: S E(2) \rightarrow[\delta, 1], \delta>0$, computed from image data. The method consists of a first step where an SR-distance map is computed as a viscosity solution of a Hamilton-Jacobi-Bellman system derived via Pontryagin's maximum principle (PMP). Subsequent backward integration, again relying on PMP, gives the SR-geodesics. For $\mathcal{C}=1$ we show that our method produces the global minimizers. Comparison with exact solutions shows a remarkable accuracy of the SR-spheres and the SR-geodesics. We present numerical computations of Maxwell points and cusp points, which we again verify for the uniform cost case $\mathcal{C}=1$. Regarding image analysis applications, trackings of elongated structures in retinal and synthetic images show that our line tracking generically deals with crossings. We show the benefits of including the SR-geometry.


Key words. roto-translation group, Hamilton-Jacobi equations, vessel tracking, sub-Riemannian geometry, morphological scale spaces

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1. Introduction. In computer vision, it is common to extract salient curves in images via minimal paths or geodesics minimizing a length functional [33]. The minimizing geodesic is defined as the curve that minimizes the length functional, which is typically weighted by a cost function with high values on image locations with high curve saliency. To compute such data-driven geodesics many authors use a two-step approach: first, a geodesic distance map to a source is computed; then steepest descent on the map gives the geodesics. In a PDE framework, the geodesic map is obtained via wavefront propagation as the viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation (the eikonal equation). For a review of this approach and applications see [38, 29, 33].

Another set of geodesic methods, partially inspired by the psychology of vision, was developed in [32, 15]. In particular, in [15] the roto-translation group $S E(2)=\mathbb{R}^{2} \rtimes S^{1}$ endowed with a sub-Riemannian (SR) metric models the functional architecture of the primary visual

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Figure 1. A: Every point in the planar curve $\gamma_{2 \mathrm{D}}(t)=(x(t), y(t))$ is lifted to a point $g=\gamma(t)=$ $(x(t), y(t), \theta(t)) \in S E(2)$ on a horizontal curve (solid line) by considering the direction of the tangent vector $\dot{\gamma}_{2 \mathrm{D}}(t)$ of the planar curve as the third coordinate. Then tangent vectors $\dot{\gamma}(t) \in \operatorname{span}\left\{\left.\mathcal{A}_{1}\right|_{\gamma(t)},\left.\mathcal{A}_{2}\right|_{\gamma(t)}\right\}=\left.\Delta\right|_{\gamma(t)}$, (2.1). B: In the lifted domain, $S E(2)$ crossing structures are disentangled. C: The $S R$-geodesic (green) better follows the curvilinear structure along the gap than the Riemannian geodesic (red).
cortex, and geodesics (stratifying a minimal surface) are used for completion of occluded contours. A stable wavelet-like approach to lift 2D images to functions on $S E(2)$ was proposed in [18]. Within the $S E(2)$ framework, images and curves are lifted to the 3 D space $\mathbb{R}^{2} \rtimes S^{1}$ of coupled positions and orientations in which intersecting curves are disentangled. The SRstructure applies a restriction to so-called horizontal curves which are the curves naturally lifted from the plane (see Figure 1A). For a general introduction to SR-geometry see [28]. For explicit formulas of SR-geodesics and optimal synthesis in $S E(2)$ see [37]. SR-geodesics in $S E(2)$ were also studied in $[8,21,10,16,25,27]$.

Here, we propose a new wavefront propagation-based method for finding SR-geodesics within $S E(2)$ with a metric tensor depending on a smooth external cost $\mathcal{C}: S E(2) \rightarrow[\delta, 1]$, $\delta>0$ fixed. Our solution is based on an HJB equation in $S E(2)$ with an SR metric including the cost. This method adapts a classical PDE approach for finding geodesics used in computer vision [38, 29, 33] to the SR-geometry case. It is of interest to interpret the viscosity solution of the corresponding HJB equation as an SR-distance map [39]. Using Pontryagin's maximum principle (PMP), we derive the HJB system with an eikonal equation providing the propagation of geodesically equidistant surfaces departing from the origin. We prove this in Theorem 3.1, and we show that SR-geodesics are computed by backtracking via PMP. In Theorem 3.2, we consider the uniform cost case (i.e., $\mathcal{C}=1$ ) and show that the surfaces coincide with the SR-spheres, i.e., the surfaces from which every tracked curve is globally optimal. This uniform cost case has been deeply studied in [37] relying on explicit ODE-integration in PMP. In this article, we will rely on a PDE approach, allowing us to extend the SR-geodesic problem to the general case where $\mathcal{C}$ is a smooth cost uniformly bounded from below and above. We will often use the results in [37] as a golden standard to verify optimality properties of the viscosity solutions and accuracy of the involved numerics of our PDE approach. We find a remarkable accuracy and convergence toward exact solutions, 1st Maxwell sets (i.e., the location where for the first time two distinct geodesics of equal length meet), and the cusp surface $[16,10]$.

Potential toward applications of the method with nonuniform cost is demonstrated by
performing vessel tracking in retinal images. Here the cost function is computed by lifting the images via oriented wavelets, as explained in section 5 . Similar ideas of computing geodesics via wavefront propagation in the extended image domain of positions and orientations, and/or scales, have been proposed in $[31,22,7]$. In addition to these interesting works we propose to rely on an SR geometry. Let us illustrate some key features of our method. In Figure 1B one can see how disentanglement of intersecting structures, due to their difference in orientations, allows us to automatically deal with crossings (a similar result can be obtained with the algorithm in [31]). The additional benefit of using an SR geometry is shown in Figure 1C, where the SR-geodesic better follows the curvilinear structure along the gap. Further benefits follow in the experimental section where the inclusion of the SR constraint helps to resolve complex configurations containing crossings, low contrast image regions, and/or near parallel vessels. More supporting tracking experiments are provided in the supplementary material (see M101846_01.pdf [local/web 30.7MB]).
1.1. Structure of the article. The article is structured as follows. First, in section 2, we give the mathematical formulation of the curve optimization problem that we aim to solve in this paper. In section 3 we describe our PDE approach that provides the SR-distance map as the viscosity solution of a boundary value problem (BVP) involving an SR-eikonal equation. Furthermore, in Theorem 3.1, we show that SR-geodesics are obtained from this distance map by backtracking (imposed by the PMP computations presented in Appendix A). In Theorem 3.2 we show that for the uniform cost case (i.e., $\mathcal{C}=1$ ) such backtracking will never pass Maxwell points nor conjugate points, and thereby our approach provides only the globally optimal solutions.

In section 4 we describe an iterative procedure on how to solve the BVP by solving a sequence of initial value problems (IVPs) for the corresponding HJB equation. Before involvement of numerics, we express the exact solutions in concatenated morphological convolutions (erosions) and time-shifts in Appendix E. Here we rely on morphological scale space PDEs $[12,2,17]$, and we show that solutions of the iterative procedure converge toward the SR-distance map. Then in section 5 we construct the external cost $\mathcal{C}$, based on a lifting of the original image to an orientation score [18]. In section 6, we describe a numerical PDE implementation of our method by using left-invariant finite differences [20] in combination with an upwind scheme [35].

In section 7 we present numerical experiments and results. In subsection 7.1 we verify the proposed method with comparisons to exact solutions for the uniform cost case. We also provide simple numerical approaches (extendable to the nonuniform cost case) to compute 1st Maxwell points and cusp points [16], which we verify for the uniform cost case with results in [37]. Finally, in subsection 7.2, we show application of the method to vessel tracking in optical images of the retina. We discuss the two main parameters that are involved: the balance between external and internal cost, and the balance between spatial and angular motion. First feasibility studies are presented on patches, and we discuss how to proceed toward automated retinal vessel tree segmentation.

This article is an extension of an SSVM conference article [6]. In addition to [6] we include the following theoretical results: the proofs of our main theorems (Theorems 3.1 and 3.2); the underlying differential geometrical tools in Appendices A, B, and C, and embedding
into geometric control theory in Appendix F; proof of our limiting procedure expressing exact solutions of the SR HJB system in terms of concatenated morphological convolutions (with offsets) in Appendix E. Regarding experiments and applications, we now include new experiments supporting the accuracy of our method, evaluation of the practical potential for vessel tree tracking, simple practical computation of specific surfaces of geometric interest (cusp surface and 1st Maxwell set), analysis of the cost function and evaluation of the parameters involved, and the roadmap toward a fast marching implementation.
2. Problem formulation. The roto-translation group $S E(2)$ carries group product:

$$
g g^{\prime}=\left(\mathbf{x}, R_{\theta}\right)\left(\mathrm{x}^{\prime}, R_{\theta^{\prime}}\right)=\left(R_{\theta} \mathrm{x}^{\prime}+\mathrm{x}, R_{\theta+\theta^{\prime}}\right)
$$

where $R_{\theta}$ is a counterclockwise planar rotation over angle $\theta$. This group can be naturally identified with the coupled space of positions and orientations $\mathbb{R}^{2} \rtimes S^{1}$ by identifying $R_{\theta} \leftrightarrow \theta$ while imposing $2 \pi$-periodicity on $\theta$. Then for each $g \in S E(2)$ we have the left multiplication $L_{g} h=g h$. Via the push-forward $\left(L_{g}\right)_{*}$ of the left-multiplication we get the left-invariant vector fields $\left\{\mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{A}_{3}\right\}$ from the Lie-algebra basis $\left\{A_{1}, A_{2}, A_{3}\right\}=\left\{\left.\partial_{x}\right|_{e},\left.\partial_{\theta}\right|_{e},\left.\partial_{y}\right|_{e}\right\}$ at the unity $e=(0,0,0)$ :

$$
\begin{align*}
& \left.\mathcal{A}_{1}\right|_{g}=\left.\cos \theta \partial_{x}\right|_{g}+\left.\sin \theta \partial_{y}\right|_{g}=\left.\left(L_{g}\right)_{*} \partial_{x}\right|_{e} \\
& \left.\mathcal{A}_{2}\right|_{g}=\left.\partial_{\theta}\right|_{g}=\left.\left(L_{g}\right)_{*} \partial_{\theta}\right|_{e}  \tag{2.1}\\
& \left.\mathcal{A}_{3}\right|_{g}=-\left.\sin \theta \partial_{x}\right|_{g}+\left.\cos \theta \partial_{y}\right|_{g}=\left.\left(L_{g}\right)_{*} \partial_{y}\right|_{e} .
\end{align*}
$$

Then all tangents $\dot{\gamma}(t) \in T_{\gamma(t)}(S E(2))$ along smooth curves $t \mapsto \gamma(t)=(x(t), y(t), \theta(t)) \in$ $S E(2)$ can be expressed as $\dot{\gamma}(t)=\left.\sum_{k=1}^{3} u^{k}(t) \mathcal{A}_{k}\right|_{\gamma(t)}$, where the contravariant components $u^{k}(t)$ of the tangents (velocities) can be considered as the control variables.

Not all curves $t \mapsto \gamma(t)$ in $S E(2)$ can be considered as a lift from a planar curve $t \mapsto$ $(x(t), y(t))$ in the sense that $\theta(t)=\arg (\dot{x}(t)+i \dot{y}(t))$. This only holds for so-called horizontal curves which have $u^{3}=0$ and thus $\dot{\gamma}(t)=\left.\sum_{k=1}^{2} u^{k}(t) \mathcal{A}_{k}\right|_{\gamma(t)}$. This means that a tangent vector to a (lifted) horizontal curve must lay in a 2D subspace of the tangent space; see the gray plane in Figure 1A. At each group element $g=(x, y, \theta)$ one has such a plane spanned by $\left.\mathcal{A}_{1}\right|_{g}$ and $\left.\mathcal{A}_{2}\right|_{g}$. When $\theta$ increases, such a plane rotates. The disjoint union of these planes forms a so-called distribution. In our case this distribution is given by

$$
\Delta:=\operatorname{span}\left\{\mathcal{A}_{1}, \mathcal{A}_{2}\right\}
$$

and vector fields that are inside the distribution are called horizontal. The commutator $\left[\mathcal{A}_{1}, \mathcal{A}_{2}\right]=\mathcal{A}_{1} \mathcal{A}_{2}-\mathcal{A}_{2} \mathcal{A}_{1}=-\mathcal{A}_{3} \notin \Delta$; thus the distribution $\Delta$ together with its commutators fill the full tangent space and thereby every two points in $S E(2)$ can be connected by a horizontal curve; see [1]. We introduce the metric tensor $G^{\mathcal{C}}: S E(2) \times \Delta \times \Delta \rightarrow \mathbb{R}$ given by

$$
\begin{equation*}
\left.G^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))=\mathcal{C}^{2}(\gamma(t))\left(\beta^{2}|\dot{x}(t) \cos \theta(t)+\dot{y}(t) \sin \theta(t)|^{2}+|\dot{\theta}(t)|^{2}\right) \tag{2.2}
\end{equation*}
$$

with $\gamma: \mathbb{R} \rightarrow S E(2)$ a smooth curve on $\mathbb{R}^{2} \rtimes S^{1}, \beta>0$ constant, $\mathcal{C}: S E(2) \rightarrow[\delta, 1]$ the given external smooth cost which is bounded from below by $\delta>0$. This brings us to the SR-manifold (commonly denoted by a triplet; cf. [28]): $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$.

Remark 1. Intuitively, a horizontal curve (cf. Figure 1A) can be seen as a lifted trajectory of a (Reeds-Shepp) car [28,37]. The metric tensor is weighted by the external $\operatorname{cost} \mathcal{C}$, and the
stiffness parameter $\beta$ puts a relative costs on hitting the gas (i.e., moving in the $\mathcal{A}_{1}$-direction) and turning the wheel (i.e., moving in the $\mathcal{A}_{2}$-direction). The connectivity property, i.e., any two group elements can be connected by a horizontal curve, reflects the intuitive fact that in an empty plane a car can be parked in any position and orientation.

Remark 2. Define $\mathcal{L}_{g} \phi(h)=\phi\left(g^{-1} h\right)$; then we have

$$
\left.G^{\mathcal{C}}\right|_{\gamma}(\dot{\gamma}, \dot{\gamma})=\left.G^{\mathcal{L}_{g} \mathcal{C}}\right|_{g \gamma}\left(\left(L_{g}\right)_{*} \dot{\gamma},\left(L_{g}\right) * \dot{\gamma}\right) .
$$

Thus, $G^{\mathcal{C}}$ is not left-invariant, but if shifting the cost as well, we can, for the computation of SR-geodesics, restrict ourselves to $\gamma(0)=e$.

We study the problem of finding SR-minimizers; i.e., for given boundary conditions $\gamma(0)=$ $e, \gamma(T)=g$, we aim to find the horizontal curve $\gamma(t)$ that minimizes the total SR-length

$$
\begin{equation*}
l=\int_{0}^{T} \sqrt{\left.G^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \mathrm{d} t \tag{2.3}
\end{equation*}
$$

If $t$ is the SR-arclength parameter, our default parameter, then $\sqrt{\left.G^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}=1$ and $l=T$. Then SR-minimizers $\gamma$ are solutions to the optimal control problem (with free $T>0$ ):

$$
\mathbf{P}_{\operatorname{mec}}^{\mathcal{C}}(S E(2)):\left\{\begin{array}{c}
\dot{\gamma}=\left.u^{1} \mathcal{A}_{1}\right|_{\gamma}+\left.u^{2} \mathcal{A}_{2}\right|_{\gamma},  \tag{2.4}\\
\gamma(0)=e, \quad \gamma(T)=g, \\
l(\gamma(\cdot))=\int_{0}^{T} \mathcal{C}(\gamma(t)) \sqrt{\beta^{2}\left|u^{1}(t)\right|^{2}+\left|u^{2}(t)\right|^{2}} \mathrm{~d} t \rightarrow \min , \\
\gamma(t) \in \operatorname{SE}(2), \quad\left(u^{1}(t), u^{2}(t)\right) \in \mathbb{R}^{2}, \quad \beta>0
\end{array}\right.
$$

In the naming of this geometric control problem we adhere to terminology in previous works [10, 16]. Stationary curves of the problem (2.4) are found via PMP [1]. Existence of minimizers follows from Chow-Rashevsky and Filippov's theorem [1], and because of the absence of abnormal trajectories (due to the two-bracket generating distribution $\Delta$ ) they are smooth.

Remark 3. The Cauchy-Schwarz inequality implies that the minimization problem for the SR-length functional $l$ with free $T$ is equivalent (see, e.g., [28]) to the minimization problem for the action functional with fixed $T$ :

$$
\begin{equation*}
J(\gamma)=\frac{1}{2} \int_{0}^{T} \mathcal{C}^{2}(\gamma(t))\left(\beta^{2}\left|u^{1}(t)\right|^{2}+\left|u^{2}(t)\right|^{2}\right) \mathrm{d} t \tag{2.5}
\end{equation*}
$$

3. Solutions via data-driven wavefront propagation. The following theorem summarizes our method for the computation of data-driven SR-geodesics in $S E(2)$. It is an extension of classical methods in the Euclidean setting $[29,38,33]$ to the SR-manifold $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$. The idea is illustrated in Figure 2.

Theorem 3.1. Let $W(g)$ be a solution of the following BVP with eikonal equation:

$$
\left\{\begin{array}{l}
\sqrt{(\mathcal{C}(g))^{-2}\left(\beta^{-2}\left|\mathcal{A}_{1} W(g)\right|^{2}+\left|\mathcal{A}_{2} W(g)\right|^{2}\right)}-1=0 \text { for } g \neq e  \tag{3.1}\\
W(e)=0
\end{array}\right.
$$

Then the isosurfaces

$$
\begin{equation*}
\mathcal{S}_{t}=\{g \in S E(2) \mid W(g)=t\} \tag{3.2}
\end{equation*}
$$



Figure 2. A, B: Our method provides both geodesically equidistant surfaces $\mathcal{S}_{t}(3.2)$ (depicted in A) and SR-geodesics. As depicted in B, geodesic equidistance holds with unit speed for all SR-geodesics passing through the surface; see Theorem 3.1. Via Theorem 3.2 we have that $W(g)=d(g, e)$ and $\left\{\mathcal{S}_{t}\right\}_{t \geq 0}$ is the family of $S R$-spheres with radius $t$ depicted in this figure for the uniform cost case.
are geodesically equidistant with unit speed $\mathcal{C}(\gamma(t)) \sqrt{\beta^{2}\left|u^{1}(t)\right|^{2}+\left|u^{2}(t)\right|^{2}}=1$, and they provide a specific part of the $S R$-wavefronts departing from $e=(0,0,0)$. An SR-geodesic ending at $g \in S E(2)$ is found by backward integration:

$$
\begin{equation*}
\dot{\gamma}_{b}(t)=-\left.\frac{\left.\mathcal{A}_{1} W\right|_{\gamma_{b}(t)}}{\left(\beta \mathcal{C}\left(\gamma_{b}(t)\right)\right)^{2}} \mathcal{A}_{1}\right|_{\gamma_{b}(t)}-\left.\frac{\left.\mathcal{A}_{2} W\right|_{\gamma_{b}(t)}}{\left(\mathcal{C}\left(\gamma_{b}(t)\right)\right)^{2}} \mathcal{A}_{2}\right|_{\gamma_{b}(t)}, \quad \gamma_{b}(0)=g . \tag{3.3}
\end{equation*}
$$

Proof. The definition of geodesically equidistant surfaces is given in Definition B. 1 in Appendix B. Furthermore, in Appendix B we provide two lemmas needed for the proof. In Lemma B.2, we connect the Fenchel transform on $\Delta$ to the Fenchel transform on $\mathbb{R}^{2}$ to obtain the result on geodesically equidistant surfaces in $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$. Then, in Lemma B. 3 in Appendix B, we derive the HJB equation for the homogeneous Lagrangian as a limit from the HJB equation for the squared Lagrangian. The backtracking result follows from application of PMP to the equivalent action functional formulation (2.5), as done in Appendix A. Akin to the $\mathbb{R}^{d}$ case [11], characteristics are found by integrating the ODEs of the PMP, where according to the proof of Lemma B. 2 we must set $p=\mathrm{d}^{S R} W$; see Remark 4 below.

The next theorem provides our main theoretical result. Recall that Maxwell points are $S E(2)$ points where two distinct geodesics with the same length meet. The 1st Maxwell set corresponds to the set of Maxwell points where the distinct geodesics meet for the first time. In the subsequent theorem we will consider a specific solution to (3.1), namely, the viscosity solution as defined in Definition C. 3 in Appendix C.

Theorem 3.2. Let $\mathcal{C}=1$. Let $W(g)$ be the viscosity solution of the $B V P$ (3.1). Then $\mathcal{S}_{t}$, (3.2), equals the $S R$-sphere of radius $t$. Backward integration via (3.3) provides globally optimal geodesics reaching e at $t=W(g)=d(g, e):=$

$$
\begin{equation*}
\min _{\substack{\gamma \in C^{\infty}\left(\mathbb{R}^{+}, S E(2)\right), T \geq 0, \dot{\gamma} \in \Delta, \gamma(0)=e, \gamma(T)=g}} \int_{0}^{T} \sqrt{|\dot{\theta}(\tau)|^{2}+\beta^{2}|\dot{x}(\tau) \cos \theta(\tau)+\dot{y}(\tau) \sin \theta(\tau)|^{2}} \mathrm{~d} \tau, \tag{3.4}
\end{equation*}
$$



Figure 3. A: $S R$-sphere $\mathcal{S}_{t}$ for $t=4$ obtained by the method in Theorem 3.1 using $\mathcal{C}=1$ and $\delta_{e}^{M}$ as initial condition via viscosity solutions of the HJB equation (4.1) implemented according to section 6. B: The full $S R$ wavefront departing from e via the method of characteristics and formulas in [27] giving rise to interior folds (corresponding to multiple valued nonviscosity solutions of the HJB equation). The Maxwell set $\mathcal{M}$ consists precisely of the dashed line on $x \cos \frac{\theta}{2}+y \sin \frac{\theta}{2}=0$ and the red circles at $|\theta|=\pi$. The dots are 2 (of the 4) conjugate points on $\mathcal{S}_{t}$ which are limits of 1 st Maxwell points (but not Maxwell points themselves). In B we see the astroidal structure of the conjugate locus $[36,13]$. In A we see that the unique viscosity solutions stop at the 1st Maxwell set. Comparison of A and B shows the global optimality and accuracy of our method at A .
and $\gamma_{b}(t)=\gamma^{\min }(d(g, e)-t)$. The SR-spheres $\mathcal{S}_{t}=\{g \in S E(2) \mid d(g, e)=t\}$ are nonsmooth at the 1 st Maxwell set $\mathcal{M}$ (cf. [37]), contained in

$$
\begin{equation*}
\mathcal{M} \subset\left\{\left.(x, y, \theta) \in S E(2)\left|x \cos \frac{\theta}{2}+y \sin \frac{\theta}{2}=0 \vee\right| \theta \right\rvert\,=\pi\right\}, \tag{3.5}
\end{equation*}
$$

and the backtracking (3.3) does not pass the 1st Maxwell set.
Proof of Theorem 3.2 can be found in Appendix D. The global optimality and nonpassing of the 1st Maxwell set can be observed in Figure 3. For the geometrical idea of the proof see Figure 4.

Remark 4. The Hamiltonian $H^{\text {fixed }}$ for the equivalent fixed time problem (2.5) equals

$$
\begin{equation*}
H^{\mathrm{fixed}}(g, p)=\frac{1}{2} \frac{1}{(\mathcal{C}(g))^{2}}\left(\beta^{-2} h_{1}^{2}+h_{2}^{2}\right)=\frac{1}{2}, \tag{3.6}
\end{equation*}
$$

with momentum covector $p=h_{1} \omega^{1}+h_{2} \omega^{2}+h_{3} \omega^{3}$ expressed in dual basis $\left\{\omega^{i}\right\}_{i=1}^{3}$ given by

$$
\begin{equation*}
\left\langle\omega^{i}, \mathcal{A}_{j}\right\rangle=\delta_{j}^{i} \Leftrightarrow \omega^{1}=\cos \theta \mathrm{d} x+\sin \theta \mathrm{d} y, \omega^{2}=\mathrm{d} \theta, \omega^{3}=-\sin \theta \mathrm{d} x+\cos \theta \mathrm{d} y . \tag{3.7}
\end{equation*}
$$

$$
\begin{aligned}
& \text { Maxwell-point } \\
& g^{*}=\left(x^{*}, y^{*}, \theta^{*}\right) \\
&=\left(-4,4, \frac{\pi}{2}\right)
\end{aligned}
$$



A


Figure 4. Maxwell point $g^{*}=(-4,4, \pi / 2)$ (in white) on $S R$-sphere $\mathcal{S}_{t}$ (in orange) for $\mathcal{C}=1$. At $g^{*}$ two SR-geodesics $\gamma_{1} \not \equiv \gamma_{2}$ with equal SR-length $t$ meet $\left(\gamma_{1}(t)=\gamma_{2}(t)\right.$. From left to right: A: projection of $\gamma_{1}$ and $\gamma_{2}$ on the plane $(x, y)$, B: 2D slices $\left(x=x^{*}, y=y^{*}\right)$ of level sets of $W(g)$ with distinguished value $W(g)=t$ (again in orange). On top we plotted the Maxwell point, the intersection of surface $x \cos \frac{\theta}{2}+y \sin \frac{\theta}{2}=0$ (in purple; this set contains a part of the 1 st Maxwell set) with the $2 D$ slices. C: The SR-sphere $\mathcal{S}_{t}$ in $S E(2)$. D: Section around $g^{*}$ revealing the upward kink due to the viscosity solution. From this kink we see that the tracking (3.3) does not cross a 1st Maxwell point as indicated in red, yielding global optimality in Theorem 3.2.

The Hamiltonian $H^{\text {free }}$ for the free-time problem (2.4) minimizing $l$ equals

$$
\begin{equation*}
H^{\text {free }}(g, p)=\sqrt{2 H^{\text {fixed }}(g, p)}-1=0 \tag{3.8}
\end{equation*}
$$

For details see Appendices A and B. These two Hamiltonians play a central role in the remainder of this article. For example, the SR-eikonal equation, (3.1), can be written as $H^{\text {free }}(g, p)=0$ with momentum ${ }^{1}$

$$
p=\mathrm{d}^{S R} W:=\sum_{i=1}^{2}\left(\mathcal{A}_{i} W\right) \omega^{i}
$$

Remark 5. SR-geodesics lose their optimality either at a Maxwell point or at a conjugate point (where the integrator of the canonical ODEs, mapping initial momentum $p_{0}$ and time $t>0$ to end-point $\gamma(t)$, is degenerate [1]). Some conjugate points are limits of Maxwell points; see Figure 3, where the 1st astroidal conjugate locus coincides with the void regions (cf. [3, Fig. 1]) after 1st Maxwell set $\mathcal{M}$. When setting a Maxwell point as initial condition, the initial derivative $\left.\mathrm{d}^{S R} W\right|_{\gamma_{b}(0)}$ is not defined. Here there are 2 horizontal directions with minimal slope; taking these directions, our algorithm produces the results in Figures 4A and 13.

[^1]4. An iterative IVP procedure to solve the SR-eikonal BVP. To obtain an iterative implementation to obtain the viscosity solution of the SR-eikonal BVP given by (3.1), we rely on viscosity solutions of the IVP. In this approach we put a connection between morphological scale spaces $[12,2]$ and morphological convolutions with morphological kernels, on the SRmanifold ( $S E(2), \Delta, G^{\mathcal{C}}$ ) and the SR-eikonal BVP.

In order to solve the SR-eikonal BVP (3.1) we resort to subsequent auxiliary IVPs on $S E(2)$ for each $r \in\left[r_{n}, r_{n+1}\right]$, with $r_{n}=n \epsilon$ at step $n \in \mathbb{N} \cup\{0\}, \epsilon>0$ fixed:

$$
\left\{\begin{align*}
\frac{\partial W_{n+1}^{\epsilon}}{\partial r}(g, r) & =1-\sqrt{(\mathcal{C}(g))^{-2}\left(\beta^{-2}\left|\mathcal{A}_{1} W_{n+1}^{\epsilon}(g, r)\right|^{2}+\left|\mathcal{A}_{2} W_{n+1}^{\epsilon}(g, r)\right|^{2}\right)},  \tag{4.1}\\
W_{n+1}^{\epsilon}\left(g, r_{n}\right) & =W_{n}^{\epsilon}\left(g, r_{n}\right) \text { for } g \neq e, \\
W_{n+1}^{\epsilon}\left(e, r_{n}\right) & =0
\end{align*}\right.
$$

for $n=1,2, \ldots$, and

$$
\left\{\begin{align*}
\frac{\partial W_{1}^{\epsilon}}{\partial r}(g, r) & =1-\sqrt{(\mathcal{C}(g))^{-2}\left(\beta^{-2}\left|\mathcal{A}_{1} W_{1}^{\epsilon}(g, r)\right|^{2}+\left|\mathcal{A}_{2} W_{1}^{\epsilon}(g, r)\right|^{2}\right)}  \tag{4.2}\\
W_{1}^{\epsilon}(g, 0) & =\delta_{e}^{M}(g)
\end{align*}\right.
$$

for $n=0$, where $\delta_{e}^{M}$ is the morphological delta (i.e., the analogue of the Dirac delta in morphological scale space methods $[12,2]$ ) given by

$$
\delta_{e}^{M}(g)=\left\{\begin{array}{c}
0 \text { if } g=e,  \tag{4.3}\\
+\infty \text { else }
\end{array}\right.
$$

Let $W_{n+1}^{\epsilon}$ denote the viscosity solution of (4.1) carrying the following support:

$$
\operatorname{supp}\left(W_{n+1}^{\epsilon}\right)=S E(2) \times\left[r_{n}, r_{n+1}\right], \text { with } r_{n}=n \epsilon
$$

So in (4.1) at the $n$th iteration ( $n \geq 1$ ) we use, for $g \neq e$, the end condition $W_{n}^{\epsilon}\left(g, r_{n}\right)$ of the $n$th evolution as an initial condition $W_{n+1}^{\epsilon}\left(g, r_{n}\right)$ of the $(n+1)$ th evolution. Only for $g=e$ we set initial condition $W_{n+1}^{\epsilon}\left(e, r_{n}\right)=0$. Then we define the pointwise limit

$$
\begin{equation*}
W^{\infty}(g):=\lim _{\epsilon \rightarrow 0}\left(\lim _{n \rightarrow \infty} W_{n+1}^{\epsilon}(g,(n+1) \epsilon)\right) \tag{4.4}
\end{equation*}
$$

Finally, regarding the application of our optimality results, it is important that each IVP solution $W_{n+1}^{\epsilon}(g, r)$ is the unique viscosity solution of (4.1), as then via (4.4) the viscosity property for the viscosity solutions of the HJB-IVP problem naturally carries over to the viscosity property of the viscosity solutions of system (3.1). Thus we obtain $W=W^{\infty}$ as the unique viscosity solution of the SR-eikonal BVP.

Details on the limit (4.4), which takes place in the continuous setting before numeric discretization is applied, can be found in Appendix E. In Appendix E we provide solutions of (4.1) by a time-shift in combination with a morphological convolution ${ }^{2}$ with the corresponding morphological kernel, and show why the double limit is necessary. A quick intuitive explanation is given in Figure 5, where we see that for $\epsilon>0$ we obtain staircasing (due to a discrete rounding of the distance/value function) and where in the limit $\epsilon \downarrow 0$ the solution $W^{\infty}(g)=W(g)=d(g, e)$ is obtained.

[^2]

Figure 5. Illustration of the pointwise limits in (4.4). Top: plot of $g \mapsto \lim _{n \rightarrow \infty} W_{n+1}^{\epsilon}\left(g, r_{n+1}\right)$ (from left to right, for $\epsilon=1, \epsilon=0.5$, and $\epsilon \downarrow 0$ ) which is a piecewise step-function; see Corollary E. 2 in Appendix E. Along the red axis $\{(x, 0,0) \mid x \in \mathbb{R}\}$ we have $x=d(g, e)$. Bottom: the corresponding graph of $x \mapsto W_{n+1}^{\epsilon}\left((x, 0,0), r_{n+1}\right)$. As $n$ grows the staircase grows; as $\epsilon \rightarrow 0$ the size of the steps in the staircase vanishes, and we see $W^{\infty}(g)=d(g, e)$ in the rightmost column.

Remark 6. The choice of our initial condition in (4.2) comes from the relation between linear and morphological scale spaces [2, 12]. Here, for linear $S E(2)$-convolutions over the $(\cdot,+)$-algebra one has $\delta_{e} *_{S E(2)} U=U$. For morphological $S E(2)$-convolutions (erosions) over the (min,+ )-algebra [16] one has a similar property:

$$
\begin{equation*}
\left(\delta_{e}^{M} \ominus U\right)(g):=\inf _{q \in S E(2)}\left\{\delta_{e}^{M}\left(q^{-1} g\right)+U(q)\right\}=U(g) \tag{4.5}
\end{equation*}
$$

where we recall (4.3). This is important for representing viscosity solutions of left-invariant HJB equations on $S E(2)$ by Lax-Oleinik [19] type of formulas (akin to the $S E(3)$ case [17]). This is, for example, employed in Appendix E.

Remark 7. The staircasing limit depicted in Figure 5 is similar to the basic eikonal BVP on $\mathbb{R}$ with solution $d(x, 0)=|x|$. On $\mathbb{R}$ the approach (4.1), (4.2), and (4.4) provides pointwise limit:

$$
|x|=\lim _{\epsilon \rightarrow 0} \sum_{m=0}^{\infty} r_{m+1} 1_{\left[r_{m}, r_{m}+\epsilon\right]}(|x|)=\lim _{\epsilon \rightarrow 0} \sum_{m=0}^{\left\lceil\frac{|x|}{\epsilon}\right\rceil} r_{m+1} 1_{\left[r_{m}, r_{m}+\epsilon\right]}(|x|)
$$

with $r_{m}=m \epsilon$.
Remark 8. By general semigroup theory [2], one cannot impose both the initial condition and a boundary condition $W^{\epsilon}(e, r)=0$ at the same time, which forced us to update the initial condition (at $g=e$ ) in our iteration scheme (4.1). The separate updating with value 0 for $g=e$ in (4.1) is crucial for the convergence in (4.4).
5. Construction of the nonuniform cost. The cost should have low values on locations with high curve saliency, and high values otherwise. Based on image $f$ we define the cost function $\delta \leq \mathcal{C} \leq 1$ via

$$
\begin{equation*}
\mathcal{C}(x, y, \theta)=\frac{1}{1+\lambda\left|\frac{\left(\mathcal{A}_{3}^{2} U_{f}\right)+(x, y, \theta)}{\left\|\left(\mathcal{A}_{3}^{2} U_{f}\right)_{+}\right\|_{\infty}}\right|^{p}} \tag{5.1}
\end{equation*}
$$

where $\lambda \geq 0, p \in \mathbb{N} ; U_{f}: S E(2) \rightarrow \mathbb{R}$ is a lift of the image, with $\|\cdot\|_{\infty}$ the sup-norm; and

$$
\left(\mathcal{A}_{3}^{2} U_{f}\right)_{+}(x, y, \theta)=\max \left\{0,\left(-\sin \theta \partial_{x}+\cos \theta \partial_{y}\right)^{2} U_{f}(x, y, \theta)\right\}
$$

is a ridge-detector [23] where we use spatially isotropic Gaussian derivatives [20]. The ridgedetector, which is based on a second order derivative in the $\mathcal{A}_{3}$-direction, gives responses only if there are convex variations orthogonal to the elongated structures of interest in $U_{f}(x, y, \theta)$. Note that by (5.1) we have $\delta=\frac{1}{1+\lambda} \leq \mathcal{C} \leq 1$.

The lifting is done using real-valued anisotropic wavelets $\psi$ :

$$
\begin{equation*}
U_{f}(\mathbf{x}, \theta)=\int_{\mathbb{R}^{2}} \psi\left(R_{\theta}^{-1}(\mathbf{y}-\mathbf{x})\right) f(\mathbf{y}) d \mathbf{y} \tag{5.2}
\end{equation*}
$$

See Figure 6. In this work we use the real part of so-called cake wavelets [18] to do the lifting. These wavelets have the property that they allow stable reconstruction and do not tamper with data evidence before processing takes place in the $S E(2)$ domain. Other types of 2D wavelets could be used as well. In related work by Péchaud, Peyré, and Keriven [31] the cost $\mathcal{C}$ was obtained via normalized cross correlation with a set of templates.

In (5.1) two parameters, $\lambda$ and $p$, are introduced. Parameter $\lambda$ can be used to increase the contrast in the cost function. For example, by choosing $\lambda=0$ one creates a uniform cost function, and by choosing $\lambda>0$ data-adaptivity is included. Parameter $p>1$ controls the steepness of the cost function, and in our experiments it is always set to $p=3$.


Figure 6. Illustration of the cost function $\mathcal{C}$. Left: retinal image patch $f$. Middle: corresponding function $U_{f}$ ("invertible orientation score") using the real part of a cake wavelet $\psi$ [18]. Right: the cost function $\mathcal{C}$ computed via (5.1) visualized via volume rendering. The orange corresponds to locations where $\mathcal{C}$ has a low value.
6. Implementation. To compute the SR-geodesics with given boundary conditions we first construct the value function $W$ in (3.1), implementing the iterations at (4.1), after which we obtain our geodesic $\gamma$ via a gradient descent on $W$ from $g$ back to $e$; recall Theorem 3.1 (and Theorem 3.2). Throughout this section, we keep using the continuous notation $g \in S E(2)$, although within all numerical procedures $g$ is sampled on the following $(2 N+1) \times(2 N+1) \times$ $\left(2 N_{\theta}\right)$ equidistant grid:

$$
\begin{equation*}
\left\{\left(x_{i}, y_{j}, \theta_{k}\right) \mid x_{i}=s_{x y} i, y_{j}=s_{x y} j, \theta_{k}=s_{\theta} k, \text { with } i, j=-N, \ldots, N, k=-N_{\theta}+1, \ldots, N_{\theta}\right\} \tag{6.1}
\end{equation*}
$$

with step-sizes $s_{\theta}=\frac{\pi}{N_{\theta}}, s_{x y}=\frac{x_{\max }}{N}$, with $N, N_{\theta} \in \mathbb{N}$. As a default we set $N=70, x_{\text {max }}=7$, $N_{\theta}=64$. The time-discretization grid is also chosen to be equidistant with time steps $\Delta r=\epsilon$.

On this grid we compute an iterative upwind scheme to obtain the viscosity solution $W^{\epsilon}$ at iteration (4.1). Here we initialize $W^{\epsilon}(\cdot, 0)=\delta_{e}^{M D}(\cdot)$, with the discrete morphological delta, given by $\delta_{e}^{M D}(g)=0$ if $g=e$ and 1 if $g \neq e$, and iterate

$$
\left\{\begin{array}{l}
W^{\epsilon}(g, r+\Delta r)=W^{\epsilon}(g, r)-\Delta r H_{D}^{\text {free }}\left(g, d W^{\epsilon}(g, r)\right) \text { for } g \neq e,  \tag{6.2}\\
W^{\epsilon}(e, r+\Delta r)=0,
\end{array}\right.
$$

with free-time Hamiltonian (see Appendix A, (A.4)) given by

$$
H_{D}^{\mathrm{free}}\left(g, d W^{\epsilon}(g, r)\right)=\left(\frac{1}{\mathcal{C}(g)} \sqrt{\beta^{-2}\left(\mathcal{A}_{1} W^{\epsilon}(g, r)\right)^{2}+\left(\mathcal{A}_{2} W^{\epsilon}(g, r)\right)^{2}}-1\right)
$$

until convergence. We set $\Delta r=\epsilon$ in (4.1). In the numerical upwind scheme, the left-invariant derivatives are calculated via

$$
\mathcal{A}_{i} W^{\epsilon}(g, r)=\max \left\{\mathcal{A}_{i}^{-} W^{\epsilon}(g, r),-\mathcal{A}_{i}^{+} W^{\epsilon}(g, r), 0\right\},
$$

where $\mathcal{A}_{i}^{+}$and $\mathcal{A}_{i}^{-}$denote, respectively, the forward and backward finite difference approximations of $\mathcal{A}_{i}$. Note that $W^{\epsilon}$ in (6.2) is a first order finite difference approximation of $W_{n+1}^{\epsilon}$ in (4.1) at time interval $r \in[n \epsilon,(n+1) \epsilon]$, and we iterate until the subsequent $\mathbb{L}_{\infty}$-norms differ less than $10^{-6}$. This upwind scheme is a straightforward extension of the scheme proposed in [35] for HJB systems on $\mathbb{R}^{n}$. It produces sharp ridges at the 1st Maxwell set (cf. Figure 3) as it is consistent at local maxima. For numerical accuracy and left-invariance we applied finite differences in the moving frame of left-invariant vector fields, using $B$-spline interpolation. This is favorable over finite differences in the fixed coordinate grid $\{x, y, \theta\}$. For details on these kinds of left-invariant finite differences, and comparisons to other finite difference implementations (e.g., in fixed coordinate grid), see [20].

In our implementation the origin $e$ is treated separately as our initial condition is not differentiable. We apply the update $W^{\epsilon}(e, r)=0$ for all $r \geq 0$. We set step-size $\epsilon=$ $0.1 \min \left(s_{x y} \beta, s_{\theta}\right)$ with $s_{x y}$ and $s_{\theta}$ step-sizes in, respectively, the $x$ - $y$-directions and the $\theta$ direction.

## 7. Experiments and results.

7.1. Verification for the uniform cost case. Throughout the paper we have illustrated the theory with figures obtained via our new wavefront propagation technique. As problem (2.4) for $\mathcal{C}=1$ was solved [37, 16], we use this as a golden standard for comparison. In this subsection we present experiments that support the accuracy of our method.


Figure 7. A: SR-geodesic example for the uniform cost case shows that our PDE discretizations (with 12 and 64 sampled orientations in red and green, respectively) are accurate in comparison to analytic solutions in [37, 16] (in black). B: The blue surface represents the cusp surface numerically computed via the proposed HJB system (with $\mathcal{C}=1$ ) and subsequent calculation of the zero-crossings of $\mathcal{A}_{1} W(x, y, \theta)$. Indeed if an SR-geodesic (in green) passes this surface, it passes in the $\theta$-direction (with infinite curvature [10, 16]), yielding a cusp on the spatial ground plane. The same blue surface is computed in [16, Fig. 11]. We even see the additional fold (top left, passing the gray plane), as some globally optimal SR-geodesics even exhibit 2 cusps.
7.1.1. Comparison of BVP solutions and the cusp surface. Let us consider Figure 7A. Here an arbitrary SR-geodesic between the $S E(2)$ points $\gamma(0)=e$ and $\gamma(T)=(6,3, \pi / 3)$ is found via the IVP in [37] with end-time $T=7.11$ and initial momentum

$$
p_{0}=h_{1}(0) \mathrm{d} x+h_{2}(0) \mathrm{d} y+h_{3}(0) \mathrm{d} \theta,
$$

with $h_{1}(0)=\sqrt{1-\left|h_{2}(0)\right|^{2}}, h_{2}(0)=0.430$, and $h_{3}(0)=-0.428$. This geodesic is used for reference (and is depicted in black in Figure 7A). Using the semianalytic approach for solving the BVP in [16], an almost identical result is obtained. The curves computed with our method with $s_{x y}=0.1$, and with angular step-sizes of $s_{\theta}=2 \pi / 12$ and $s_{\theta}=2 \pi / 64$, are shown in Figure 7A in red and green, respectively. Already at low resolution, we observe accurate results. In Figure 3 we compare one SR-sphere for $T=4$ (Figure 3A) found via our method with the SR-wavefront departing from $e$ (Figure 3B) computed by the method of characteristics [27]. We observe that our solution is nonsmooth at the 1st Maxwell set $\mathcal{M}$ (3.5) and that the unique viscosity solution stops precisely there, confirming Theorem 3.2. Finally, the blue surface in Figure 7B represents the cusp surface, i.e., the surface consisting of all cusp points. Cusps are points that can occur on geodesics when they are projected into the image plane (see Figure 7B). This happens at points $g$ where the geodesic is tangent to $\left.\partial_{\theta}\right|_{g}=\left.\mathcal{A}_{2}\right|_{g}$. Then the cusp surface $\mathfrak{S}_{\text {cusp }}$ is easily computed as a zero-crossing:

$$
\begin{equation*}
\mathfrak{S}_{\text {cusp }}:=\left\{g \in S E(2) \mid \mathcal{A}_{1} W(g)=0\right\} . \tag{7.1}
\end{equation*}
$$

It is in agreement with the exact cusp surface analytically computed in [16, Fig. 11].
The geometric idea behind (7.1) is that we have a cusp at time $t$ if $u^{1}(t)=\frac{1}{C^{2}(\gamma(t))} h_{1}(t)=$ $\frac{1}{c^{2}(\gamma(t))} \mathcal{A}_{1} W(\gamma(t))=0$, which directly follows from Appendix A. For further details on the set of end-conditions reachable without cusps, see Appendix F.
7.1.2. Comparison and computation of SR-spheres. In order to validate the solutions obtained with our PDE method, we compare them with the exact SR-distance map. This exact SR-distance map was computed by explicit formulas for SR-geodesics (given on p. 386 in [27]) in combination with explicit formulas for the cut time, which coincides with the 1st Maxwell time, given by (5.18)-(5.19) in [27]. The experiments were done in the following way:

1. Compute a set of end-points,

$$
E P(T)=\left\{\left(x_{i}, y_{i}, \theta_{i}\right)=\operatorname{Exp}\left(p_{i}, T\right) \mid p_{i} \in C, T \leq t_{1}^{\operatorname{MAX}}\left(p_{i}\right), i=1, \ldots, i_{\max }\right\}
$$

lying on the SR-sphere of fixed radius $T$ using analytic formulas for the exponential map (cf. Remark 9 below) and 1st Maxwell time $t_{1}^{\mathrm{MAX}}$ [27]. The number of end-points was chosen as $i_{\max }=72 T^{2}$, and $C$ is the cylinder in momentum space given by

$$
C=\left\{p \in T_{e}^{*}(S E(2)) \mid \quad H^{\text {fixed }}(e, p)=1 / 2\right\}
$$

where we recall (3.6). The sampling points $p_{i}$ are taken by a uniform grid on the rectifying coordinates $(\varphi, k)$ of the mathematical pendulum (the ODE that arises in the PMP procedure; cf. [27, Chap. 3.2]), both for the rotating pendulum case ( $p_{i} \in C_{2}$, yielding $S$-curves) and the oscillating pendulum case ( $p_{i} \in C_{1}$, yielding $U$-curves), where we note that $C=\overline{C_{1} \cup C_{2}}$.
2. Evaluate the distance function $W\left(g_{i}\right)=W\left(x_{i}, y_{i}, \theta_{i}\right)$ obtained by our numerical PDE approach in section 6 for every point of the set $E P(T)$. We use third order Hermite interpolation for $W\left(x_{i}, y_{i}, \theta_{i}\right)$ at $g=g_{i} \in E P(T)$ in between the grid (6.1).
3. Compute the maximum absolute error $\max _{g_{i} \in E P(T)}\left|W\left(g_{i}\right)-T\right|$ and the maximum relative error $\max _{g_{i} \in E P(T)}\left(\left|W\left(g_{i}\right)-T\right| / T\right)$.
Remark 9. The exponential map Exp : $C \times \mathbb{R}^{+} \rightarrow S E(2)$ provides the end-point $(x(t), y(t)$, $\theta(t))=\gamma(t)=\operatorname{Exp}\left(p_{0}, t\right)$ of the SR-geodesic $\gamma$, given SR-arclength $t$ and initial momentum $p_{0} \in T_{e}^{*}(S E(2))$. This exponential map integrates the PMP ODEs in Appendix A and should not be confused with the exponential map from the Lie algebra to the Lie group.

In Figure 8 we have depicted a comparison of SR-spheres obtained by our numerical PDE algorithm and the set of points $E P(T)$ lying on exact SR-spheres obtained by analytic formulas.

The absolute and relative errors of the SR-distance computations for each of the endpoints located on SR-spheres of radii $T$ are presented in Figure 9. The red graph corresponds to a sampling of $\left(N, N_{\theta}\right)=(50,64)$, recall (6.1), used in the SR-distance computation by our numerical PDE approach, and the blue graph corresponds to the finer sampling $\left(N, N_{\theta}\right)=$ $(140,128)$. We see that the maximum absolute error does not grow, and that the relative error decreases when increasing the radius of the SR-sphere. An increase in sampling rate improves the result. For the finer sampling case, neither absolute nor relative errors exceed 0.1.
7.1.3. Comparison and computation of 1st Maxwell set. We can compute the 1st Maxwell set (recall (3.5); see also Appendix D) as the set of points where forward and backward left-invariant derivatives acting on the SR-distance map have different signs:

$$
\begin{equation*}
\mathcal{M}_{\mathrm{num}}=\bigcup_{i=1}^{2}\left\{(x, y, \theta) \in S E(2) \mid \mathcal{A}_{i}^{+} W(x, y, \theta)>0, \mathcal{A}_{i}^{-} W(x, y, \theta)<0\right\} \tag{7.2}
\end{equation*}
$$



Figure 8. Comparison of $S R$-spheres obtained by our numerical PDE approach and the set of points $E P(T)$ lying on exact $S R$-spheres obtained by analytic formulas. From left to right: the $S R$-sphere with radius $t=T=3, T=4$, and $T=5$. The color indicates the difference between the exact and the numerical values of the SR-distance (blue for smallest, green for middle, and red for highest differences). Thus, we see our algorithm is accurate, in particular along the fixed coordinate grid directions along the $x$ - and $\theta$-axes.


Figure 9. Maximum error in computing of $S R$-distance for end-points located on $S R$-spheres of different radiit $=T$ (from 1 to 7 with step 0.1 ), with number of end-points $i_{\text {max }}=72 T^{2}$. In red: errors are computed on a courser grid $\left(N, N_{\theta}\right)=(50,64)$; in blue: errors on a finer grid $\left(N, N_{\theta}\right)=(140,128)$, with step-sizes $s_{\theta}=\frac{2 \pi}{N_{\theta}}$ and $s_{x y}=\frac{7}{N}$.

Here $i=1$ corresponds to the local component of the 1st Maxwell set, and $i=2$ corresponds to the global component of the 1st Maxwell set. The local component consists of two connected components lying on the surface given by $x \cos \frac{\theta}{2}+y \sin \frac{\theta}{2}=0$ (i.e., the purple surface in Figure 4), and the global component is the plane given by equation $\theta=\pi$ (for details, see [37]). In Figure 10 we compare the local component of $\mathcal{M}_{\text {num }}$ computed by our PDE approach with its exact counterpart $\mathcal{M}$, presented in [37, Thm. 3.5]. It shows that $\mathcal{M}_{\text {num }}$ is close to the exact $\mathcal{M}$. Although not shown here, a similar picture was obtained for the global component, where $\mathcal{M}_{\text {num }}$ indeed covers the plane $\theta=\pi$. Summarizing, this experiment verifies the correctness of the proposed method, but it also shows that the method allows us to observe the behavior of the 1st Maxwell set. Equation (7.2) allows us to numerically compute the Maxwell set for the data-driven cases $\mathcal{C} \neq 1$ where exact solutions are out of reach.
7.2. Feasibility study for application in retinal imaging. As a feasibility study for the application of our method in retinal images we tested the method on numerous image patches exhibiting both crossings, bifurcations, and low contrast (see Figures 11 and 12). For each seed point $g_{0}$ the value function $g \mapsto W\left(g_{0}^{-1} g\right)$ was calculated according to the implementation


Figure 10. Comparison of the 1 st Maxwell set obtained by our numerical PDE approach with the exact 1 st Maxwell set [37]. Note that the local components of the 1 st Maxwell set are part of the purple surface in Figure 4. Left: Local component of the exact Maxwell set $\mathcal{M}$ obtained by [37, Thm. 3.5] (where we recall that the cut locus coincides with the closure $\overline{\mathcal{M}}$ of the first Maxwell set [36, Thm. 3.3]). Middle: Local components of the Maxwell set $\mathcal{M}_{\text {num }}$ computed numerically by (7.2). Right: Single case of a Maxwell point on the local part of the Maxwell set.
details in section 6, after which multiple end-points were traced back to the seed point. The image dimensions of the patches is $180 \times 140$.

For the construction of the cost function (see, e.g., Figure 6) we set $p=3$, and the lifting was done using cake wavelets with angular resolution $\pi / 16$. More precisely we used a cake wavelet with standard parameters $\left(N=8, N_{\theta}=32, s_{\theta}=\frac{\pi}{8}, \sigma_{s}=20 p x, \gamma=0.8\right)$; for details see [4, Chap. 2]. The precise choice of anisotropic wavelet is not decisive for the algorithm (so other types of anisotropic wavelets and cost constructions could have been applied as well).

In all experiments we run with 4 settings for the two parameters $(\beta, \lambda)$ determining the SR-geodesics; we set $\beta_{\text {small }}=0.05, \beta_{\text {large }}=0.1, \lambda_{\text {small }}=10, \lambda_{\text {large }}=100$. The idea of these settings is to see the effect of the parameters, where we recall that $\beta$ controls global stiffness of the curves, and $\lambda$ controls the influence of the external cost. We also include comparisons to a Riemannian wavefront propagation method on $\mathbb{R}^{2}$, and a Riemannian wavefront propagation method on $S E(2)$. These comparisons clearly show the advantage of including the SR-geometry in the problem. For results on two representative patches, see Figure 11. For results on 25 other patches, see the supplementary material (M101846_01.pdf [local/web $30.7 \mathrm{MB}]$ ) of this article. Here, for fair and basic comparison of the geometries, we rely on the same cost function $\mathcal{C}$. That is, we compare

- Riemannian geodesics $\gamma(t)=(x(t), y(t), \theta(t))$ in $\left(S E(2), G_{\text {full }}^{\mathcal{C}}\right)$ with

$$
\left.G_{\text {full }}^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))=(\mathcal{C}(\gamma(t)))^{2}\left(|\dot{\theta}(t)|^{2}+\beta^{2}|\dot{x}(t)|^{2}+\beta^{2}|\dot{y}(t)|^{2}\right) ;
$$

- Riemannian geodesics $\mathbf{x}(s)=(x(s), y(s))$ in $\left(\mathbb{R}^{2}, G_{\mathbb{R}^{2}}^{\mathcal{C}}\right)$ with metric tensor

$$
\left.G_{\mathbb{R}^{2}}^{\mathcal{C}}\right|_{\mathbf{x}(s)}(\dot{\mathbf{x}}(s), \dot{\mathbf{x}}(s))=(c(\mathbf{x}(s)))^{2}\left(|\dot{x}(s)|^{2}+|\dot{y}(s)|^{2}\right),
$$

with $c(x(s), y(s))=\min _{\theta \in[0,2 \pi)} \mathcal{C}(x(s), y(s), \theta)$.


Figure 11. Data-adaptive $S R$-geodesics obtained via our proposed tracking method (Theorem 3.1), with external cost (5.1), with $p=3, \beta$ equals $\beta_{\text {small }}=0.01, \beta_{\text {large }}=0.1$, and $\lambda$ equals $\lambda_{\text {small }}=10, \lambda_{\text {large }}=100$. We applied tracking from two seed points, each with several end-points (to test the crossings/bifurcations). To distinguish between tracks from the two seed points we plotted tracts of different lighting intensity. We indicated the valid cases only if all trajectories are correctly dealt with.

Typically, the wavefront propagation tracking methods on $\left(\mathbb{R}^{2}, G_{\mathbb{R}^{2}}\right)$ produce incorrect shortcuts at crossings and very nonsmooth curves. The Riemannian wavefront propagation tracking method (with spatial isotropy) $\left(S E(2), G_{\text {full }}^{\mathcal{C}}\right)$ often deals correctly with crossings, but typically suffers from incorrect jumps toward nearly parallel neighboring vessels. Also it yields nonsmooth curves. This can be corrected when including extreme anisotropy; see Remark 10


Figure 12. Tractography (again for $\lambda=\lambda_{\text {small }}=10, \lambda=\lambda_{\text {large }}=100$, and $\beta=0.01, p=3$ ) in a patch of a challenging low-contrast retinal image of a diabetic patient. In case of low contrast (and less reliable cost) it is better to keep $\lambda$ small, in contrast to high contrast cases depicted in Figure 11. To distinguish between tracks from the two seed points, we plotted tracts of different lighting intensity. We indicated the valid cases only if all trajectories are correctly dealt with.
below. The SR-wavefront propagation method produces smooth curves that appropriately deal with crossings. For high-contrast images with reliable cost $\mathcal{C}$, best results are obtained with low $\beta$ and large $\lambda$. However, in low-contrast images and/or patient data with severe abnormalities, low $\lambda$ is preferable, as in these cases the cost function is less reliable. This can be observed in Figure 12.

Remark 10. It is possible to construct a family of anisotropic Riemannian metric tensors (recall (3.7)): $G_{\epsilon}^{\mathcal{C}}=\mathcal{C}^{2}\left(\beta^{2} \omega^{1} \otimes \omega^{1}+\omega^{2} \otimes \omega^{2}+\beta^{2} \epsilon^{-2} \omega^{3} \otimes \omega^{3}\right)$, which bridges the SR-metric $G^{\mathcal{C}}$ of our method (obtained by $\epsilon \rightarrow 0$ ) to the full Riemannian metric tensor $G_{\text {full }}^{\mathcal{C}}$ (obtained by $\epsilon \rightarrow 1$ ). For the values of $\beta$ considered here, Riemannian geodesics and smooth Riemannian spheres for highly anisotropic cases $\epsilon \leq 0.1$ approximate SR-geodesics and nonsmooth SRspheres. In fact, with such extreme anisotropy in the Riemannian setting, the nonsmooth ridges $\mathcal{M}$ in the SR -spheres (see, e.g., the 1st Maxwell sets in Figure 3) are only a little smoothed, and also the cusp surface hardly changes. This observation allows us to use the anisotropic fast-marching [26] as an alternative fast method for computing the solution of (3.1), instead of the iterative upwind finite difference approach in section 4.

The experiments indicate that $\beta=0.01$ (small) in combination with $\lambda=100$ (large) are preferable on our patches. This typically holds for good quality retinal images of healthy volunteers. In lower quality retinal images of diabetic patients, however, the cost function is less reliable and here $\lambda=10$ (small) can be preferable; see Figure 12. However, it might not be optimal to set the $\beta$ parameter globally, as we did in these experiments, as smaller vessels are often more tortuous and therefore require more flexibility; see, e.g., [6, Fig. 7]. Furthermore, we do not include precise centerline extraction, which could, e.g., be achieved by considering the vessel width as an extra feature (as in $[7,31,22]$ ).

In future work we will pursue an SR fast-marching implementation of our method for fully automated vascular tree extraction starting from an automatically detected optic nerve head via a state-of-the art method [5] followed by SR-geodesics (comparable to [14] in a Riemannian setting) in between boundary points detected by an $S E(2)$-morphological approach. First experiments show that such an SR fast-marching method leads to considerable decrease in CPU time, hardly reduces the accuracy of the method, and can be used to perform accurate and fast automatic full retinal tree segmentation. The advantage of such an approach over our previous work on automated vascular tree detection [4] is that each curve is a global minimizer of a formal geometric control curve optimization problem. However, the SR fast-marching and automatic detection of the complete vascular tree via SR-geodesics is beyond the scope of this theoretically oriented paper.
8. Conclusion. In this paper we propose a novel, flexible, and accurate numerical method for computing solutions to the optimal control problem (2.4), i.e., finding SR-geodesics in $S E(2)$ with nonuniform cost. The method generalizes classical approach [38, 29, 33] for finding cost adaptive geodesics in Euclidean settings to the SR case. It consists of a wavefront propagation of geodesically equidistant surfaces computed via the viscosity solution of an HJB system in $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$, and subsequent backward integration, which gives the optimal tracks. We used PMP to derive both the HJB equation and the backtracking. We have shown global optimality for the uniform cost case $(\mathcal{C}=1)$ and that our method generates SR-spheres. Compared to previous works regarding SR-geodesics in (SE(2), $\Delta, G^{1}$ ) [37, 16, 25], we solve the boundary value problem without shooting techniques, using a computational method that always provides the optimal solution. Compared with wavefront propagation methods on the extended domain of positions and orientations in image analysis [31, 30], we consider an SR-metric instead of a Riemannian metric. Results in retinal vessel tracking are promising, and by our data-adaptive approach, it now follows that SR-geometry can make a considerable difference in real medical imaging applications.

Fast, efficient implementation using ordered upwind schemes (such as the anisotropic fastmarching method presented in [26]) is planned as future work, as is adaptation to other Lie groups such as $S E(3), S O(3)$. Of particular interest in neuroimaging is application to high angular resolution diffusion imaging (HARDI) by considering the extension to $S E(3)[17,30]$.

Appendix A. Application of PMP for canonical equations for cost-adaptive SR-geodesics. We study optimal control problem (2.4). Recall Remark 3. Next we apply PMP to the action functional $J$ (2.5) with fixed total time $T>0$. Since $\left[\mathcal{A}_{i}, \mathcal{A}_{j}\right]=\sum_{k=1}^{3} c_{i j}^{k} \mathcal{A}_{k}$, with nonzero coefficients $c_{12}^{3}=-c_{21}^{3}=-1, c_{23}^{1}=-c_{32}^{1}=-1$, we have $[\Delta, \Delta]=T(S E(2))$ and only need to consider normal trajectories. Then the control-dependent Hamiltonian of PMP expressed via left-invariant Hamiltonians $h_{i}(p, g)=\left\langle p, \mathcal{A}_{i}(g)\right\rangle, i=1,2,3$, with momentum $p \in T_{g}^{*}(S E(2))$, and $g=(x, y, \theta) \in \mathbb{R}^{2} \times S^{1}$ reads as

$$
H_{u}(p, g)=u^{1} h_{1}(p, g)+u^{2} h_{2}(p, g)-\frac{1}{2} \mathcal{C}^{2}(g)\left(\beta^{2}\left|u^{1}\right|^{2}+\left|u^{2}\right|^{2}\right)
$$

Optimization over all controls produces the (maximized) Hamiltonian

$$
H^{\mathrm{fixed}}(g, p)=\frac{1}{2 \mathcal{C}^{2}(g)}\left(\frac{h_{1}^{2}}{\beta^{2}}+h_{2}^{2}\right)
$$

and gives expression for extremal controls $u^{1}(t)=\frac{h_{1}(t)}{\mathcal{C}^{2}\left(\gamma(t) \beta^{2}\right.}, u^{2}(t)=\frac{h_{2}(t)}{C^{2}(\gamma(t))}$. Using SRarclength parametrization $\left.G^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))=1$ implies $H^{\text {fixed }}=\frac{1}{2}$ along extremal trajectories. We have the Poisson brackets

$$
\begin{equation*}
\left\{H, h_{1}\right\}=\frac{\mathcal{A}_{1} \mathcal{C}}{\mathcal{C}}+\frac{h_{2} h_{3}}{\mathcal{C}^{2}},\left\{H, h_{2}\right\}=\frac{\mathcal{A}_{2} \mathcal{C}}{\mathcal{C}}-\frac{h_{1} h_{3}}{\beta^{2} \mathcal{C}^{2}},\left\{H, h_{3}\right\}=\frac{\mathcal{A}_{3} \mathcal{C}}{\mathcal{C}}-\frac{h_{2} h_{1}}{\mathcal{C}^{2}} \tag{A.1}
\end{equation*}
$$

where $H=H^{\text {fixed }}$ and with $\{F, G\}=\sum_{i=1}^{3} \frac{\partial F}{\partial h_{i}} \mathcal{A}_{i} G-\frac{\partial G}{\partial h_{i}} \mathcal{A}_{i} F$. By (A.1), by $\left\{h_{i}, h_{j}\right\}=$ $\mathcal{A}_{i} h_{j}-\mathcal{A}_{j} h_{i}=\sum_{k=1}^{3} c_{i j}^{k} h_{k}$, and by $\dot{h}_{i}=\left\{H, h_{i}\right\}$, PMP gives us

$$
\begin{align*}
& \text { - vertical part (for adjoint variables), } \\
& \dot{\gamma}(\cdot)=\left.\sum_{i=1}^{2} u^{i}(\cdot) \mathcal{A}_{i}\right|_{\gamma(\cdot)} \text { and }\left\{\begin{array}{l}
\dot{x}=\frac{h_{1}}{C^{2}\left(\gamma \cdot(\cdot) \beta^{2}\right.} \cos \theta, \\
\dot{y}=\frac{h_{1}}{\left.C^{2}(\gamma \cdot)\right) \beta^{2}} \sin \theta, \\
\dot{\theta}=\frac{h_{2}}{C^{2}(\gamma(\cdot))}
\end{array}\right.  \tag{A.3}\\
& \text { - horizontal part (for state variables), }
\end{align*}
$$

with dual basis $\left\{\omega^{i}\right\}$ for $T^{*}(S E(2))$ defined by $\left\langle\omega^{i}, \mathcal{A}_{j}\right\rangle=\delta_{j}^{i}$.
For a consistency check, we also apply the PMP technique directly to problem (2.4) with free terminal time $T$, where typically [1] the Hamiltonian vanishes. Then, using SR-arclength parameter $t$, the control-dependent Hamiltonian of PMP equals

$$
H_{u}(g, p)=u^{1} h_{1}(p, g)+u^{2} h_{2}(p, g)-\mathcal{C}(g) \sqrt{\beta^{2}\left|u^{1}\right|^{2}+\left|u^{2}\right|^{2}} .
$$

Optimization over all controls under SR-arclength parametrization constraint $\mathcal{C} \sqrt{\beta^{2}\left|u^{1}\right|^{2}+\left|u^{2}\right|^{2}}$ $=1$ produces via Euler-Lagrange optimization with respect to (w.r.t.) ( $u^{1}, u^{2}$ ) (via unit Lagrange multiplier) the (maximized) Hamiltonian:

$$
\begin{equation*}
H^{\text {free }}(g, p)=\frac{1}{\mathcal{C}(g)} \sqrt{\frac{\left|h_{1}\right|^{2}}{\beta^{2}}+\left|h_{2}\right|^{2}}-1=0 \text { with } p=\sum_{i=1}^{3} h_{i} \omega^{i} . \tag{A.4}
\end{equation*}
$$

By straightforward computations one can verify that both the horizontal part and the vertical part of PMP (but now applied to $H^{\text {free }}$ ) is exactly the same as (A.3) and (A.2).

Remark 11. The two approaches produce the same curves and equations, but their Hamiltonians are different. Nevertheless, we have $H^{\text {free }}=0 \Leftrightarrow H^{\text {fixed }}=\frac{1}{2}$.

Appendix B. Lemmas applied in the proof of Theorem 3.1. In this section we consider preliminaries and lemmas needed for Theorem 3.1. Before we can make statements on SRspheres we need to explain the notion of geodesically equidistant surfaces, and their connection to HJB equations. In fact, propagation of geodesically equidistant surfaces in $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$ is described by an HJB system on this SR-manifold.

Recall Remark 3. Also recall that in Appendix A we applied PMP to this problem, yielding constant Hamiltonian $H^{\text {fixed }}=\frac{1}{2 C^{2}}\left(\beta^{-2} h_{1}^{2}+h_{2}^{2}\right)=\frac{1}{2}$ relating to $H^{\text {free }}=\frac{1}{\mathcal{C}} \sqrt{\beta^{-2} h_{1}^{2}+h_{2}^{2}}-1=0$ via $H^{\text {free }}=\sqrt{2 H^{\text {fixed }}}-1$.

In our analysis of geodesically equidistant surface propagation we first resort to the nonhomogeneous viewpoint on the Lagrangian and Hamiltonian (with fixed time) and then obtain the results on the actual homogeneous problem (with free time) via a limiting procedure.

Definition B.1. Given $V: S E(2) \times \mathbb{R}^{+} \rightarrow \mathbb{R}$ continuous. Given a Lagrangian $L(\gamma(r), \dot{\gamma}(r))$ on the $S R$ manifold $\left(S E(2), \Delta, G^{\mathcal{C}}\right)$, with $L(\gamma, \cdot): \Delta \rightarrow \mathbb{R}^{+}$convex. Then the family of surfaces

$$
\begin{equation*}
\mathcal{S}_{r}:=\left\{g \in S E(2) \mid V(g, r)=W_{0}(r)\right\}, \text { with } \tag{B.1}
\end{equation*}
$$

$W_{0}: \mathbb{R} \rightarrow \mathbb{R}$ monotonic, smooth, is geodesically equidistant if $L(\gamma(r), \dot{\gamma}(r))=W_{0}^{\prime}(r)$ for an $S R$-geodesic $\gamma$ in (SE(2), $\left.\Delta, G^{\mathcal{C}}\right)$.

Remark 12. The motivation for this definition is

$$
\frac{d}{d R} \int_{0}^{R} L(\gamma(r), \dot{\gamma}(r)) \mathrm{d} r=L(\gamma(R), \dot{\gamma}(R))=\frac{d W_{0}}{d r}(R)
$$

Lemma B.2. Let $L$ be nonhomogeneous and $\lim _{|v| \rightarrow \infty} \frac{L(\cdot, v)}{|v|}=\infty$. Then the family of surfaces $\left\{\mathcal{S}_{r}\right\}_{r \in \mathbb{R}}$ is geodesically equidistant if and only if $V$ satisfies the following HJB equation (where $r$ may be monotonically reparameterized):

$$
\begin{equation*}
\frac{\partial V}{\partial r}(g, r)=-H\left(g, \mathrm{~d}^{S R} V(g, r)\right), \text { with } \mathrm{d}^{S R} V(g, r)=\mathbb{P}_{\Delta}^{*} \mathrm{~d} V(g, r)=\left.\sum_{i=1}^{2} \mathcal{A}_{i} V(g, r) \omega^{i}\right|_{g} \tag{B.2}
\end{equation*}
$$

Here $\mathbb{P}_{\Delta}^{*}(p)=\sum_{i=1}^{2} h_{i} \omega^{i}$ for all $p=\sum_{i=1}^{3} h_{i} \omega^{i}$ is a dual projection expressed in dual basis $\omega^{i}$ given by $\left\langle\omega^{i}, \mathcal{A}_{j}\right\rangle=\delta_{j}^{i}$, and Hamiltonian $H(g, p)=\max _{v \in T_{g}(S E(2))}\{\langle p, v\rangle-L(g, v)\}$.

Proof. Substitute an arbitrary transversal minimizer $\gamma(r)$ into $V(\cdot, r)$ and take the total derivative w.r.t. $r$ :

$$
\frac{d}{d r} V(\gamma(r), r)=\frac{\partial}{\partial r} V(\gamma(r), r)+\left\langle\left.\mathrm{d} V\right|_{\gamma(r)}, \dot{\gamma}(r)\right\rangle
$$

Now point $\gamma(r)$ lies on $S_{r}$, with tangent $\dot{\gamma}(r)=\left.\sum_{i=1}^{2} u^{i}(r) \mathcal{A}_{i}\right|_{\gamma(r)}$, and thereby we have

$$
\frac{d}{d r} V(\gamma(r), r)=L(\gamma(r), \dot{\gamma}(r))=\frac{\partial}{\partial r} V(\gamma(r), r)+\left.\sum_{i=1}^{2} u^{i}(r) \mathcal{A}_{i}\right|_{\gamma(r)} V(\gamma(r), r)
$$

As a result we have

$$
\begin{gather*}
-L(\gamma(r), \dot{\gamma}(r))+\left.\sum_{i=1}^{2} u^{i}(r) \mathcal{A}_{i}\right|_{\gamma(r)} V(\gamma(r), r)=-\frac{\partial V}{\partial r}(\gamma(r), r) \stackrel{(1)}{\mapsto} \\
\sup _{\left(u^{1}(r), u^{2}(r) \in \mathbb{R}^{2}\right.} \sum_{i=1}^{2} u^{i}(r) h_{i}(r)-L(\gamma(r), \dot{\gamma}(r))=-\frac{\partial V}{\partial r}(\gamma(r), r) \stackrel{(2)}{\Leftrightarrow}  \tag{B.3}\\
H\left(\gamma(r), \mathbb{P}_{\Delta}^{*} \mathrm{~d} V(\gamma(r), r)\right)=-\frac{\partial V}{\partial r}(\gamma(r), r),
\end{gather*}
$$

with components $h_{i}(r)=\left.\mathcal{A}_{i}\right|_{\gamma(r)} V(\gamma(r), r)$ of projected momentum covector

$$
\mathbb{P}_{\Delta}^{*} p(r)=\left.\sum_{i=1}^{2} h_{i}(r) \omega^{i}\right|_{\gamma(r)}=\mathbb{P}_{\Delta}^{*} \mathrm{~d} V(\gamma(r), r) .
$$

Now every point $g \in S_{r}$ is part of a transversal minimizing curve $\gamma(r)$ and the result follows. So the " $\Rightarrow$ " is proven. Conversely, if the HJB equation is satisfied, it follows by the same computations (in reverse order) that $L(\gamma(r), \dot{\gamma}(r))=\frac{d}{d r} V(\gamma(r), r)$, which equals $W_{0}^{\prime}(r)$.

Remark 13. In PMP [1] (see also Appendix A) the controls are optimized to obtain the Hamiltonian $H$ from the control-dependent Hamiltonian $H_{u}$. The first equivalence in (B.3) is due to the maximum condition of PMP. The second equivalence in (B.3) is by definition of the Hamiltonian, where by the convexity assumption of the Lagrangian the supremum is actually a maximum [19, Chap. 8].

Next we apply the limiting procedure to obtain HJB equations for geodesically equidistant surfaces in the actual homogeneous case of interest. The actual homogeneous Lagrangian case with $T$-free can be obtained as a limit $(1 \leq \eta \rightarrow \infty)$ from nonhomogeneous Lagrangian cases:

$$
\begin{equation*}
L_{\eta}(\gamma(t), \dot{\gamma}(t))=\frac{2 \eta-1}{2 \eta}\left(\left.G^{\mathcal{C}}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\right)^{\frac{\eta}{2 \eta-1}} \tag{B.4}
\end{equation*}
$$

and the corresponding Hamiltonian (see Remark 15 below) equals

$$
\begin{equation*}
H_{\eta}(\gamma(t), p(t))=\frac{1}{2 \eta}\left(\beta^{-2} h_{1}^{2}+h_{2}^{2}\right)^{\eta}|\mathcal{C}(\gamma(t))|^{-2 \eta} \tag{B.5}
\end{equation*}
$$

and setting $r=t=W_{0}(t)$. Thus $\frac{\partial V}{\partial r}(\gamma(r), r)=\frac{\partial V}{\partial t}(\gamma(t), t)=W_{0}^{\prime}(t)=L(\gamma(t), \dot{\gamma}(t))=$ $\sqrt{\left.G^{C}\right|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}=1$ in (B.3). Next we replace $V$ by $W$ to distinguish between the homogeneous and the nonhomogeneous cases.

Lemma B.3. The family of surfaces given by (B.1) is geodesically equidistant w.r.t. homogeneous Lagrangian $L_{\infty}(\gamma, \dot{\gamma})=\sqrt{\left.G^{\mathcal{C}}\right|_{\gamma}(\dot{\gamma}, \dot{\gamma})}$, with $r=t=W_{0}(t)$, if and only if $W$ satisfies the HJB equation

$$
\begin{equation*}
\frac{1}{\mathcal{C}} \sqrt{\beta^{-2}\left|\mathcal{A}_{1} W\right|^{2}+\left|\mathcal{A}_{2} W\right|^{2}}=1 \Leftrightarrow H=0 \tag{B.6}
\end{equation*}
$$

where $H=\lim _{\eta \rightarrow \infty} H_{\eta}=H^{\text {free }}$ the vanishing free-time Hamiltonian in Appendix A. Defining Hamiltonian $\tilde{H}$ by

$$
\begin{equation*}
\tilde{H}(g, p):=\mathcal{C}^{-1}(g) \sqrt{\beta^{-2} h_{1}^{2}+h_{2}^{2}} \tag{B.7}
\end{equation*}
$$

puts (B.6) in eikonal form $\tilde{H}\left(g, \mathrm{~d}^{S R} W(g, t)\right)=1$.
Proof. The proof is tangential to the proof of Lemma B.2. For $1 \leq \eta<\infty$ we can apply Lemma B. 2 to Lagrangian $L_{\eta}$ given by (B.4) whose associated Hamiltonian $H_{\eta}$ is given by (B.5) due to PMP (or just the Fenchel transform on $\mathbb{R}^{2}$ ). In the limiting case $\eta \rightarrow \infty$, where
the Lagrangian is homogeneous and the Hamiltonian vanishes. Finally we note that now we have

$$
\frac{\partial W}{\partial r}(\gamma(r), r)=\frac{\partial W}{\partial t}(\gamma(t), t)=W_{0}^{\prime}(t)=L(\gamma(t), \dot{\gamma}(t))=1
$$

from which the result follows.
Remark 14. The relation between the various Hamiltonians is

$$
H_{\eta \rightarrow \infty}=H^{\text {free }}=\sqrt{2 H^{\text {fixed }}}-1=\sqrt{2 H_{\eta=1}}-1=\tilde{H}-1=0 .
$$

Remark 15. The relation between the Lagrangian $L_{\eta}$ given by (B.4) and the Hamiltonian (B.5) is the (left-invariant, SR) Fenchel transform on $S E(2)$. Due to left-invariance this Fenchel transform actually boils down to an ordinary Fenchel transform on $\mathbb{R}^{2}$ when expressing velocity and momentum in the left-invariant frame. Indeed we have

$$
\begin{align*}
H_{\eta}(\gamma, p)= & {\left[\mathfrak{F}_{\mathcal{L}(S E(2)) \cap \Delta}\left(L_{\eta}(\gamma, \cdot)\right)\right](p) }  \tag{B.8}\\
& :=\sup _{\left(u^{1}, u^{2}\right) \in \mathbb{R}^{2}}\left\{-\frac{2 \eta-1}{2 \eta}(\mathcal{C}(\gamma))^{\frac{2 \eta}{2 \eta-1}}\left(\beta^{2}\left|u^{1}\right|^{2}+\left|u^{2}\right|^{2}\right)^{\frac{\eta}{2 \eta-1}}+h_{1} u^{1}+h_{2} u^{2}\right\}
\end{align*}
$$

with horizontal velocity $v=u^{1} \mathcal{A}_{1}+u^{2} \mathcal{A}_{2}$ and momentum $p=\sum_{i=1}^{3} h_{i} \omega^{i}$.
Appendix C. Viscosity solutions for HJB systems in SE(2).
Definition C.1. The (Cauchy problem) for an HJB equation (akin to [19, Chap. 10.1]) on $S E(2)$ is given by

$$
\left\{\begin{array}{l}
\frac{\partial W}{\partial t}=-H\left(g, \mathrm{~d}^{S R} W\right) \text { in } S E(2) \times(0, T),  \tag{C.1}\\
W(g, 0)=W_{0}
\end{array}\right.
$$

whereas a BVP for an HJB equation is given as

$$
\begin{equation*}
H\left(g, \mathrm{~d}^{S R} W\right)=0 \text { on } S E(2) \backslash\{e\}, \quad W(e)=0 \tag{C.2}
\end{equation*}
$$

where $T>0$ is prescribed, $W_{0}$ is a given function (or a cost measure [2]), $H(g, p)=H^{\text {free }}(g, p)$ is the free-time Hamiltonian given by (3.8), and $\mathrm{d}^{S R} W=\left.\sum_{i=1}^{2} \mathcal{A}_{i} W(g, t) \omega^{i}\right|_{g}$.

Remark 16. Combined Cauchy-Dirichlet problems exist [39], but they are defined on (analytic) open and bounded domains. Thereby they cannot be applied to our set of interest $S E(2) \backslash\{e\}$ as this would violate semigroup theory [2, 19, 40, 12]. This is also clear in view of the Cramer transform [2], putting an isomorphism between HJB and diffusion systems.

Remark 17. In (C.2) it is crucial that the free-time Hamiltonian is used. In the definition of viscosity solutions of the Cauchy problem (C.1), one can set both $H=H^{\text {free }}$ (as done in the body of the article) or $H=H^{\text {fixed }}$, as done in Appendix B.

HJB systems in general do not have unique solutions. To avoid multiple (nondesirable) solutions, one must impose the viscosity condition [19, 24] commonly applied in wavefront methods acting directly in the image domain $\mathbb{R}^{2}[29,38]$. The viscosity solution is obtained by the vanishing viscosity method [24]. The idea of this method is to add to the HJB equation a term $\varepsilon \Delta$ and to pass to the limit when $\varepsilon$ goes to 0 . Here $\Delta$ denotes the Laplacian, which
in our case (for $\mathcal{C}=1$ ) equals $\Delta^{S R}=\sum_{i=1}^{2} \mathcal{A}_{i}\left(\beta_{i}\right)^{-2} \mathcal{A}_{i}$, with $\beta_{1}=\beta, \beta_{2}=1$. Here the name "viscosity solutions" comes from fluid dynamics, where typically the term $\varepsilon \Delta$ represents a physical viscosity. For an intuitive illustration of the geometric property of such solutions see [11, Fig. 30]. The viscosity solution of the IVP can be defined alternatively as follows.

Definition C.2. Let $H(g, \cdot)$ be a convex Hamiltonian for all $g \in S E(2)$ such that $H(g, p) \rightarrow$ $\infty$ if $p \rightarrow \infty$. The function $W: S E(2) \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity solution of $\frac{\partial W}{\partial t}=-H\left(g, \mathrm{~d}^{S R} W\right)$ if it is a weak solution ${ }^{3}$ such that for all functions $V \in C^{1}(S E(2) \times \mathbb{R}, \mathbb{R})$ one has that

- if $W-V$ attains a local maximum at $\left(g_{0}, t_{0}\right)$, then $\left.\left(\frac{\partial V}{\partial t}+H\left(g, \mathrm{~d}^{S R} V\right)\right)\right|_{\left(g_{0}, t_{0}\right)} \leq 0$;
- if $W-V$ attains a local minimum at $\left(g_{0}, t_{0}\right)$, then $\left.\left(\frac{\partial V}{\partial t}+H\left(g, \mathrm{~d}^{S R} V\right)\right)\right|_{\left(g_{0}, t_{0}\right)} \geq 0$.

Similarly, the viscosity solution of the BVP (that is equivalent to the eikonal equation, when $t$ is an SR-arclength) can be defined as follows.

Definition C.3. A solution $W: S E(2) \rightarrow \mathbb{R}$ of (C.2) is called a viscosity solution if for all functions $V \in C^{1}(S E(2), \mathbb{R})$ one has that

- if $W-V$ attains a local maximum at $g_{0}$, then $\left.H^{\text {free }}\left(g_{0}, \mathrm{~d}^{S R} V\right)\right) \leq 0$;
- if $W-V$ attains a local minimum at $g_{0}$, then $\left.H^{\text {free }}\left(g_{0}, \mathrm{~d}^{S R} V\right)\right) \geq 0$.

Appendix D. Proof of Theorem 3.2. The backtracking (3.3) is a direct result of Lemma B. 3 in Appendix B and PMP in Appendix A. According to these results one must set

$$
u^{1}(t)=\frac{h_{1}(t)}{(C(\gamma(t)))^{2} \beta^{2}}=\frac{\left.\mathcal{A}_{1} W\right|_{\gamma(t)}}{(C(\gamma(t)))^{2} \beta^{2}} \quad \text { and } \quad u^{2}(t)=\frac{h_{2}(t)}{(C(\gamma(t)))^{2}}=\frac{\left.\mathcal{A}_{2} W\right|_{\gamma(t)}}{(C(\gamma(t)))^{2}}
$$

from which the result follows. Then we recall from Theorem 3.1 that $\mathcal{S}_{t}$ given by (3.2) are geodesically equidistant surfaces propagating with unit speed from the origin. So $\mathcal{S}_{t}$ are candidates for SR -spheres, but it remains to be shown that the backtracking (3.3) will pass neither a Maxwell point nor a conjugate point, i.e., $t \leq t_{\text {cut }}$. Here $t_{\text {cut }}$ denotes cut time, where a geodesic loses its optimality.

At Maxwell points $g^{*}$ induced by the 8 reflectional symmetries [27], two distinct SRgeodesics meet with the same SR-distance. As SR-geodesics in (SE(2), $\Delta, G^{1}$ ) are analytic [27], these two SR-geodesics do not coincide on an open set around end condition $g^{*}$, and the SR-spheres are nonsmooth at $g^{*}$. Regarding the set $\mathcal{M}$, we note that the Maxwell sets related to the $i$ th reflectional symmetry $\epsilon_{i}$ are defined by

$$
\begin{aligned}
& \operatorname{MAX}^{i}=\left\{\left(p_{0}, t\right) \in T_{e}^{*}(S E(2)) \times \mathbb{R}^{+} \left\lvert\, H\left(p_{0}\right)=\frac{1}{2}\right. \text { and } \operatorname{Exp}\left(p_{0}, t\right)=\operatorname{Exp}\left(\epsilon_{i} p_{0}, t\right)\right\}, \\
& \max ^{i}=\operatorname{Exp}\left(\operatorname{MAX}^{i}\right), i=1, \ldots, 8
\end{aligned}
$$

where we may discard indices $i=3,4,6$ as they are contained in $\left\{\max _{1}, \max _{2}, \max _{5}, \max _{7}\right\}$. Now with $\widetilde{\max }^{i}$ we denote the Maxwell set with minimal positive Maxwell times over all symmetries (i.e., we include the constraint $t \leq \min \left\{t_{\max }^{i}\right\}$, where the minimum is taken over all positive Maxwell times along each trajectory), then we find $\mathcal{M}$ to be contained within the

[^3]union of the following sets: ${ }^{4}$
$$
\widetilde{\max }^{2} \subset\{(x, y, \theta) \in S E(2) \mid y \sin \theta / 2+x \cos \theta / 2=0\}, \widetilde{\max }^{5}=\{(x, y, \theta) \in S E(2) \mid \theta=\pi\}
$$
where [27, Thm. 5.2] shows we must discard the first reflectional symmetry $\epsilon_{1}$ as it does not produce Maxwell points. Now for generic geodesics (not passing the special conjugate points that are limit points of Maxwell points and not Maxwell points themselves) $t_{\mathrm{cut}}=t_{\mathrm{MAX}}^{1}$, as proven in [36, Thm. 3.3], where $t_{\mathrm{MAX}}^{1}>0$ denotes the first Maxwell time associated to the 8 discrete reflectional symmetries.

During the backtracking the set $\mathcal{M}$ is never reached at internal times (only when starting at them; recall Remark 5), since they are "uphill" from all possible directions during dual steepest descent tracking (3.3), as we will show next. As a result we have $t \leq t_{\text {cut }}=t_{\text {MAX }}^{1}$. Consider Figure 4. At Maxwell points $g^{*} \in \mathcal{M}$, due to the reflectional symmetries, there exist two distinct directions in the 2D horizontal part $\Delta_{g^{*}}$ of the tangent space $T_{g^{*}}(S E(2))$ where the directional derivative is positive. If there were a direction in the tangent space where the directional derivative is negative, then there would be a direction in $\Delta_{g^{*}}$ with zero directional derivative of $W(\cdot)$ at $g^{*}$ toward the interior of the sphere, yielding a contradiction. Here we note that due to the viscosity property of the HJB solution, kinks at the Maxwell points are pointing upward (see Figures 4 and 13) in the backward minimization tracking process [11, Fig. 30]. Furthermore, we note that SR-spheres $\mathcal{S}_{t}$ are continuous [37] and compact, as they are the preimage $\mathcal{S}_{t}=d(\cdot, e)^{\leftarrow}(\{t\})$ of compact set $\{t\}$ under continuous mapping $d(\cdot, e)$. Continuity of $d(\cdot, e)$ implies the spheres are equal to the 2D boundaries of the SR-balls (w.r.t. the normal product topology on $\mathbb{R}^{2} \times S^{1}$ ).

The algorithm also cannot pass conjugate points that are limits of 1st Maxwell points, but not Maxwell points themselves. See Figure 3. Such points exist on the surface $R_{2}=0$ and are by definition within $\overline{\mathcal{M}} \backslash \mathcal{M}$. Suppose the algorithm would pass such a point at a time $t>0$ (e.g., there exist 4 such points on the sphere with radius 4; see Figure 4); then due to the astroidal shape of the wavefront at such a point (cf. [36, Fig. 11]), there is a close neighboring tract that would pass a 1st Maxwell point which was already shown to be impossible (due to the upward kink property of viscosity solutions).

Remark 18. The SR-spheres are nonsmooth only at the 1 st Maxwell set $\mathcal{M}$. They are smooth at the conjugate points in $\overline{\mathcal{M}} \backslash \mathcal{M}$ (where the reflectional symmetry no longer produces two curves/fronts). In the other points on $\mathcal{S}_{t} \backslash \overline{\mathcal{M}}$ the SR-spheres are locally smooth (by the Cauchy-Kovalevskaya theorem and the semigroup property of the HJB equations).

Appendix E. The limiting procedure (4.4) for the SR-eikonal equation. In this section we study the limit procedure (4.4), illustrated in Figure 5. To this end we first provide a formal representation of the viscosity solutions of system (4.1), where we rely on viscosity solutions of morphological scale spaces obtained by superposition over the ( $\min ,+$ ) algebra, i.e., obtained by morphological convolution (erosions) with the morphological impulse response; cf. [12]. Now as the HJB equations of such morphological scale spaces do not involve a global offset

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Figure 13. Overview of Maxwell points. Two Maxwell points on the purple surface $x \cos \frac{\theta}{2}+y \sin \frac{\theta}{2}=0$ and two red Maxwell points on the surface $|\theta|=\pi$ (recall Figure 4). In all cases we see that local kinks in the viscosity solutions are upward, and the backtracking algorithm cannot pass these points.
by 1 on the right-hand side of the PDE, we need to combine such erosions with a time-shift in order to take into account the global offset. It turns out that the combination of these techniques provides staircases with steps of size $\epsilon$, so that we obtain the appropriate limit by taking the limit $\epsilon \rightarrow 0$ afterward, as done in (4.4).

Morphological convolutions over the $S E(2)$ group are obtained by replacing in linear leftinvariant convolutions (likewise the $S E(3)$ case studied in [17]) the usual (,$+ \cdot$ )-algebra by the (min, +)-algebra. Such erosions on $S E(2)$ are given by

$$
\begin{equation*}
(k \ominus f)(g):=\inf _{h \in S E(2)}\left\{k\left(h^{-1} g\right)+f(h)\right\} \tag{E.1}
\end{equation*}
$$

Furthermore, to include the updating of the initial condition in (4.1) we define

$$
\tilde{W}(g):= \begin{cases}W(g) & \text { if } g \neq e,  \tag{E.2}\\ 0 & \text { if } g=e\end{cases}
$$

Lemma E.1. Let $\epsilon>0, n \in \mathbb{N}$. The viscosity solution of (4.1) is given by

$$
\begin{equation*}
W_{n+1}^{\epsilon}(g, r)=\left(k_{r-n \epsilon} \ominus \tilde{W}_{n}^{\epsilon}\right)(g)+(r-n \epsilon) \tag{E.3}
\end{equation*}
$$

for $r \in\left[r_{n}, r_{n+1}\right]=[n \epsilon,(n+1) \epsilon]$, and the morphological kernel $k_{v}(g), v \geq 0$, is given by

$$
k_{v}(g)= \begin{cases}0 & \text { if } d(g, e) \leq v \\ \infty & \text { else }\end{cases}
$$

where $d(g, e)$ denotes the Carnot-Carathéodory distance (3.4) between $g \in S E(2)$ and $e=$ $(0,0,0)$. For $n=0$ we have that the viscosity solution of (4.2) is given by $W_{1}^{\epsilon}(g, r)=k_{r}(g)+r$.

Proof. In order to account for the constant offset in the HJB equations of (4.1) and (4.2), we set $r=r_{\text {new }}+r_{n}$ and define for $n=0,1,2, \ldots$ the functions $V_{n+1}^{\epsilon}: S E(2) \times[0, \epsilon] \rightarrow \mathbb{R}$ by

$$
V_{n+1}^{\epsilon}\left(g, r_{\text {new }}\right):=W_{n+1}^{\epsilon}\left(g, r_{\text {new }}+r_{n}\right)-r_{\text {new }},
$$

with $r_{\text {new }} \in[0, \epsilon]$ and $V_{n+1}^{\epsilon}$ the viscosity solution of

$$
\left\{\begin{array}{l}
\frac{\partial V_{n+1}^{\epsilon}}{\partial r_{\text {new }}}\left(g, r_{\text {new }}\right)=-1+1-\tilde{H}\left(g, \mathrm{~d}^{S R} V_{n+1}^{\epsilon}\left(g, r_{\text {new }}\right)\right)=-\tilde{H}\left(g, \mathrm{~d}^{S R} V_{n+1}^{\epsilon}\left(g, r_{\text {new }}\right)\right), \\
\text { for } g \neq e \text { we have } V_{n+1}^{\epsilon}(g, 0)= \begin{cases}\infty & \text { if } n=0 \\
W_{n}^{\epsilon}\left(g, r_{n}\right) & \text { if } n \in \mathbb{N},\end{cases} \\
\text { for } g=e \text { we have } V_{n+1}^{\epsilon}(g, 0)=V_{n+1}^{\epsilon}(e, 0)=0,
\end{array}\right.
$$

where we use short notation for the SR-derivative $\mathrm{d}^{S R} V:=\sum_{i=1}^{2} \mathcal{A}_{i} V \omega^{i}$ (recall (3.7) in Remark 4), and where Hamiltonian $\tilde{H}$ is given by (B.7).

Now let us first consider the case $n=0$. By the results in Appendix B the Hamiltonian system (4.2) provides geodesically equidistant wavefront propagation traveling with unit speed and departing directly from the unity element. As a result, we find

$$
V_{1}^{\epsilon}\left(g, r_{\text {new }}\right)=k_{r_{\text {new }}}(g)= \begin{cases}0 & \text { if } d(g, e) \leq r_{\text {new }}, \\ \infty & \text { else }\end{cases}
$$

and by left-invariant "superposition" over the ( $\min ,+$ )-algebra we find for $n=1,2, \ldots$ that $V_{n+1}^{\epsilon}\left(g, r_{\text {new }}\right)=\left(k_{r_{\text {new }}} \ominus \tilde{W}_{n}^{\epsilon}\left(\cdot, r_{n}\right)\right)(g)$, where we recall (E.2). Finally, we have

$$
W_{n+1}^{\epsilon}(g, r)=V_{n+1}^{\epsilon}(g, r-n \epsilon)+r-n \epsilon=\left(k_{r-n \epsilon} \ominus \tilde{W}_{n}^{\epsilon}\left(\cdot, r_{n}\right)\right)(g)+r-n \epsilon .
$$

Corollary E.2. Let $n \in \mathbb{N}$, and let $\epsilon>0$. The following identity holds:

$$
\begin{align*}
& W_{n+1}^{\epsilon}\left(g, r_{n+1}\right)=\left(k_{\epsilon} \ominus \tilde{W}_{n}^{\epsilon}\left(\cdot, r_{n}\right)\right)(g)+\epsilon \\
& = \begin{cases}\sum_{m=0}^{n}(m+1) \epsilon 1_{\left[r_{m}, r_{m+1}\right]}(d(g, e)) & \text { if } d(g, e) \leq r_{n+1}=(n+1) \epsilon, \\
\infty & \text { if } d(g, e)>r_{n+1},\end{cases} \tag{E.4}
\end{align*}
$$

where $1_{\left[r_{m}, r_{m+1}\right]}$ denotes the indicator function on set $\left[r_{m}, r_{m+1}\right]$.
Proof. The first part follows by Lemma E. 1 for $r=r_{n+1}$ (i.e., $r_{\text {new }}=\epsilon$ ). The second part follows by induction. Recall from Lemma E. 1 that $W_{1}^{\epsilon}(g, r)=k_{r}(g)+r$. Now application of (E.3) for $n=1$ yields

$$
\begin{align*}
& W_{2}^{\epsilon}\left(g, r_{2}\right)=\left(k_{\epsilon} \ominus \tilde{W}_{1}^{\epsilon}\left(\cdot, r_{1}\right)\right)(g)+\epsilon=\epsilon+\underset{h \in \inf _{g, \epsilon}}{ } \begin{cases}k_{\epsilon}(h)+\epsilon & \text { if } h \neq e, \\
0 & \text { if } h=e\end{cases} \\
& =\epsilon+\left\{\begin{array}{lll}
0 & \text { if } d(g, e) \leq \epsilon, \\
\epsilon & \text { if } \epsilon<d(g, e) \leq 2 \epsilon, \\
\infty & \text { else }
\end{array}= \begin{cases}\sum_{m=0}^{1}(m+1) \epsilon 1_{\left[r_{m}, r_{m+1}\right]}(d(g, e)) & \text { if } d(g, e) \leq r_{2}, \\
\infty & \text { else },\end{cases} \right. \tag{E.5}
\end{align*}
$$

with $\overline{B_{g, \epsilon}}=\{h \in S E(2) \mid d(g, h) \leq \epsilon\}$. This can intuitively be seen from the geometric meaning of an erosion $\tilde{W}_{1}^{\epsilon} \mapsto k_{\epsilon} \ominus \tilde{W}_{1}^{\epsilon}$ where one drops cylinders from below on the graph of $\tilde{W}_{1}^{\epsilon}\left(\cdot, r_{n}\right)$, and considering the new hull where cylinders get stuck. Equation (E.5) can also be seen directly from the definition of $k_{\epsilon}$. Let us verify each case separately:

- If $d(g, e)>2 \epsilon$, we have that the value must be infinite; otherwise, supposing it were finite, by the definition of the morphological kernel $k_{\epsilon}$ we would need to have that $d(g, e) \leq d(g, h)+d(h, e) \leq 2 \epsilon$, yielding a contradiction.
- If $d(g, e) \leq \epsilon$, then in the erosion-minimization we can take $h=e$ and obtain $\epsilon+0$.
- If $\epsilon<d(g, e) \leq 2 \epsilon$, then in the erosion-minimization we cannot take $h=e$, but for allowed choices we obtain $k_{\epsilon}(e)=0$ and $\epsilon+\epsilon$ as output.
Similarly we have, by inserting induction hypothesis for $n$ and recursion (E.3),

$$
\begin{aligned}
& W_{n+2}^{\epsilon}\left(g, r_{n+2}\right)=\left(k_{\epsilon} \ominus \tilde{W}_{n+1}^{\epsilon}\left(\cdot, r_{n+1}\right)\right)(g)+\epsilon=\epsilon+\sum_{m=0}^{n+1}(m+1) \epsilon 1_{\left[r_{m+1}, r_{m+2}\right]}(d(g, e)) \\
& =\epsilon+\sum_{m^{\prime}=1}^{n+2} m^{\prime} \epsilon 1_{\left[r_{m^{\prime}}, r_{m^{\prime}+1}\right]}(d(g, e))=\sum_{m^{\prime}=0}^{n+1}\left(m^{\prime}+1\right) \epsilon 1_{\left[r_{m^{\prime}}, r_{m^{\prime}+1}\right]}(d(g, e))
\end{aligned}
$$

for $d(g, e) \leq r_{n+2}$. Here we applied $m^{\prime}=m+1$ so that the result follows for $n+1$.
Theorem E.3. Let $g \in S E(2)$ be given. We have the following limit:

$$
\lim _{\epsilon \rightarrow 0} \lim _{n \rightarrow \infty} W_{n+1}^{\epsilon}(g,(n+1) \epsilon)=d(g, e) .
$$

Proof. Application of Corollary E. 2 gives

$$
\lim _{n \rightarrow \infty} W_{n+1}^{\epsilon}(g,(n+1) \epsilon)=\sum_{k=0}^{\infty}(k+1) \epsilon 1_{\left[r_{k}, r_{k+1}\right]}(d(g, e))=\sum_{k=0}^{N^{*}(g, \epsilon)}(k+1) \epsilon 1_{\left[r_{k}, r_{k+1}\right]}(d(g, e)),
$$

with $N^{*}(g, \epsilon)=\left\lceil\frac{d(g, e)}{\epsilon}\right\rceil$, i.e., the smallest integer $\geq \frac{d(g, e)}{\epsilon} \in \mathbb{R}^{+}$. As a result we have

$$
\begin{aligned}
\lim _{\epsilon \downarrow 0} \lim _{n \rightarrow \infty} W_{n+1}^{\epsilon}\left(g, r_{n+1}\right) & =\lim _{\epsilon \downarrow 0} W_{N^{*}(g, \epsilon)+1}^{\varepsilon}(g,(n+1) \epsilon) \\
& =\lim _{\epsilon \downarrow 0} \sum_{k=0}^{N^{*}(g, \epsilon)}(k+1) \epsilon 1_{\left[r_{k}, r_{k+1}\right]}(d(g, e))=d(g, e),
\end{aligned}
$$

where the size of the steps in the staircase toward $d(g, e)$ vanishes as $\epsilon \rightarrow 0$. Recall Figure 5 .

Appendix F. Embedding into geometric control theory. As mentioned in Remark 1 the problem $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$ given by (2.4) actually comes from a mechanical problem in geometric control, where a so-called Reeds-Shepp car [34] can proceed both forward and backward in the path optimization. As pointed out in [9] such a problem, for certain end conditions, cannot be considered as a curve optimization problem on the plane. The underlying difficulty is that for certain boundary conditions, the smooth minimizers of problem $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$ have the property that their spatial projections exhibit a cusp and cannot be parameterized by spatial arclength, since the control variable $u^{1}$ switches sign at the cusp. See Figure 14.

In 2D image analysis applications, solutions without cusps may be required. In this appendix, we propose problem $\mathbf{P}_{\text {contour }}^{\mathcal{C}}(S E(2))$ as a modification of problem $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$, which considers only the end conditions such that cusps do not occur.

Let us denote $\mathbf{x}=(x, y) \in \mathbb{R}^{2}$; then for $g=(\mathbf{x}, \theta) \in \mathfrak{R} \subset S E(2)$ and $\mathcal{C}=1$ the following problem on the spatial plane is well-posed:

$$
\mathbf{P}_{\text {curve }}^{\mathcal{C}}\left(\mathbb{R}^{2}\right):\left\{\begin{array}{l}
\gamma(0)=\mathbf{0}, \gamma(L)=\mathbf{x}  \tag{F.1}\\
\dot{\gamma}(0)=(1,0)^{T}, \dot{\gamma}(L)=(\cos \theta, \sin \theta)^{T}, \\
l(\gamma(\cdot))=\int_{0}^{L} \mathcal{C}(\gamma(s)) \sqrt{\beta^{2}+\kappa^{2}(s)} \mathrm{d} s \rightarrow \min , \\
\gamma:[0, L] \rightarrow \mathbb{R}^{2}, \beta>0,
\end{array}\right.
$$

where $L$ denotes spatial length and $\kappa$ curvature of the curve $\gamma \in C^{\infty}\left([0, L], \mathbb{R}^{2}\right)$, and where $\mathfrak{R} \subset S E(2)$ denotes the set of allowable end conditions. In [17] this set is explicitly determined and partially depicted in Figure 14C. In this article we studied $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$ (and not (F.1)),


Figure 14. Three $S R$-geodesics for uniform $\operatorname{cost} \mathcal{C}=1$ to reveal the differences of the three geometric control problems (2.4), (F.1), and (F.2) for $\beta=1$. A: Plots of spatial projections of $S R$-geodesics in $\mathbb{R}^{2}$. B: $S R$-geodesics in $S E(2)$. C: A part of $\mathfrak{R}$ (the set of end conditions for which $\boldsymbol{P}_{\text {curve }}\left(\mathbb{R}^{2}\right)$ is well-defined [16]) depicted as reachable cones around the origin. End condition $g_{1}=\left(x_{1}, y_{1}, \theta_{1}\right) \in \mathfrak{\Re}$ yields the minimizing curve of $\boldsymbol{P}_{\text {contour }}, \boldsymbol{P}_{\text {mec }}$, and $\boldsymbol{P}_{\text {curve }}$. End condition $g_{2}=\left(-x_{1}, y_{1},-\theta_{1}\right)$ yields the minimizing curve in $\boldsymbol{P}_{\text {contour }}, \boldsymbol{P}_{\text {mec }}$, and it is invalid for $\boldsymbol{P}_{\text {curve }}$. End condition $g_{3}$ is invalid for both $\boldsymbol{P}_{\text {curve }}$ and $\boldsymbol{P}_{\text {contour }}$, as it induces an internal cusp. For $\mathcal{C}=1$ the set of allowable end conditions for $\boldsymbol{P}_{\text {contour }}$ equals $\mathfrak{R}^{\mathcal{C}=1}=\mathfrak{R} \cup \mathfrak{Q}$; recall (F.3). For $\mathcal{C} \neq 1$ this set $\mathfrak{R}^{\mathcal{C}}$ differs and can be computed; cf. Remark 20.
and we look both forward and backward. Then, to avoid cusps, we must consider the problem

$$
\mathbf{P}_{\text {contour }}^{\mathcal{C}}(S E(2)):\left\{\begin{array}{l}
\dot{\gamma}(t)=\left.u^{1}(t) \mathcal{A}_{1}\right|_{\gamma(t)}+\left.u^{2}(t) \mathcal{A}_{2}\right|_{\gamma(t)} \text { for } t \in[0, T],  \tag{F.2}\\
\gamma(0)=e, \quad \gamma(T)=g=(\mathbf{x}, \theta) \in \mathfrak{R}^{\mathcal{C}}, \\
l(\gamma(\cdot))=\int_{0}^{T} \mathcal{C}(\gamma(t)) \sqrt{\beta^{2}\left|u^{1}(t)\right|^{2}+\left|u^{2}(t)\right|^{2}} \mathrm{~d} t \rightarrow \min , \\
\text { with curve } \gamma:[0, T] \rightarrow \mathrm{SE}(2), \text { with controls: } \\
\left(u^{1}(t), u^{2}(t)\right) \in \mathbb{R}^{2}, \text { and } u^{1}(t) \text { does not change sign, }
\end{array}\right.
$$

where $\mathfrak{R}^{\mathcal{C}}$ is the set of all $g \in S E(2)$ such that the minimizing SR-geodesic $(\mathrm{s}) \gamma(\cdot)=(\mathbf{x}(\cdot), \theta(\cdot))$ do not exhibit a cusp in their spatial projections $\mathbf{x}(\cdot)$. We distinguish between three cases for the end condition $g$ (see Figure 14):

- If $g$ is chosen such that the optimal control $u^{1} \geq 0$, then the lift of problem $\mathbf{P}_{\text {curve }}^{\mathcal{C}}\left(\mathbb{R}^{2}\right)$ coincides with $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$ and also with $\mathbf{P}_{\text {contour }}^{\mathcal{C}}(S E(2))$.
- If $g$ is chosen such that the optimal control $u^{1} \leq 0$, then problem $\mathbf{P}_{\text {mec }}^{\mathcal{C}}(S E(2))$ and problem $\mathbf{P}_{\text {contour }}^{C}(S E(2))$ coincide.
- If $g$ is chosen such that the optimal control $u^{1}(t)$ switches sign at some internal time $t \in(0, T)$, then $g \in S E(2) \backslash \mathfrak{R}^{\mathcal{C}}$ and the spatial projection of the corresponding minimizing SR-geodesic(s) has an internal cusp, which we consider not desirable in our applications of interest.
Remark 19. Geodesics in $\mathbf{P}_{\text {contour }}^{\mathcal{C}}(S E(2))$ can depart forward or backward from the origin. Then, for $\mathcal{C}=1$, the set $\mathfrak{R}^{\mathcal{C}=1}$ of allowable end conditions can be obtained from the set
$\mathfrak{R} \subset\{(x, y, \theta) \in S E(2) \mid x \geq 0\}$ by a reflectional symmetry:

$$
\begin{equation*}
\mathfrak{R}^{\mathcal{C}=1}=\mathfrak{R} \cup \mathfrak{Q} \text { with } \mathfrak{Q}=\{(x, y, \theta) \in S E(2) \mid(-x, y,-\theta) \in \mathfrak{R}\} \tag{F.3}
\end{equation*}
$$

Remark 20. In section 7.1 we provided a very simple numerical tool to compute the surface in $S E(2)$ where cusps appear also for $\mathcal{C} \neq 1$; recall (7.2). This surface is a boundary of a volume in $S E(2)$ that contains the set $\mathfrak{R}^{\mathcal{C}}$.

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[^1]:    ${ }^{1}$ Note that the SR-gradient $\nabla^{S R} W=\left(G^{\mathcal{C}}\right)^{-1} \mathrm{~d} W=\mathcal{C}^{-2} \sum_{i=1}^{2} \beta_{i}^{-2} \mathcal{A}_{i} W \mathcal{A}_{i}$, with $\beta_{1}=\beta$, $\beta_{2}=1$, by definition is the Riesz representative (being a vector) of this SR-derivative (being a covector).

[^2]:    ${ }^{2}$ In fact, an "erosion" according to the terminology in morphological scale space theory; see, e.g., [12].

[^3]:    ${ }^{3}$ By weak solution we mean a not necessarily differentiable Lipschitz function, satisfying the equation almost everywhere (for further details see [19]).

[^4]:    ${ }^{4}$ In [37, eq. 3.13] it is shown that $\widetilde{\max }^{2}=\{(x, y, \theta) \in S E(2) \mid y \sin \theta / 2+x \cos \theta / 2=0$ and $\mid-x \sin (\theta / 2)+$ $y \cos (\theta / 2)\left|>\left|R_{1}^{1}(\theta)\right|\right\}$ with $R_{1}^{1}$ defined in [37, Lem. 2.5]. We also observed such a loss of the Maxwell point property in our numerical algorithm, as kinks in $W(g)=t$ can disappear when moving on the set $y \sin \theta / 2+$ $x \cos \theta / 2=0$. See Figure 10.

