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MATHEMATICS

Optimality of Euler's Elasticae[1]

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Euler's problem of stationary configurations of an elastic rod with fixed endpoints and tangents at the endpoints is considered. The existence and boundedness of optimal controls are proved. Bounds on the cut points and conjugate points are obtained.

1. PROBLEM STATEMENT

In 1744 Leonhard Euler considered the following problem on stationary configurations of an elastic rod [1]. Given an elastic rod in the plane with fixed endpoints and tangents at the endpoints, one should determine possible profiles of the rod under the boundary conditions specified. Euler obtained differential equations for stationary configurations of the rod and described their possible qualitative types. These configurations are called Euler elasticae.

Euler elasticae are critical points of the elastic energy functional on the space of curves with fixed endpoints and tangents at the endpoints. The question as to which of the critical points are minima (local or global) remained open. This work is devoted to the study of this question.

The elastic problem is formalized as the following optimal control problem (see [2]):

$$\dot{q} = X_1(q) + uX_2(q),$$
$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u \in \mathbb{R}; \tag{1}$$

$$X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial\theta}; \tag{2}$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1),$$
$$t_1 \text{ is fixed}; \tag{3}$$

$$J = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min; \tag{4}$$

$$u(\cdot) \in L_2[0, t_1], \quad q(\cdot) \in AC[0, t_1]. \tag{5}$$

Euler's problem is a left-invariant problem on the group of motions of a plane, so we can assume that the initial point is $q_0 = (0, 0, 0)$.

2. ATTAINABLE SET AND EXISTENCE OF OPTIMAL CONTROLS

Consider a control system $\dot{q} = f(q, u)$. Let $u = u(t)$ be an admissible control, and let q_0 be a point of the state space. Denote by $q(t; u, q_0)$ the trajectory of the system corresponding to the control $u(t)$ and satisfying the initial condition $q(0; u, q_0) = q_0$. The attainable set of the control system from the point q_0 in time t_1 is defined as the set $\mathcal{A}_{q_0}(t_1) = \{q(t_1; u, q_0) | u = u(t)$ is an admissible control, $t \in [0, t_1]\}$.

Theorem 1. *For problem* (1)–(5), *the attainable set from the point* $q_0 = (0, 0, 0)$ *in time* $t_1 > 0$ *has the form* $\mathcal{A}_{q_0}(t_1) = \{(x, y, \theta) \in M | x^2 + y^2 < t_1^2$ or $(x, y, \theta) = (t_1, 0, 0)\}$.

Theorem 2. *Let* $q_1 \in \mathcal{A}_{q_0}(t_1)$. *Then there exists an optimal control in problem* (1)–(5). *Moreover, this optimal control is essentially bounded.*

3. EXTREMALS

We introduce, linear on fibers of the cotangent bundle T^*M, Hamiltonians $h_i(\lambda) = \langle\lambda, X_i\rangle$, $\lambda \in T^*M$, corresponding to the fields X_1, X_2 (see (2)), and the family of Hamiltonians

$$h_u^\nu(\lambda) = \langle\lambda, X_1 + uX_2\rangle + \frac{\nu}{2}u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2}u^2,$$

$$\lambda \in T^*M, \quad u \in \mathbb{R}, \quad \nu \in \mathbb{R}.$$

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Denote by $\vec{h}_i$ the Hamiltonian vector field on T^*M corresponding to the Hamiltonian h_i.

We apply the Pontryagin maximum principle in invariant form [3] to Euler's problem. Let $u(t)$ and $q(t)$, $t \in [0, t_1]$, be an optimal control and the corresponding optimal trajectory in problem (1)–(5). Then there exists a Lipschitzian curve $\lambda_t \in T^*M$, $\pi(\lambda_t) = q(t)$, and a number $\nu \le 0$ for which the following conditions hold for almost all $t \in [0, t_1]$:

$$\dot{\lambda}_t = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t),$$

$$h^{\nu}_{u(t)}(\lambda_t) = \max_{u \in \mathbb{R}} h^{\nu}_u(\lambda_t), \quad (\nu, \lambda_t) \neq 0.$$

In the abnormal case ($\nu = 0$), the extremal trajectories are $\theta \equiv 0$, $x = t$, $y \equiv 0$; i.e., the elasticae are straight lines. Straight line segments provide a solution to Euler's problem under appropriate boundary conditions (zero forces are applied at the endpoints of the rod).

In the normal case ($\nu = -1$), the extremals are trajectories of the Hamiltonian system with the maximized Hamiltonian $H = h_1 + \frac{1}{2}h_2^2$:

$$\dot{h}_1 = -h_2h_3, \quad \dot{h}_2 = h_3, \quad \dot{h}_3 = h_1h_2, \tag{6}$$

$$\dot{q} = X_1 + h_2X_2. \tag{7}$$

In the coordinates (β, c, r) in the fiber T^*_qM of the cotangent bundle defined by the formulas $h_1 = -r\cos\beta$, $h_3 = -r\sin\beta$, $h_2 = c$, the vertical subsystem (6) takes the form of the pendulum equation

$$\dot{\beta} = c, \quad \dot{c} = -r\sin\beta, \quad \dot{r} = 0. \tag{8}$$

Euler elasticae are parametrized by Jacobi's functions [4]. Depending on the value of the total energy of the pendulum $E = \frac{c^2}{2} - r\cos\beta \in [-r, +\infty)$, elasticae have different qualitative types, which were discovered by Euler. For $E \in (-r, r)$, $r \neq 0$, elasticae have inflection points and are called inflectional. For $E \in (r, +\infty)$, $r \neq 0$, elasticae do not have inflection points and are called noninflectional. In the critical case $E = r \neq 0$, an elastica either has one loop or is a segment. In the case $E = -r \neq 0$, an elastica is a segment. Finally, for $r = 0$, an elastica is a segment or an arc of a circle. Some typical shapes of elasticae are presented in [5].

4. MAXWELL POINTS

Consider an optimal control problem of the form

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U, \tag{9}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ is fixed}, \tag{10}$$

$$J_{t_1}[u] = \int_0^{t_1} \varphi(q(t), u(t))dt \to \min, \tag{11}$$

where M and U are finite-dimensional analytic manifolds, $f(q, u)$ and $\varphi(q, u)$ are an analytic vector field and an analytic function, respectively, depending on a control parameter u. The normal Hamiltonian of the Pontryagin maximum principle for this problem is defined as $h_u(\lambda) = \langle \lambda, f(q, u)\rangle - \varphi(q, u)$, $\lambda \in T^*M$, $q = \pi(\lambda) \in M$, $u \in U$. Suppose that all normal extremals in problem (9)–(11) satisfy the strong Legendre condition [3]. Let the maximized Hamiltonian $H(\lambda) = \max_{u \in U} h_u(\lambda)$ be an analytic function. We assume that the corresponding Hamiltonian vector field $\vec{H}$ is complete. Denote the normal extremal trajectories corresponding to covectors $\lambda, \tilde{\lambda} \in T^*_{q_0}M$ by $q(s)$ and $\tilde{q}(s)$, and the corresponding extremal controls by $u(s)$ and $\tilde{u}(s)$.

The Maxwell set for time t in the preimage of the exponential mapping $N = T^*_{q_0}M$ is defined as follows:

$$\mathrm{MAX}_t = \{\lambda \in N \mid \exists \tilde{\lambda} \in N: \tilde{q}(s) \not\equiv q(s),$$
$$s \in [0, t], \quad \tilde{q}(t) = q(t), \quad J_t[u] = J_t[\tilde{u}]\}.$$

The point $q(t)$ is called a Maxwell point of the trajectory $q(s)$, $s \in [0, t_1]$, and the instant t is called a Maxwell time.

Proposition 1. *If a normal extremal trajectory $q(s)$, $s \in [0, t_1]$, contains a Maxwell point $q(t)$, $t \in (0, t_1)$, then the trajectory $q(s)$ is not optimal for problem* (9)–(11).

In problems with a large group of symmetries, the Maxwell set can often be found via the study of fixed points of the symmetry group (see, e.g., [6–10].)

The cut time t_{cut} for a trajectory $q(s)$ is defined as follows:

$t_{\mathrm{cut}} = \sup\{t_1 > 0 \mid q(s)$ is optimal at the segment $[0, t_1]\}$.

For normal extremal trajectories $q(s)$, the cut time is a function of the initial covector λ:

$$t_{\mathrm{cut}}: N = T^*_{q_0}M \to [0, +\infty], \quad t = t_{\mathrm{cut}}(\lambda).$$

The short arcs of regular extremal trajectories ore optimal; thus, $t_{\mathrm{cut}}(\lambda) > 0$ for any $\lambda \in N$. On the other hand, some extremal trajectories may be optimal on an arbitrarily long segment $[0, t_1]$, $t_1 \in (0, +\infty)$; in this case, $t_{\mathrm{cut}} = +\infty$.

According to Proposition 1, a normal extremal trajectory $q(s)$ cannot be optimal after a Maxwell point. Using this relation, one can obtain the following upper bound on the cut time in terms of the period $T(\lambda)$ of oscillations of pendulum (8).

Theorem 3. *In Euler's elastic problem,*

$$t_{\mathrm{cut}}(\lambda) \le T(\lambda), \quad \lambda \in N.$$

5. CONJUGATE POINTS

Extremal trajectories lose their local optimality at the first conjugate point, see [3]. Denote by $t^1_{\text{conj}}(\lambda)$ the first conjugate time on the normal extremal trajectory $q(s)$ corresponding to the initial covector $\lambda \in N$.

The conjugate points in Euler's elastic problem are bounded as follows.

Theorem 4. (1) *On any inflectional elastica, there are an infinite number of isolated conjugate points. The first conjugate time* $t^1_{\text{conj}}(\lambda)$ *admits the following bound in terms of the period of oscillations of the pendulum* $T(\lambda)$:

$$t^1_{\text{conj}} \in \left[\frac{T}{2}, \frac{3T}{2}\right].$$

The first conjugate point is contained between the first and third inflection points on the elastica.

(2) *All the remaining elasticae do not admit conjugate points.*

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