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CONJUGATE POINTS IN THE EULER ELASTIC PROBLEM

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Abstract. For the classical Euler elastic problem, conjugate points are described. Inflexional elasticas admit the first conjugate point between the first and third inflexion points. All other elasticas do not have conjugate points. As a result, the problem of stability of Euler elasticas is solved.

1. Introduction

This work is devoted to the study of the following problem considered by L. Euler [7]. Given an elastic rod in the plane with fixed endpoints and tangents at the endpoints, one should determine possible profiles of the rod under the given boundary conditions. The Euler problem can be stated as the following optimal control problem:

$$\dot{x} = \cos\theta, \tag{1.1}$$

$$\dot{y} = \sin\theta, \tag{1.2}$$

$$\dot{\theta} = u, \tag{1.3}$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u \in \mathbb{R}, \tag{1.4}$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1), \quad t_1 \text{ is fixed}, \tag{1.5}$$

$$J = \frac{1}{2}\int_0^{t_1} u^2(t)\, dt \to \min, \tag{1.6}$$

where the integral J evaluates the elastic energy of the rod.

This paper is an immediate continuation of our previous work [11], which contained the following material: history of the problem, description of

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attainable set, proof of existence and boundedness of optimal controls, parametrization of extremals by the Jacobi functions, description of discrete symmetries and the corresponding Maxwell points. In this work we widely use the notation, definitions, and results of work [11].

Euler described extremal trajectories of problem (1.1)–(1.6), their projections to the plane (x, y) being called Euler elasticas. However, the question of optimality of elasticas remained open. Our aim is to characterize global and local optimality of Euler elasticas. Short segments of elasticas are optimal. The main result of the previous work [11] in this direction was an upper bound on cut points, i.e., points where elasticas lose their global optimality. In this work, we describe conjugate points along elasticas; we obtain precise bounds for the first conjugate point, where the elasticas lose their local optimality.

Each inflexional elastica contains an infinite number of conjugate points. The first conjugate point occurs between Maxwell points; visually, the first conjugate point is located between the first and third inflexion points of the elastica.

All other elasticas do not contain conjugate points.

Note that Max Born proved in his thesis [5] that if an elastic arc is free of inflexion points, then it does not contain conjugate points; therefore, in this part we repeated Max Born's result. However, our method of proving is more flexible, and we believe that it will be useful for the study of conjugate points in other optimal control problems.

This work has the following structure. In Sec. 2, we recall some basic facts of the theory of conjugate points along regular extremals of optimal control problems. These facts are rather well known, but are scattered through the literature. The main facts of this theory necessary for us are as follows: (1) an instant $t > 0$ is a conjugate point iff the exponential mapping for the time t is degenerate; (2) the Morse index of the second variation of the endpoint mapping along an extremal is equal to the number of conjugate points taking into account their multiplicity; (3) the Morse index is equal to the Maslov index of the curve in a Lagrange Grassmanian obtained by the linearization of the flow of the Hamiltonian system of the Pontryagin maximum principle; (4) the Maslov index is invariant with respect to homotopies of extremals provided that their endpoints are not conjugate. We apply this theory for description of conjugate points in the Euler problem. In Sec. 3, we obtain estimates for the first conjugate point on inflexional elasticas. Moreover, we improve our result of work [11] on the upper bound of the cut time on inflexional elasticas. In Sec. 4 we show that all other elasticas do not contain conjugate points. In Sec. 5, we summarize results obtained in this paper and [11], and discuss their possible consequences for future work.

In this work, we use extensively the Jacobi functions (see [8, 14]). We apply the system "Mathematica" [15] to carry out complicated calculations and to produce illustrations.

2. Conjugate points, Morse index, and Maslov index

In this section, we recall some basic facts from the theory of conjugate points in optimal control problems (see [1–4, 13]).

2.1. **The optimal control problem and Hamiltonians.** We consider an optimal control problem of the form

$$\dot{q} = f(q,u), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{2.1}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ is fixed}, \tag{2.2}$$

$$J^{t_1}[u] = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, \tag{2.3}$$

where M is a finite-dimensional analytic manifold, $f(q,u)$ and $\varphi(q,u)$ are analytic in (q,u) families of vector fields and functions on M depending on the control parameter $u \in U$, and U is an open subset of $\mathbb{R}^m$. Admissible controls are $u(\cdot) \in L_\infty[0,t_1]$, and admissible trajectories $q(\cdot)$ are Lipschitzian. Let

$$h_u(\lambda) = \langle \lambda, f(q,u) \rangle - \varphi(q,u), \quad \lambda \in T^*M, \quad q = \pi(\lambda) \in M, \quad u \in U,$$

be the *normal Hamiltonian of the PMP* for problem (2.1)–(2.3). Fix a triple $(\widetilde{u}(t), \lambda_t, q(t))$ consisting of a normal extremal control $\widetilde{u}(t)$, the corresponding extremal λ_t, and the extremal trajectory $q(t)$ for problem (2.1)–(2.3).

In the sequel, we suppose that the following hypothesis holds:

(**H1**) for all $\lambda \in T^*M$ and $u \in U$, the quadratic form $\dfrac{\partial^2 h_u}{\partial u^2}(\lambda)$ is negative definite.

Note that condition (**H1**) implies the strong Legendre condition along an extremal pair $(\widetilde{u}(t), \lambda_t)$:

$$\left. \frac{\partial^2 h_u}{\partial u^2} \right|_{u=\widetilde{u}(t)} (\lambda_t)(v,v) < -\alpha |v|^2, \quad t \in [0,t_1], \quad v \in \mathbb{R}^m, \quad \alpha > 0,$$

i.e., the extremal λ_t is regular [2].

Moreover, we also assume that the following condition is satisfied:

(**H2**) for any $\lambda \in T^*M$, the function $u \mapsto h_u(\lambda)$, $u \in U$, has a maximum point $\bar{u}(\lambda) \in U$:

$$h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \quad \lambda \in T^*M.$$

In terms of work [1], condition (**H2**) means that T^*M is a regular domain of the Hamiltonian $h_u(\lambda)$. Condition (**H1**) means that the function $u \mapsto h_u(\lambda)$ has no maximum points in addition to $\bar{u}(\lambda)$. At the maximum point $\left.\dfrac{\partial h}{\partial u}\right|_{u=\bar{u}(\lambda)}(\lambda) = 0$ for all $\lambda \in T^*M$. By the implicit-function theorem, the mapping $\lambda \mapsto \bar{u}(\lambda)$ is analytic. The *maximized Hamiltonian* $H(\lambda) = h_{\bar{u}(\lambda)}(\lambda)$, $\lambda \in T^*M$, is also analytic. The extremal λ_t is a trajectory of the corresponding Hamiltonian vector field: $\dot{\lambda}_t = \vec{H}(\lambda_t)$, and the extremal control is $\widetilde{u}(t) = \bar{u}(\lambda_t)$.

2.2. **The second variation and its Morse index.** Consider the *endpoint mapping* for problem (2.1)–(2.3):

$$F_t : \mathcal{U} = L_\infty([0,t], U) \to M, \quad u(\cdot) \mapsto (q_u(t), J^t[u]), \tag{2.4}$$

where $q_u(\cdot)$ is the trajectory of the control system (2.1) with the initial condition $q_u(0) = q_0$ corresponding to the control $u = u(\cdot)$. Since $\widetilde{u} \in \mathcal{U}$ is an extremal control, it follows that the differential (*first variation*) $D_{\widetilde{u}} : T_{\widetilde{u}}\mathcal{U} \to T_{q_{\widetilde{u}(t)}}M$ is degenerate, i.e., not surjective, for all $t \in (0, t_1]$ (see [2]).

Introduce one more important hypothesis:

(**H3**) the extremal control $\widetilde{u}(\cdot)$ is a corank one critical point of the endpoint mapping F_t, i.e.,

$$\operatorname{codim} \operatorname{Im} D_{\widetilde{u}} F_t = 1, \qquad t \in (0, t_1].$$

Condition (**H3**) means that there exists a unique, up to a nonzero factor, extremal λ_t corresponding to the extremal control $\widetilde{u}(t)$.

For any extremal control $u \in \mathcal{U}$, there exists a well-defined Hessian (*second variation*; see [2]) of the endpoint mapping — a quadratic mapping

$$\operatorname{Hess}_u F_t : \operatorname{Ker} D_u F_t \to \operatorname{Coker} D_u F_t = T_{q_u(t)}M / \operatorname{Im} D_u F_t.$$

Condition (**H3**) means that $\dim \bigl(T_{q_{\widetilde{u}}(t)}M / \operatorname{Im} D_{\widetilde{u}} F_t\bigr) = 1$ for all $t \in (0, t_1]$ and, therefore, the quadratic form

$$Q_t = \lambda_t \operatorname{Hess}_{\widetilde{u}} F_t : \operatorname{Ker} D_{\widetilde{u}} F_t \to \mathbb{R}, \quad t \in (0, t_1], \tag{2.5}$$

the projection of the second variation to the extremal λ_t, is defined uniquely up to a positive factor.

The *Morse index* of a quadratic form Q defined in a Banach space $\mathcal{L}$ is the maximal dimension of the negative space of the form Q:

$$\operatorname{ind} Q = \max\{\dim L \mid L \subset \mathcal{L},\ Q|_{L \setminus \{0\}} < 0\}.$$

The kernel of the quadratic form $Q(x)$ is the space

$$\operatorname{Ker} Q = \{x \in \mathcal{L} \mid Q(x, y) = 0 \ \forall y \in \mathcal{L}\},$$

where $Q(x, y)$ is the symmetric bilinear form corresponding to the quadratic form $Q(x)$. A quadratic form is said to be degenerate if it has a nonzero kernel. The multiplicity of degeneration of the form Q is equal to the dimension of its kernel: $\operatorname{dgn} Q = \dim \operatorname{Ker} Q$.

Now we return to the quadratic form Q_t given by (2.5) — the second variation of the endpoint mapping for the extremal pair $\widetilde{u}(t), \lambda_t$ of the optimal control problem (2.1)–(2.3). We continue the quadratic form Q_t from the space L_∞ to the space L_2 by continuity, and denote by K_t the closure of the space $\operatorname{Ker} D_{\widetilde{u}} F_t$ in $L_2[0,t]$.

Proposition 2.1 (see [2, Proposition 20.2], [13, Theorem 1])**.** *Under hypotheses* (**H1**) *and* (**H3**)*, the quadratic form* $Q_t|_{K_t}$ *is positive for small* $t > 0$*. In particular,* $\operatorname{ind} Q_t|_{K_t} = 0$ *for small* $t > 0$.

An instant $t_* \in (0, t_1]$ is called a *conjugate time* (for the initial instant $t = 0$) along the extremal λ_t if the quadratic form $Q_{t_*}|_{K_{t_*}}$ is degenerate. In this case the point $q_u(t_*) = \pi(\lambda_{t_*})$ is said to be *conjugate* for the initial point q_0 along the extremal trajectory $q_u(\cdot)$.

Proposition 2.2 (see [13, Theorem 1])**.** *Under hypotheses* (**H1**) *and* (**H3**):

(1) *conjugate points along the extremal* λ_t *are isolated*: $0 < t_*^1 < \cdots < t_*^N \le t_1$;
(2) *the Morse index of the second variation is expressed by the formula*

$$\operatorname{ind} Q_t|_{K_t} = \sum \{\operatorname{dgn} Q_{t_*^i} \mid 0 < t_*^i < t\}.$$

The *local optimality* of extremal trajectories is characterized in terms of conjugate points. Speaking about local optimality of extremal trajectories in the calculus of variations and optimal control, one distinguishes the strong optimality (in the norm of the space $C([0,t_1], M)$) and the weak optimality (in the norm of the space $C^1([0,t_1], M)$). Under hypotheses (**H1**)–(**H3**), normal extremal trajectories lose their local optimality (both strong and weak) at the first conjugate point [2]. Thus, in the sequel, when speaking about local optimality, we mean both strong and weak optimality.

Proposition 2.3 (see [2, Proposition 21.2, Theorem 21.3])**.** *Let conditions* (**H1**)–(**H3**) *be satisfied.*

(1) *If the interval* $(0, t_1]$ *does not contain conjugate points, then the extremal trajectory* $q(t)$*,* $t \in [0, t_1]$*, is locally optimal.*
(2) *If the interval* $(0, t_1)$ *contains a conjugate point, then the extremal trajectory* $q(t)$*,* $t \in [0, t_1]$*, is not locally optimal.*

2.3. **The exponential mapping.** We will add to hypotheses (**H1**)–(**H3**) one more condition:

(**H4**) All trajectories of the Hamiltonian vector field $\vec{H}(\lambda)$, $\lambda \in T^*M$, are continued to the segment $t \in [0, t_1]$.

Consider the *exponential mapping for the time* t:

$$\mathrm{Exp}_t : N = T^*_{q_0}M \to M, \quad \mathrm{Exp}_t(\lambda) = \pi \circ e^{t\vec{H}}(\lambda) = q(t), \quad t \in [0, t_1].$$

One can construct a theory of conjugate points in terms of the family of the subspaces

$$\Lambda(t) = e_*^{-t\vec{H}} T_{\lambda_t}(T^*_{q(t)}M) \subset T_{\lambda_0}(N),$$

via the linearization of the flow of the Hamiltonian vector field $\vec{H}$ along the extremal λ_t.

2.4. **The Maslov index of a curve in the Lagrange Grassmanian.** First, we recall some basic facts of the symplectic geometry (see details in [1, 4]). Let (Σ, σ) be a *symplectic space*, i.e., Σ is a $2n$-dimensional linear space and σ is a nondegenerate skew-symmetric bilinear form on Σ. The *skew-orthogonal complement* to a subspace $\Gamma \subset \Sigma$ is the subspace $\Gamma^{\angle} = \{x \in \Sigma \mid \sigma(x, \Gamma) = 0\}$. Since σ is nondegenerate, it follows that $\dim \Gamma^{\angle} = 2n - \dim \Gamma$. A subspace $\Gamma \subset \Sigma$ is said to be *Lagrangian* if $\Gamma = \Gamma^{\angle}$; in this case $\dim \Gamma = n$. The set of all Lagrangian subspaces in Σ is called the *Lagrange Grassmanian* and is denoted by $L(\Sigma)$; it is a smooth manifold of dimension $n(n+1)/2$ in the Grassmanian $G_n(\Sigma)$ of all n-dimensional subspaces in Σ.

Fix an element $\Pi \in L(\Sigma)$. Define an open set

$$\Pi^{\pitchfork} = \{\Lambda \in L(\Sigma) \mid \Lambda \cap \Pi = 0\}.$$

The subset

$$\mathcal{M}_\Pi = L(\Sigma) \setminus \Pi^{\pitchfork} = \{\Lambda \in L(\Sigma) \mid \Lambda \cap \Pi \neq 0\}$$

is called the train for Π. The set $\mathcal{M}_\Pi$ is not a smooth submanifold in $L(\Sigma)$, but it is represented by the union of smooth strata:

$$\mathcal{M}_\Pi = \bigcup_{k \geq 1} \mathcal{M}_\Pi^{(k)},$$

where

$$\mathcal{M}_\Pi^{(k)} = \{\Lambda \in L(\Sigma) \mid \dim(\Lambda \cap \Pi) = k\}$$

is a smooth submanifold of $L(\Sigma)$ of codimension $k(k+1)/2$.

Consider a smooth curve $\Lambda(t) \in L(\Sigma)$, $t \in [t_0, t_1]$, i.e., a family of Lagrangian subspaces in Σ smoothly depending on t. Assume that $\Lambda(t_0), \Lambda(t_1) \in \Pi^{\pitchfork}$. The *Maslov index* $\mu_\Pi(\Lambda(\cdot))$ of the curve $\Lambda(\cdot)$ is the intersection index of this curve with the set $\mathcal{M}_\Pi$.

In more detail, let the curve $\Lambda(\cdot)$ do not intersect $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$; this can always be achieved by a small perturbation of this curve. For the smooth hypersurface $\mathcal{M}_\Pi^{(1)} \subset L(\Sigma)$, one can define its coorientation in an invariant way as follows. Any tangent vector to $L(\Sigma)$ at a point $\Lambda \in L(\Sigma)$ can naturally be identified with a certain quadratic form on Λ. Take a tangent vector $\dot{\Lambda}(t) \in T_{\Lambda(t)}L(\Sigma)$ to a smooth curve $\Lambda(t) \in L(\Sigma)$. Choose a point

$x \in \Lambda(t)$ of the n-dimensional space $\Lambda(t) \subset \Sigma$. Choose any smooth curve $\tau \mapsto x(\tau)$ in Σ such that $x(\tau) \in \Lambda(\tau)$ for all τ, and $x(\tau) = x$. Then the quadratic form $\underline{\dot{\Lambda}}(t)(x)$, $x \in \Lambda(t)$, is defined by the formula $\underline{\dot{\Lambda}}(t)(x) = \sigma(x, \dot{x}(t))$. One can show that $\sigma(x, \dot{x}(t))$ does not depend on the choice of the curve $x(\tau)$, i.e., one obtains a well-defined quadratic form $\underline{\dot{\Lambda}}(t)$ on the space $\Lambda(t)$. Moreover, the correspondence $\dot{\Lambda} \mapsto \underline{\dot{\Lambda}}$, $\dot{\Lambda} \in T_\Lambda L(\Sigma)$, defines an isomorphism of the tangent space $T_\Lambda L(\Sigma)$ and the linear space of quadratic forms on Λ, see [1].

The Maslov index $\mu_\Pi(\Lambda(\cdot))$ is defined as the number of transitions of the curve $\Lambda(\cdot)$ from the negative side of the manifold $\mathcal{M}_\Pi^{(1)}$ (i.e., with $\underline{\dot{\Lambda}}(t) > 0$) minus the number of reverse transitions (with $\underline{\dot{\Lambda}}(t) < 0$), taking into account multiplicity.

The fundamental property of the Maslov index is its homotopy invariance [3]: for any homotopy $\Lambda^s(t)$, $t \in [t_0^s, t_1^s]$, $s \in [0,1]$, such that $\Lambda^s(t_0^s), \Lambda^s(t_1^s) \in \Pi^{\pitchfork}$ for all $s \in [0,1]$, we have $\mu_\Pi(\Lambda^0(\cdot)) = \mu_\Pi(\Lambda^1(\cdot))$. This fact is proved in the same way as the homotopy invariance of the usual intersection index of a curve with a smooth cooriented surface.

For monotone curves in Lagrange Grassmanian $L(\Sigma)$, the following way of evaluation of the Maslov index can be used.

Proposition 2.4 (see [1, Corollary I.1]). *Let* $\underline{\dot{\Lambda}}(t) \leq 0$, $t \in [t_0, t_1]$, *and let* $\{t \in [t_0, t_1] \mid \Lambda(t) \cap \Pi \neq 0\}$ *be a finite subset of the open interval* (t_0, t_1). *Then*

$$\mu_\Pi(\Lambda(\cdot)) = -\sum_{t \in (t_0, t_1)} \dim(\Lambda(t) \cap \Pi). \tag{2.6}$$

In fact, in [1, Corollary I.1], a statement for a nondecreasing curve ($\underline{\dot{\Lambda}}(t) \geq 0$) is given and, therefore, in the right-hand side of formula (2.6) the minus sign is absent. As was pointed out in the remark after Corollary I.1 in [1], the passage from nondecreasing curves to nonincreasing ones is obtained by the inversion of the direction of time $t \mapsto t_0 + t_1 - t$.

The theory of the Maslov index can be used for the computation of the Morse index for regular extremals in optimal control problems.

2.5. **The Morse index and the Maslov index.** Let λ_t, $t \in [0, t_1]$, be a normal extremal of the optimal control problem (2.1)–(2.3), and let hypotheses (**H1**)–(**H4**) be satisfied. Consider the family of quadratic forms Q_t given by (2.5).

The extremal λ_t determines a smooth curve

$$\Lambda(t) = e_*^{-t\vec{H}} T_{\lambda_t}(T^*_{q(t)}M) \in L(\Sigma), \quad t \in [0, t_1],$$

in the Lagrange Grassmanian $L(\Sigma)$, where $\Sigma = T_{\lambda_0}(T^*M)$. The initial point of this curve is the tangent space to the fiber $\Lambda(0) = \Pi = T_{\lambda_0}(T^*_{q_0}M)$. The strong Legendre condition (see (**H1**)) implies the monotone decreasing of the curve $\Lambda(t)$: the quadratic forms $\underline{\dot{\Lambda}}(t) < 0$, $t \in [0, t_1]$ (see [1, Lemma I.4]) and, therefore, its Maslov index can be computed via Proposition 2.4.

On the other hand, the following important statement establishes a relation between the Morse index of the second variation Q_t and Maslov index of the curve $\Lambda(t)$.

Proposition 2.5 (see [1, Theorem I.3, Corollary I.2]). *Let hypotheses* (**H1**)–(**H4**) *be satisfied. Then*:

(1) *An instant $t \in (0, t_1]$ is a conjugate time iff $\Lambda(t) \cap \Pi \neq 0$.*
(2) *If $\Lambda(t_1) \cap \Pi = 0$, then there exists $\bar{t} > 0$ such that*

$$\operatorname{ind} Q_{t_1}|_{K_{t_1}} = -\mu_\Pi(\Lambda(\cdot)|_{[t_0,t_1]}) \quad \forall t_0 \in (0, \bar{t}).$$

(3) *If $\{t \in (0, t_1] \mid \Lambda(t) \cap \Pi \neq 0\}$ is a finite subset of the open interval $(0, t_1)$, then*

$$\operatorname{ind} Q_{t_1}|_{K_{t_1}} = \sum_{t \in (0,t_1)} \dim(\Lambda(t) \cap \Lambda(0)).$$

Item (1) of Proposition 2.5 obviously implies the following statement.

Corollary 2.1. *Let hypotheses* (**H1**)–(**H4**) *hold. An instant $t \in (0, t_1)$ is a conjugate time iff the mapping Exp_t is degenerate.*

Proof. The condition $\Lambda(t) \cap \Pi \neq 0$ means that $e_*^{t\vec{H}}(\Pi) \cap T_{\lambda_t}(T^*_{q(t)}M) \neq 0$, which is equivalent to degeneracy of the mapping $\operatorname{Exp}_t = \pi \circ e^{t\vec{H}}$. □

Due to Proposition 2.5, we obtain a statement on homotopy invariance of the Maslov index of the second variation.

Proposition 2.6. *Let $(u^s(t), \lambda_t^s)$, $t \in [0, t_1^s]$, $s \in [0, 1]$, be a continuous in parameter s family of normal extremal pairs in the optimal control problem* (2.1)–(2.3) *satisfying conditions* (**H1**)–(**H4**)*. Assume that, for any $s \in [0, 1]$, the terminal instant $t = t_1^s$ is not a conjugate time along the extremal λ_t^s. Then*

$$\operatorname{ind} Q_{t_1^1}\Big|_{K_{t_1^1}} = \operatorname{ind} Q_{t_1^0}\Big|_{K_{t_1^0}}. \tag{2.7}$$

Proof. It follows from the continuity and strict monotonicity of the curves $\Lambda^s(t) = e_*^{-t\vec{H}} T_{\lambda_t^s}(T^*_{q^s(t)}M)$, $q^s(t) = \pi(\lambda_t^s)$ that there exists $\bar{t} > 0$ such that $\bar{t} < t_s$ for all $s \in [0, 1]$ and any instant $t \in (0, \bar{t})$ is not a conjugate time along the extremal λ_t^s.

According to item (2) of Proposition 2.5, we have

$$\operatorname{ind} Q_{t_1^s}|_{K_{t_1^s}} = -\mu_\Pi(\Lambda^s(\cdot)|_{[t_0,t_1^s]}) \quad \forall t_0 \in (0, \bar{t}) \quad \forall s \in [0, 1]. \tag{2.8}$$

For all $s \in [0, 1]$, we have $\Lambda^s(t_0) \cap \Pi = \Lambda^s(t_1^s) \cap \Pi = 0$. Then the homotopy invariance of the Maslov index implies that the function $s \mapsto \mu_\Pi(\Lambda^s(\cdot)|_{[t_0,t_1^s]})$

is constant on the segment $s \in [0,1]$. Thus, Eq. (2.8) implies the required equality (2.7). $\square$

The following statements can be useful for the proof of absence of conjugate points under homotopy or limit passage.

Corollary 2.2. *Let all hypotheses of Proposition* 2.6 *be satisfied. If an extremal trajectory* $q^0(t) = \pi(\lambda_t^0)$, $t \in (0, t_1^0]$, *does not contain conjugate points, then the extremal trajectory* $q^1(t) = \pi(\lambda_t^1)$, $t \in (0, t_1^1]$, *also does not contain conjugate points.*

Proof. A regular extremal does not contain conjugate points iff its Maslov index is zero and, therefore, the statement follows from Proposition 2.6. $\square$

Corollary 2.3. *Let* $(u^s(t), \lambda_t^s)$, $t \in [0, +\infty)$, $s \in [0,1]$, *be a continuous in parameter s family of normal extremal pairs in the optimal control problem* (2.1)–(2.3) *satisfying hypotheses* (**H1**)–(**H4**). *Let for any* $s \in [0,1]$ *and* $T > 0$ *the extremal* λ_t^s *have no conjugate points for* $t \in (0, T]$. *Then for any* $T > 0$, *the extremal* λ_t^1 *also has no conjugate points for* $t \in (0, T]$.

Proof. Fix any $T > 0$. By Proposition 2.2, conjugate points along the extremal λ_t^1 are isolated and, therefore, there exists an instant $t_1 > T$ that is not a conjugate time along λ_t^1. Consider the family of extremals λ_t^s, $t \in [0, t_1]$, $s \in [0,1]$. Corollary 2.2 implies that the extremal λ_t^1 has no conjugate points for $t \in (0, t_1]$ and, therefore, also for $t \in (0, T]$. $\square$

2.6. **Preliminary remarks on the Euler problem.** In this section we show that the Euler elastic problem satisfies all hypotheses required for the general theory of conjugate points described in Secs. 2.1–2.5.

Recall (see [11]) that the Euler problem is stated as follows:

$$\dot{q} = X_1(q) + uX_2(q), \quad q \in M = \mathbb{R}^2 \times S^1, \quad u \in \mathbb{R}, \tag{2.9}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ is fixed}, \tag{2.10}$$

$$J = \frac{1}{2} \int_0^{t_1} u^2 dt \to \min, \tag{2.11}$$

where

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta},$$

$$[X_1, X_2] = X_3 = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

This problem has the form (2.1)–(2.3), and the regularity conditions for M, f, and φ are satisfied.

In terms of the Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$, $i = 1, 2, 3$, the normal Hamiltonian of the PMP for the Euler problem is

$$h_u(\lambda) = h_1(\lambda) + u h_2(\lambda) - \frac{1}{2}u^2.$$

We have $\dfrac{\partial^2 h_u}{\partial u^2} = -1 < 0$, i.e., hypothesis (**H1**) holds.

Condition (**H2**) obviously holds.

Let $u(t)$ be a normal extremal control in the Euler problem. The corank of the control $u(t)$ is equal to the dimension of the space of solutions of the linear Hamiltonian system of the PMP $\dot\lambda_t = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t)$, i.e., to the number of distinct nonzero solutions of the Hamiltonian system corresponding to the maximized Hamiltonian $H = h_1 + h_2^2/2$:

$$\dot\lambda_t = \vec{h}_1(\lambda_t) + h_2\vec{h}_2(\lambda_t), \quad u(t) = h_2(\lambda_t). \tag{2.12}$$

We are interested in the number of distinct nonzero solutions of the vertical subsystem of system (2.12):

$$\begin{cases} \dot h_1 = -h_2 h_3, \\ \dot h_2 = h_3, \\ \dot h_3 = h_1 h_2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot\beta = c, \\ \dot c = -r\sin\beta, \\ \dot r = 0, \end{cases} \tag{2.13}$$

where $h_1 = -r\cos\beta$, $h_2 = c$, and $h_3 = -r\sin\beta$ (see [11]).

To the extremal control $u(t) \equiv 0$, there correspond two distinct nonzero extremals $(h_1, h_2, h_3)(\lambda_t) = (\pm r, 0, 0)$, $r \neq 0$; therefore, in this case $\operatorname{corank} u = 2$.

If $u(t) \not\equiv 0$, then $c_t = u(t) \not\equiv 0$. Then the function c_t uniquely determines the functions $r\sin\beta_t = -\dot c_t$ and $r\cos\beta_t = -\ddot c_t/c_t$ via system (2.13). Therefore, the curve $(h_1, h_2, h_3)(\lambda_t) \not\equiv 0$ is uniquely determined. Consequently, $\operatorname{corank} u = 1$ in the case $u(t) \not\equiv 0$.

Note that the control $u(t) \equiv 0$ is optimal and, therefore, in the sequel, we can assume in the study of optimality of extremal controls that their corank is equal to 1, i.e., hypothesis (**H3**) is satisfied.

Finally, hypothesis (**H4**) is also satisfied since the Hamiltonian field $\vec{H}$ is complete (its trajectories are parametrized by the Jacobi functions determined for all $t \in \mathbb{R}$).

Summing up, all hypotheses (**H1**)–(**H4**) are satisfied for the Euler elastic problem and, therefore, the theory of conjugate points stated in this section is applicable.

3. Conjugate points on inflexional elasticas

In this section, we describe conjugate points on inflexional elasticas in the Euler problem. We perform explicit computations and estimates on the basis of parametrization of extremal trajectories obtained in [11].

The consideration is based on the decomposition of the preimage of the exponential mapping $T_{q_0}^* M = N = \bigcup_{i=1}^{7} N_i$ introduced in [11]. In this section, we consider the case $\lambda \in N_1$. In [11, Sec. 8.2], a parametrization of the exponential mapping in the Euler problem $\mathrm{Exp}_t : (\varphi, k, r) \mapsto (x_t, y_t, \theta_t)$ was obtained in terms of elliptic coordinates in the domain N_1. By virtue of Corollary 2.1, an instant t is a conjugate time iff the mapping Exp_t is degenerate, i.e., iff its Jacobian $J = \dfrac{\partial(x_t, y_t, \theta_t)}{\partial(\varphi, k, r)}$ vanishes. A direct computation using the parametrization of the exponential mapping obtained in [11, Sec. 8.2] yields the following:

$$J = \frac{\partial(x_t, y_t, \theta_t)}{\partial(\varphi, k, r)} = \frac{1}{\sqrt{r}\cos(\theta_t/2)} \frac{\partial(x_t, y_t, \sin(\theta_t/2))}{\partial(\varphi, k, \sqrt{r})} = -\frac{32k}{(1-k^2) r^{3/2} \Delta^2} J_1, \tag{3.1}$$

$$J_1 = a_0 + a_1 z + a_2 z^2, \quad z = \operatorname{sn}^2 \tau \in [0,1], \tag{3.2}$$

$$a_2 = -k^2 \operatorname{sn} p \, x_1, \tag{3.3}$$

$$a_2 + a_1 + a_0 = (1-k^2) \operatorname{sn} p \, x_1, \tag{3.4}$$

$$a_0 = f_1(p,k) \, x_2, \tag{3.5}$$

$$\begin{aligned} x_1 = &- \operatorname{dn} p (2 \operatorname{sn} p \operatorname{dn} p \operatorname{E}^3(p) + ((4k^2 - 5) p \operatorname{sn} p \operatorname{dn} p \\ &+ \operatorname{cn} p (3 - 6k^2 \operatorname{sn}^2 p)) \operatorname{E}^2(p) + ((4k^2-5) \operatorname{cn} p (1 - 2k^2 \operatorname{sn}^2 p) p \\ &+ \operatorname{sn} p \operatorname{dn} p (4p^2 - 1 + k^2(6 \operatorname{sn}^2 p - 4 - 4p^2))) \operatorname{E}(p) \\ &+ p \operatorname{sn} p \operatorname{dn} p (1 - (1-k^2)p^2 + k^2(4k^2-5) \operatorname{sn}^2 p) \\ &+ 2 \operatorname{cn} p (k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p + (1-k^2)(1 - 2k^2 \operatorname{sn}^2 p) p^2)), \end{aligned} \tag{3.6}$$

$$\begin{aligned} x_2 = &\operatorname{cn} p (2(1-k^2) p \operatorname{E}(p) - \operatorname{E}^2(p) - (1-k^2)p^2) \\ &+ \operatorname{sn} p \operatorname{dn} p (\operatorname{E}(p) - (1-k^2)p), \end{aligned} \tag{3.7}$$

$$f_1(p,k) = \operatorname{sn} p \operatorname{dn} p - (2 \operatorname{E}(p) - p) \operatorname{cn} p, \tag{3.8}$$

$$p = \sqrt{r} t/2, \qquad \tau = \sqrt{r}(\varphi + t/2), \quad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau .$$

Here cn, sn, dn, and E are the Jacobi functions (see details in [11]).

3.1. **Preliminary lemmas.** In this section, we describe roots and signs of the functions a_0 and $a_2 + a_1 + a_0$ that essentially evaluate the numerator of the Jacobian J at the extreme points $z = 0$ and 1 respectively (see (3.1), (3.2)).

3.1.1. *Roots of the function a_0.* Roots of the function $f_1(p)$ defined in (3.8) were described in [10]. For completeness, we cite the statements we will need in the sequel.

Proposition 3.1 (see [10, Lemma 2.1]). *The equation $2E(k) - K(k) = 0$, $k \in [0,1)$, has a unique root $k_0 \in (0,1)$. Moreover,*

$$k \in [0, k_0) \ \Rightarrow \ 2E - K > 0,$$
$$k \in (k_0, 1) \ \Rightarrow \ 2E - K < 0.$$

Here and below, $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds (see [8, 11, 14]). Numerical computations yield the approximate value $k_0 \approx 0.909$.

Proposition 3.2 (see [10, Proposition 2.1]). *For any $k \in [0,1)$, the function $f_1(p,k)$ has a countable number of roots p_n^1, $n \in \mathbb{Z}$, localized as follows*: $p_0^1 = 0$ *and*

$$p_n^1 \in (-K + 2Kn, \ K + 2Kn), \quad n \in \mathbb{Z}.$$

Moreover, for $n \in \mathbb{N}$

$$k \in [0, k_0) \ \Rightarrow \ p_n^1 \in (2Kn, K + 2Kn),$$
$$k = k_0 \ \Rightarrow \ p_n^1 = 2Kn,$$
$$k \in (k_0, 1) \ \Rightarrow \ p_n^1 \in (-K + 2Kn, 2Kn),$$

where k_0 is the unique root of the equation $2E(k) - K(k) = 0$ (see Proposition 3.1).

Now we establish the signs of the function $f_1(p)$ between its zeros p_n^1.

Lemma 3.1. *For any $m = 0, 1, 2, \dots$, we have*:

$$p \in (p_{2m}^1, p_{2m+1}^1) \ \Rightarrow \ f_1(p) > 0,$$
$$p \in (p_{2m+1}^1, p_{2m+2}^1) \ \Rightarrow \ f_1(p) < 0.$$

Proof. By virtue of the equality

$$\left(\frac{f_1(p)}{\operatorname{cn} p}\right)' = \frac{\operatorname{sn}^2 p \, \operatorname{dn}^2 p}{\operatorname{cn}^2 p},$$

the function $f_1(p)/\operatorname{cn} p$ increases on the segments of the form $[-K + 2Kn, K + 2Kn]$, $n \in \mathbb{Z}$. Therefore, the function $f_1(p)/\operatorname{cn} p$, as well as $f_1(p)$, changes its sign at the points $p_n^1 \in (-K + 2Kn, K + 2Kn)$. It remains to verify that $f_1(p)$ is positive on the first interval $(p_0^1, p_1^1) = (0, p_1^1)$. We have $f_1(p) = p^3/3 + o(p^3) > 0$, $p \to 0$, and the statement follows. □

Now we describe zeros of the function x_2 that enters factorization (3.5) of the function a_0.

Lemma 3.2. *The function $x_2(p)$ given by (3.7) has a countable number of roots $p = p_n^{x_2} \geq 0$. We have $p_0^{x_2} = 0$ and $p_n^{x_2} \in (2Kn, K + 2Kn)$ for $n \in \mathbb{N}$, and, moreover,*

$$k < k_0 \ \Rightarrow \ p_n^{x_2} \in (p_n^1, K + 2Kn). \tag{3.9}$$

Further,

$$p \in (p^{x_2}_{2m}, p^{x_2}_{2m+1}) \;\Rightarrow\; x_2(p) > 0, \tag{3.10}$$

$$p \in (p^{x_2}_{2m+1}, p^{x_2}_{2m+2}) \;\Rightarrow\; x_2(p) < 0, \quad m = 0, 1, 2, \dots . \tag{3.11}$$

Proof. First, we show that the function

$$\frac{x_2(p)}{\operatorname{sn} p \operatorname{dn} p} \text{ increases when } p \in (2Kn, 2K + 2Kn). \tag{3.12}$$

A direct computation yields

$$\left(\frac{x_2(p)}{\operatorname{sn} p \operatorname{dn} p}\right)' = \frac{x_3(p)}{\operatorname{sn}^2 p \operatorname{dn}^2 p}, \tag{3.13}$$

$$x_3 = k^2(\operatorname{cn}^2 p \operatorname{E}(p) + \alpha)^2 + (1 - k^2)(\operatorname{E}(p) + \beta)^2, \tag{3.14}$$

$$\alpha = (1 - k^2)p \operatorname{sn}^2 p - \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p, \quad \beta = -p \operatorname{dn}^2 p\,.$$

Since

$$\operatorname{E}(p) + \beta = \frac{2}{3}k^2 p^3 + o(p^3), \quad \operatorname{cn}^2 p \operatorname{E}(p) + \alpha = \frac{2}{3}(1 - k^2)p^3 + o(p^3),$$

we have $\operatorname{E}(p) + \beta \not\equiv 0$, $\operatorname{cn}^2 p \operatorname{E}(p) + \alpha \not\equiv 0$. Therefore, the function $x_3(p)$ given by (3.14) is nonnegative and vanishes only at isolated points. By virtue of Eq. (3.13), assertion (3.12) follows.

Further, we have

$$x_2|_{p=2Kn} = \operatorname{cn} p\, x_4(p),$$

$$x_4 = -((1 - k^2)(\operatorname{E}(p) - p)^2 + k^2 \operatorname{E}^2(p)) < 0 \text{ for all } p \neq 0.$$

Therefore,

$$p = 2K + 4Kn \;\Rightarrow\; \operatorname{cn} p < 0,\ x_2 > 0,$$

$$p = 4Kn \;\Rightarrow\; \operatorname{cn} p > 0,\ x_2 < 0.$$

Consequently, $x_2/(\operatorname{sn} p \operatorname{dn} p) \to \pm\infty$ as $p \to 2Kn \mp 0$, $n \in \mathbb{N}$. Moreover, it follows from the asymptotics

$$x_2 = (4/45)\, k^2(1 - k^2)p^6 + o(p^6), \quad p \to 0, \tag{3.15}$$

that $x_2/(\operatorname{sn} p \operatorname{dn} p) \to +0$ as $p \to +0$.

Thus,

$$p \in (0, 2K) \;\Rightarrow\; \frac{x_2}{\operatorname{sn} p \operatorname{dn} p} > 0,$$

$$p \in (2Kn, 2K + 2Kn) \;\Rightarrow\; \frac{x_2}{\operatorname{sn} p \operatorname{dn} p} \text{ increases from } -\infty \text{ to } +\infty.$$

Therefore, there exists a unique root of $x_2(p)/(\operatorname{sn} p \operatorname{dn} p)$ and hence of $x_2(p)$ at the interval $(2Kn, 2K + 2Kn)$. We denote it by $p^{x_2}_n$.

Now we localize $p_n^{x_2}$ with respect to the point $K+2Kn$. We have

$$x_2|_{p=K+2Kn} = \operatorname{dn} p \operatorname{sn} p(\mathrm{E}(p) - (1-k^2)p),$$
$$\mathrm{E}(p) - (1-k^2)p = k^2 \int_0^p \operatorname{cn}^2 t\, dt > 0, \quad p > 0.$$

Now

$$p = K + 4Kn \;\Rightarrow\; \operatorname{sn} p = 1,\; x_2 > 0 \;\Rightarrow\; \frac{x_2}{\operatorname{sn} p} > 0,$$
$$p = 3K + 4Kn \;\Rightarrow\; \operatorname{sn} p = -1,\; x_2 < 0 \;\Rightarrow\; \frac{x_2}{\operatorname{sn} p} > 0.$$

Consequently, $p_n^{x_2} \in (2Kn, K+2Kn)$ for all $n \in \mathbb{N}$.

Let $k < k_0$; then $p_n^1 \in (2Kn, K+2Kn)$. Now we clarify the mutual disposition of the points p_n^1 and $p_n^{x_2}$ in this case. By virtue of (3.8),

$$f_1(p) = 0 \;\Leftrightarrow\; \mathrm{E}(p) = (\operatorname{dn} p \operatorname{sn} p / \operatorname{cn} p + p)/2.$$

A direct computation yields

$$x_2|_{\mathrm{E}(p)=(\operatorname{dn} p \operatorname{sn} p/\operatorname{cn} p+p)/2} = 8 \operatorname{cn}^2 p\,(\operatorname{sn} p \operatorname{dn} p - p \operatorname{cn} p)\,\mathrm{E}(p).$$

Since for $p = p_n^1$ we have $\operatorname{cn} p \neq 0$, it follows that for $p = p_n^1$, the functions x_2 and $\operatorname{sn} p \operatorname{dn} p - p \operatorname{cn} p$ have the same sign. Then both for $p = p_{2l-1}^1 \in (4Kl-2K, 4Kl-K)$ and for $p = p_{2l}^1 \in (4Kl, 4Kl+K)$ we obtain $x_2/\operatorname{sn} p < 0$. Consequently, $p_n^1 < p_n^{x_5}$ for all $n \in \mathbb{N}$, i.e., inclusion (3.9) is proved. The roots $p_n^{x_2}$ are localized as required.

For $p > 0$, the functions x_2 and $\operatorname{sn} p$ have distinct roots and, therefore, it follows from (3.12) that x_2 changes its sign at the points $p_n^{x_2}$, $n \in \mathbb{N}$. The distribution of signs (3.10), (3.11) follows from the fact that the function x_2 is positive on the first interval $(p_0^{x_2}, p_1^{x_2}) = (0, p_1^{x_2})$ (see (3.15)). □

For $p > 0$, the function a_0 vanishes at the points $p = p_n^1$ and $p = p_n^{x_2}$ defined and localized in Proposition 3.2 and Lemma 3.2. Now decomposition (3.5) and Lemmas 3.1, 3.2 imply the following statement about the distribution of signs of the function a_0.

Lemma 3.3. *Let $k \in (0,1)$. If $p \in (0, p_1^1)$, then $a_0 > 0$. For any $n \in \mathbb{N}$, if $p \in (p_n^1, p_n^{x_2})$, then $a_0 < 0$, and if $p \in (p_n^{x_2}, p_{n+1}^1)$, then $a_0 > 0$.*

3.1.2. *Roots of the function $a_0 + a_1 + a_2$.* In order to obtain a similar description for the function $a_0 + a_1 + a_2$, we have to describe roots of the function x_1 (see decomposition (3.4)).

Lemma 3.4. *For $p \geq 0$, the function $x_1(p)$ defined by* (3.6) *has a countable number of roots $p_0 = 0$, $p_n^{x_1} \in (p_n^1, p_{n+1}^1)$, $n \in \mathbb{N}$. Moreover,*

$$p \in (p_{2m}^{x_1}, p_{2m+1}^{x_1}) \;\Rightarrow\; x_1(p) > 0, \tag{3.16}$$
$$p \in (p_{2m+1}^{x_1}, p_{2m+2}^{x_1}) \;\Rightarrow\; x_1(p) < 0. \tag{3.17}$$

Proof. A direct computation yields

$$\left(\frac{x_1(p)}{\operatorname{dn} p f_1(p)}\right)' = \frac{x_5(p)}{4f_1^2(p)}, \tag{3.18}$$

$$x_5 = k^2(\operatorname{cn} p E_4\, p + \alpha)^2 + (1-k^2)(pE_2+\beta)^2 \geq 0,$$

$$E_2 = 2\,\mathrm{E}(p) - p, \quad E_4 = \operatorname{cn} p(2\,\mathrm{E}(p) - p) - 2\operatorname{sn} p \operatorname{dn} p,$$

$$\alpha = (1 + \operatorname{sn}^2 p - 2k^2 \operatorname{sn}^2 p)E_2^2 + 4\operatorname{cn} p \operatorname{sn} p \operatorname{dn} p(1-2k^2)E_2 + 4(2k^2-1)\operatorname{sn}^2 p \operatorname{dn}^2 p,$$

$$\beta = (2k^2 \operatorname{sn}^2 p - 1)E_2^2 + 8k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p E_2 - 8k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p.$$

Since

$$\operatorname{cn} p E_4\, p + \alpha = \frac{4}{45}(1-k^2)p^6 + o(p^6) \not\equiv 0,$$

$$pE_2 + \beta = -\frac{4}{45}k^2p^6 + o(p^6) \not\equiv 0,$$

the function $x_5(p)$ is nonnegative and vanishes at isolated points. In view of Eq. (3.18), the function $x_1(p)/(\operatorname{dn} p f_1(p))$ increases on the intervals where $f_1(p) \neq 0$.

Now we find the sign of x_1 at the points p_n^1. We have

$$x_1|_{\mathrm{E}(p)=(\operatorname{dn} p \operatorname{sn} p/\operatorname{cn} p + p)/2} = \frac{x_6(p)}{4\operatorname{cn}^3 p},$$

$$x_6(p) = x_6^0 + x_6^1\, p + x_6^2\, p^2,$$

$$x_6^2(p) = -\operatorname{cn}^2 p \operatorname{dn} p(1 - k^2 \operatorname{sn}^2 p\,(2 - \operatorname{sn}^2 p)),$$

$$x_6^1(p) = 2\operatorname{cn} p \operatorname{sn} p(1 - 2k^6 \operatorname{sn}^6 p - k^2 \operatorname{sn}^2 p\,(3 + \operatorname{sn}^2 p) + k^4 \operatorname{sn}^4 p\,(4 + \operatorname{sn}^2 p)),$$

$$x_6^0(p) = -\operatorname{dn}^3 p \operatorname{sn}^2 p\,(1 - k^2 \operatorname{sn}^2 p\,(2 - \operatorname{sn}^2 p)).$$

Note that $1 - k^2 \operatorname{sn}^2 p\,(2 - \operatorname{sn}^2 p) = \operatorname{dn}^4 p + k^2(1-k^2)\operatorname{sn}^4 p > 0$. Consider the discriminant

$$x_{6d} = (x_6^1)^2 - 4x_6^0\, x_6^2 = -16k^2(1-k^2)\operatorname{cn}^2 p \operatorname{sn}^6 p \operatorname{dn}^8 p$$

of the quadratic polynomial $x_6(p)$. If $k \neq k_0$, then for $p = p_n^1$ we have $\operatorname{cn} p \neq 0$ and $\operatorname{sn} p \neq 0$ and, therefore, $x_6^2, x_6^0, x_{6d} < 0$ and $x_6 < 0$. If $k = k_0$, then for $p = p_n^1$ we have $\operatorname{cn} p \neq 0$ and $\operatorname{sn} p \neq 0$ and, therefore, $x_6^0 = 0$, $x_6^2 < 0$, $x_{6d} = 0$, $x_6^1 = 0$, and $x_6 = x_6^2 p^2 < 0$.

Thus, for all $k \in (0,1)$, if $p = p_n^1 > 0$, then $\operatorname{sgn} x_1 = -\operatorname{sgn} \operatorname{cn} p$. If $p = p_{2l-1}^1 \in (4Kl - 2K, 4Kl - K)$, then $\operatorname{cn} p < 0$ and, therefore, $x_1 > 0$. Similarly, if $p = p_{2l}^1 \in (4Kl, 4Kl + K)$, then $\operatorname{cn} p > 0$ and $x_1 < 0$.

Consequently,

$$p \in (0, p_1^1) \;\Rightarrow\; \frac{x_1(p)}{\operatorname{dn} p f_1(p)} \text{ increases from 0 to } +\infty \;\Rightarrow\; x_1(p) > 0, \quad (3.19)$$

$$p \in (p_n^1, p_{n+1}^1),\; n \in \mathbb{N} \;\Rightarrow\; \frac{x_1(p)}{\operatorname{dn} p f_1(p)} \text{ increases from } -\infty \text{ to } +\infty$$

and, therefore, x_1 has a unique root $p_n^{x_1} \in (p_n^1, p_{n+1}^1)$.

The required signs of the function $x_1(p)$ on the intervals (3.16) and (3.17) follow from the inequality on the first interval (3.19), and from the fact that $x_1(p)/(\operatorname{dn} p f_1(p))$ and $x_1(p)$ changes its sign at the points $p_n^{x_1}$, $n \in \mathbb{N}$. □

Remark. By virtue of decomposition (3.4), we have the equality

$$\begin{aligned}\{p > 0 \mid a_0 + a_1 + a_2 = 0\} &= \{p > 0 \mid \operatorname{sn} p = 0\} \cup \{p > 0 \mid x_1 = 0\} \\ &= \{2Km \mid m \in \mathbb{N}\} \cup \{p_n^{x_1} \mid n \in \mathbb{N}\}. \quad (3.20)\end{aligned}$$

In order to obtain a complete description of roots of the function $a_0 + a_1 + a_2$, one should describe mutual disposition of the points $2Km$ and $p_n^{x_1}$. Numerical computations show that some of these points may coincide one with another. For example, numerical computations yield the following relations between the first roots in families (3.20): if $k \in (0, \bar{k})$, then $p_1^{x_1} > 2K$; if $k = \bar{k}$, then $p_1^{x_1} = 2K$; if $k \in (\bar{k}, 1)$, then $p_1^{x_1} < 2K$ for a number $\bar{k} \approx 0.998$. We do not go into details of this analysis, but in the sequel, we allow different possibilities of mutual disposition of the roots $p_n^{x_1}$ and $2Km$.

3.2. **Bounds of the conjugate time.** In this section, we estimate the first conjugate time in the Euler problem along inflexional elasticas.

We obtain from Eqs. (3.3) and (3.4) that $a_2 = -k^2/(1-k^2)(a_0+a_1+a_2)$ and, therefore, the Jacobian appearing in (3.1), (3.2) can be represented as

$$J_1(p, k, z) = (1 - z)a_0 + z(1 - k^2 z)/(1 - k^2)(a_0 + a_1 + a_2). \quad (3.21)$$

Note that $(1 - k^2 z)/(1 - k^2) > 0$. In order to describe the first conjugate point along an extremal trajectory $q(t) = \pi \circ e^{t\vec{H}}(\lambda)$, $\lambda \in N_1$, it suffices to describe the first positive root of the function J_1 for fixed k and z:

$$p_1^{\mathrm{conj}}(k, z) = \min\{p > 0 \mid J_1(p, k, z) = 0\}.$$

This minimum exists since, by virtue of the regularity of normal extremals, small intervals $p \in (0, \varepsilon)$ do not contain conjugate points. Below, in the proof of Theorem 3.1, we prove this independently on the basis of an explicit expression for the function J_1.

Theorem 3.1. *Let $\lambda \in N_1$. For all $k \in (0,1)$ and $z \in [0,1]$, the number $p_1^{\mathrm{conj}}(k,z)$ belongs to the segment bounded by the points $2K(k)$ and $p_1^1(k)$, namely*:

(1) $k \in (0, k_0) \;\Rightarrow\; p_1^{\mathrm{conj}} \in [2K, p_1^1]$;
(2) $k = k_0 \;\Rightarrow\; p_1^{\mathrm{conj}} = 2K = p_1^1$;

(3) $k \in (k_0, 1) \;\Rightarrow\; p_1^{\mathrm{conj}} \in [p_1^1, 2K]$.

Moreover, for any $k \in (0,1)$, there exists $\varepsilon = \varepsilon(k) > 0$ such that:

(1′) *if $k \in (0, k_0)$, then*

$$p \in (0, 2K) \;\Rightarrow\; J_1 > 0, \tag{3.22}$$

$$p \in (p_1^1, p_1^1 + \varepsilon) \;\Rightarrow\; J_1 < 0; \tag{3.23}$$

(2′) *if $k = k_0$, then*

$$p \in (0, 2K) \;\Rightarrow\; J_1 > 0, \tag{3.24}$$

$$p \in (2K, 2K + \varepsilon) \;\Rightarrow\; J_1 < 0; \tag{3.25}$$

(3′) *if $k \in (k_0, 1)$, then*

$$p \in (0, p_1^1) \;\Rightarrow\; J_1 > 0; \tag{3.26}$$

moreover,

(3′a) *in the case $p_1^{x_1} \in (p_1^1, 2K)$:*

$$p \in (p_1^{x_1}, p_1^{x_1} + \varepsilon) \;\Rightarrow\; J_1 < 0; \tag{3.27}$$

(3′b) *in the case $p_1^{x_1} = 2K$:*

$$p = 2K = p_1^{x_1} \;\Rightarrow\; J_1 \le 0, \tag{3.28}$$

(3′c) *in the case $p_1^{x_1} \in (2K, p_1^2)$:*

$$p \in (2K, 2K + \varepsilon) \;\Rightarrow\; J_1 < 0. \tag{3.29}$$

Proof. It is easy to see that, by virtue of the continuity of the function $J_1(p)$, items (1′)–(3′) imply items (1)–(3), respectively, and, therefore, we prove statements (1′)–(3′).

(1′) Fix any $k \in (0, k_0)$, then $2K < p_1^1$ (see Proposition 3.2).

If $p \in (0, 2K)$, then Lemmas 3.1, 3.2, and 3.4 and decompositions (3.5) and (3.4) imply the following:

$$f_1 > 0 \text{ and } x_2 > 0 \;\Rightarrow\; a_0 > 0,$$
$$\operatorname{sn} p > 0 \text{ and } x_1 > 0 \;\Rightarrow\; a_0 + a_1 + a_2 > 0.$$

Then representation (3.21) yields the inequality $J_1(p, z) > 0$ for all $z \in [0, 1]$ and all $p \in (0, 2K)$. Implication (3.22) follows.

Lemmas 3.2 and 3.4 imply that $p_1^{x_2} \in (p_1^1, 3K)$ and $p_1^{x_1} \in (p_1^1, p_2^1)$, respectively. Denote $\widehat{p}_1 = \min(p_1^{x_2}, p_1^{x_1}) > p_1^1$.

If $p \in (p_1^1, \widehat{p}_1)$, then we obtain from Lemmas 3.1, 3.2, and 3.4 and decompositions (3.5) and (3.4) the following:

$$f_1 > 0 \text{ and } x_2 > 0 \;\Rightarrow\; a_0 < 0,$$
$$\operatorname{sn} p < 0 \text{ and } x_1 > 0 \;\Rightarrow\; a_0 + a_1 + a_2 < 0.$$

Representation (3.21) implies that $J_1(p, z) < 0$ for all $z \in [0, 1]$ and all $p \in (p_1^1, \widehat{p}_1)$, i.e., implication (3.23) is proved for $\varepsilon = \widehat{p}_1 - p_1^1 > 0$.

(2′) Let $k = k_0$. Similarly to item (1′),

$$p \in (0, 2K) \;\Rightarrow\; a_0 > 0 \text{ and } a_0 + a_1 + a_2 > 0 \;\Rightarrow\; J_1 > 0,$$
$$p \in (2K, \widehat{p}_1) \;\Rightarrow\; a_0 < 0 \text{ and } a_0 + a_1 + a_2 < 0 \;\Rightarrow\; J_1 < 0,$$

where $\widehat{p}_1 = \min(p_1^{x_1}, p_1^{x_2}) > 2K$. Thus, implications (3.24) and (3.25) follow for $\varepsilon = \widehat{p}_1 - p_1^{x_2} > 0$.

(3′) Let $k \in (k_0, 1)$, then $p_1^1(k) < 2K(k)$.

Let $p \in (0, p_1^1)$. Then we have the following:

$$f_1 > 0 \text{ and } x_2 > 0 \;\Rightarrow\; a_0 > 0,$$
$$\operatorname{sn} p > 0 \text{ and } x_1 > 0 \;\Rightarrow\; a_0 + a_1 + a_2 = 0.$$

Thus, $J_1 > 0$, and implication (3.26) is proved.

(3′a) Consider the case $p_1^{x_1} \in (p_1^1, 2K)$. Let $p \in (p_1^{x_1}, 2K)$; then, since $f_1 < 0$ and $x_2 > 0$, we have $a_0 < 0$; since $\operatorname{sn} p > 0$ and $x_1 < 0$, we have $a_0 + a_1 + a_2 < 0$. Thus, $J_1 < 0$, and implication (3.27) follows for $\varepsilon = 2K - p_1^{x_1} > 0$. In this case,

$$p_1^{\mathrm{conj}}(z) \in [p_1^1, p_1^{x_1}] \subset [p_1^1, 2K) \quad \forall z \in [0, 1]. \tag{3.30}$$

(3′b) Consider the case $p_1^{x_1} = 2K$. Let $p = 2K$; then, since $f_1 < 0$ and $x_2 > 0$, we have $a_0 < 0$; since $\operatorname{sn} p = x_1 = 0$, we have $a_0 + a_1 + a_2 = 0$. Consequently, $J_1 \le 0$, and implication (3.28) follows.

(3′c) Finally, consider the case $p_1^{x_1} \in (2K, p_1^2)$. Let $p \in (2K, \min(p_1^{x_1}, p_1^{x_2}))$; then, since $f_1 < 0$ and $x_2 > 0$, we have $a_0 < 0$; since $\operatorname{sn} p < 0$ and $x_1 > 0$, we have $a_0 + a_1 + a_2 < 0$. Thus, $J_1 < 0$, and implication (3.29) is proved for $\varepsilon = \min(p_1^{x_1}, p_1^{x_2}) - 2K > 0$. □

Remark. As one can see from inclusion (3.30), for $p_1^{x_1} \in (p_1^1, 2K)$ the range of the function $p_1^{\mathrm{conj}}(z)$ is strictly less than the segment $[p_1^1, 2K]$. Judging by plots of the function $p_1^{\mathrm{conj}}(z)$, $z = \operatorname{sn}^2 \tau$, this function is smooth and strictly monotone on the segment $\tau \in [0, K]$ (see Figs. 1–4).

From decompositions (3.5) and (3.4) and Lemmas 3.2 and 3.4 we obtain the following description of all (not only the first) conjugate points for the cases $z = \operatorname{sn}^2 \tau = 0$ or 1 (i.e., for elasticas centered at its vertex or inflexion point, respectively).

Corollary 3.1. *Let $\lambda \in N_1$ and $k \in (0, 1)$.*

(1) *If $z = 0$, then $\{p > 0 \mid J_1(p, z) = 0\} = \{p_n^1 \mid n \in \mathbb{N}\} \cup \{p_m^{x_2} \mid m \in \mathbb{N}\}$.*
(2) *If $z = 1$, then $\{p > 0 \mid J_1(p, z) = 0\} = \{2Kn \mid n \in \mathbb{N}\} \cup \{p_m^{x_1} \mid m \in \mathbb{N}\}$.*

Remark. According to Lemma 3.2 and Proposition 3.2, in item (1) of Corollary 3.1 all roots p_n^1 and $p_m^{x_2}$ are pairwise distinct. However, in item (2) some of roots $2Kn$ and $p_m^{x_1}$ may coincide one with another, see the remark at the end of Sec. 3.1.

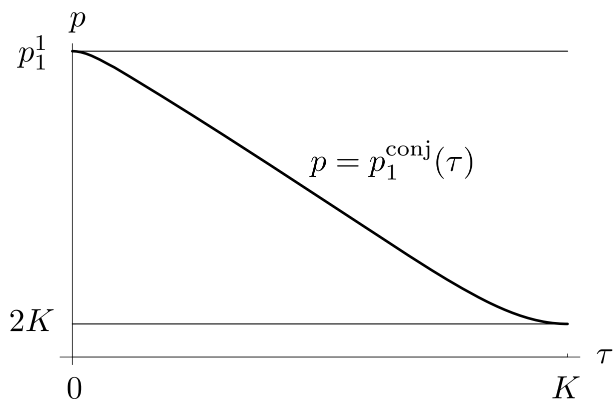

Fig. 1. $p = p_1^{\mathrm{conj}}(k, \tau)$, $k \in (0, k_0)$.

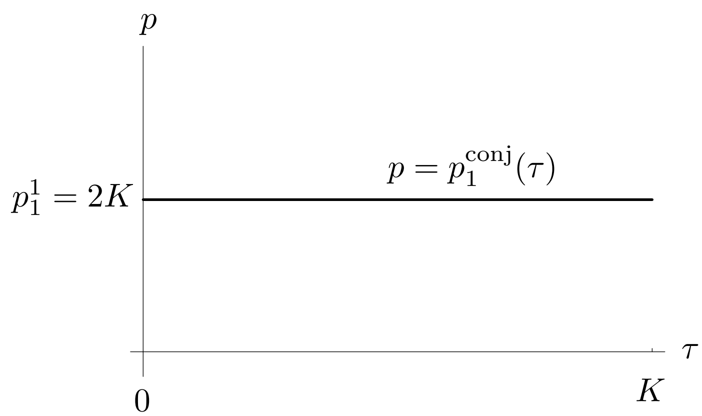

Fig. 2. $p = p_1^{\mathrm{conj}}(k, \tau)$, $k = k_0$.

Now we apply preceding results in order to bound the first conjugate time along normal extremal trajectories in the case $\lambda \in N_1$:

$$t_1^{\mathrm{conj}}(\lambda) = \min\{t > 0 \mid t \text{ is the conjugate time along the trajectory } q(s) = \mathrm{Exp}_s(\lambda)\}.$$

Theorem 3.2. *Let $\lambda = (k, \varphi, r) \in N_1$. Then the number $t_1^{\mathrm{conj}}(\lambda)$ belongs to the segment with the endpoints $\dfrac{4K(k)}{\sqrt{r}}$, $\dfrac{2p_1^1(k)}{\sqrt{r}}$, namely*:

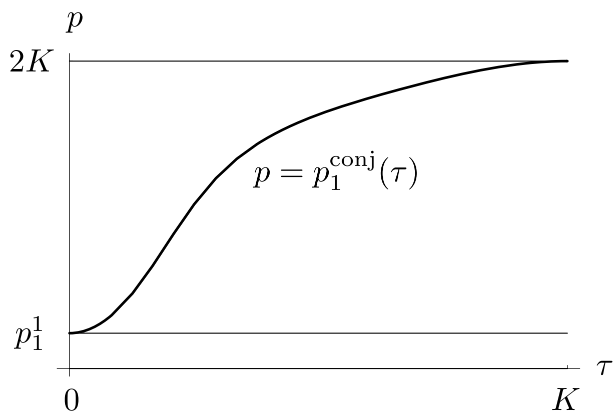

Fig. 3. $p = p_1^{\mathrm{conj}}(k,\tau)$, $k \in (k_0, 1)$, $2K \le p_1^{x_1}$.

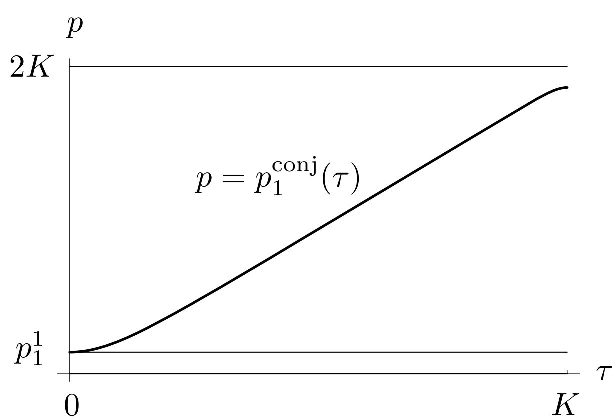

Fig. 4. $p = p_1^{\mathrm{conj}}(k,\tau)$, $k \in (k_0, 1)$, $2K > p_1^{x_1}$.

(1) $k \in (0, k_0) \;\Rightarrow\; t_1^{\mathrm{conj}} \in \left[\dfrac{4K(k)}{\sqrt{r}}, \dfrac{2p_1^1(k)}{\sqrt{r}}\right]$;

(2) $k = k_0 \;\Rightarrow\; t_1^{\mathrm{conj}} = \dfrac{4K(k)}{\sqrt{r}} = \dfrac{2p_1^1(k)}{\sqrt{r}}$;

(3) $k \in (k_0, 1) \;\Rightarrow\; t_1^{\mathrm{conj}} \in \left[\dfrac{2p_1^1(k)}{\sqrt{r}}, \dfrac{4K(k)}{\sqrt{r}}\right]$.

Proof. By Corollary 2.1, an instant $t > 0$ is a conjugate time iff

$$J(t,k,\varphi,r) = \frac{\partial(x_t, y_t, \theta_t)}{\partial(\varphi, k, r)} = -\frac{32k}{(1-k^2)r^{3/2}\Delta^2} J_1(p,k,z) = 0,$$

$$p = \sqrt{r}t/2, \quad \tau = \sqrt{r}(\varphi + t/2), \quad z = \operatorname{sn}^2\tau, \quad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau$$

(see (3.1)).

(1) Let $k \in (0, k_0)$; then $\dfrac{4K(k)}{\sqrt{r}} < \dfrac{2p_1^1(k)}{\sqrt{r}}$. According to item (1′) of Theorem 3.1, for some $\varepsilon = \varepsilon(k) > 0$, we obtain the chains:

$$t \in \left(0, \frac{4K}{\sqrt{r}}\right) \Rightarrow p \in (0, 2K) \Rightarrow J(t,k,\varphi,r) < 0 \ \forall \varphi, r,$$

$$t \in \left(\frac{2p_1^1(k)}{\sqrt{r}}, \frac{2(p_1^1(k)+\varepsilon)}{\sqrt{r}}\right) \Rightarrow p \in (p_1^1(k), p_1^1(k)+\varepsilon) \Rightarrow J(t,k,\varphi,r) > 0 \ \forall \ \varphi, r.$$

By virtue of the continuity of the function J with respect to t, we obtain the required inclusion $t_1^{\text{conj}} \in \left[\dfrac{4K(k)}{\sqrt{r}}, \dfrac{2p_1^2(k)}{\sqrt{r}}\right]$.

Statements (2) and (3) of this theorem follow similarly from items (2′) and (3′) of Theorem 3.1. □

In [11, Sec. 12], a function $\mathbf{t} : N \to (0, +\infty]$ was defined that provides an upper bound for the cut time in the Euler elastic problem (see [11, Theorem 12.1]). It follows from [11, formula (12.2)] that

$$\mathbf{t}(\lambda) = \min\left(\frac{4K(k)}{\sqrt{r}}, \frac{2p_1^1(k)}{\sqrt{r}}\right), \quad \lambda \in N_1.$$

Comparing this equality with Theorem 3.2, we obtain the following statement.

Corollary 3.2. *If* $\lambda \in N_1$, *then* $t_1^{\text{conj}}(\lambda) \geq \mathbf{t}(\lambda)$.

A natural measure of time along extremal trajectories in the Euler problem is the period of the pendulum $T(k) = 4K(k)/\sqrt{r}$. In terms of this measure, the bounds from Theorem 3.2 are rewritten as follows.

Corollary 3.3. *Let* $\lambda \in N_1$. *Then*:

(1) $k \in (0, k_0) \Rightarrow t_1^{\text{conj}} \in [T, t_1^1] \subset [T, 3T/2)$, $t_1^1 = 2p_1^1/\sqrt{r} \in (T, 3T/2)$;
(2) $k = k_0 \Rightarrow t_1^{\text{conj}} = T$;
(3) $k \in (k_0, 1) \Rightarrow t_1^{\text{conj}} \in [t_1^1, T] \subset (T/2, T]$, $t_1^1 = 2p_1^1/\sqrt{r} \in (T/2, T)$.

It is instructive to state the conditions of local optimality for elasticas in terms of their inflexion points.

Corollary 3.4. *Let* $\lambda \in N_1$, *and let* $\Gamma = \{\gamma_s = (x_s, y_s) \mid s \in [0,t]\}$, $q(s) = (x_s, y_s, \theta_s) = \operatorname{Exp}(\lambda_s)$, *be the corresponding elastica.*

(1) *If the arc Γ does not contain inflexion points, then it is locally optimal.*
(2) *If $k \in (0, k_0]$ and the arc Γ contains exactly one inflexion point, then it is locally optimal.*
(3) *If the arc Γ contains not less than three inflexion points in its interior, then it is not locally optimal.*

Proof. (1) If the elastic arc Γ does not contain inflexion points, then its curvature $c_s = 2k\sqrt{r}\,\mathrm{cn}(\sqrt{r}(\varphi + s))$ does not vanish for $s \in [0, t]$. But the Jacobi function $\mathrm{cn}(\sqrt{r}(\varphi + s))$ vanishes at any segment of length not less than half of its period and, therefore, $t < T/2$. By Corollary 3.3, we have $T/2 < t_1^{\mathrm{conj}}$, consequently, $t < t_1^{\mathrm{conj}}$. Thus, the interval $(0, t]$ does not contain conjugate points and, therefore, the corresponding extremal trajectory $q(s)$ is locally optimal (see Proposition 2.3).

(2) Let $k \in (0, k_0]$, and let the arc Γ contain exactly one inflexion point. Then the function c_s has exactly one root on the segment $s \in [0, t]$ and $t < T$. By Corollary 3.3, we have $T \leq t_1^{\mathrm{conj}}$ and, therefore, $t < t_1^{\mathrm{conj}}$, and the elastica Γ is locally optimal.

(3) Let the arc Γ contain in its interior not less than three inflexion points. Then its curvature c_s has not less than three roots on the interval $s \in (0, t)$. Consequently, the interval $(0, t)$ contains a complete period $[\widetilde{t}_0, \widetilde{t}_1]$ of the curvature c_s such that $c_s = 0$ at the endpoints $s = \widetilde{t}_0$ and $s = \widetilde{t}_1$ and, therefore, $(0, t)$ contains a greater segment with the same center:

$$\exists\, [\widetilde{t}_0 - \varepsilon, \widetilde{t}_1 + \varepsilon] \subset (0, t), \quad \varepsilon > 0,$$
$$\sqrt{r}(\varphi + \widetilde{t}_0) = K + 2Kn, \quad \sqrt{r}(\varphi + \widetilde{t}_1) = 5K + 2Kn, \quad n \in \mathbb{Z}.$$

Thus, the arc Γ contains inside itself the elastica $\widetilde{\Gamma} = \{\gamma_s \mid s \in [\widetilde{t}_0 - \varepsilon, \widetilde{t}_1 + \varepsilon]\}$. Now we show that the arc $\widetilde{\Gamma}$ is not locally optimal, this means that the arc Γ containing $\widetilde{\Gamma}$ is also not locally optimal (indeed, if a trajectory $q(s)$, $s \in [0, t]$, is locally optimal, then any its part $q(s)$, $s \in [t_0^1, t_1^1] \subset [0, t]$ is also locally optimal).

For the arc $\widetilde{\Gamma}$, we have the following:

$$(\widetilde{t}_1 + \varepsilon) - (\widetilde{t}_0 - \varepsilon) = 4K/\sqrt{r} + 2\varepsilon = T + 2\varepsilon,$$
$$\tau = ((\sqrt{r}(\varphi + \widetilde{t}_0 - \varepsilon) + \sqrt{r}(\varphi + \widetilde{t}_1 + \varepsilon))/2 = 3K + 2Kn,$$
$$z = \mathrm{sn}^2\,\tau = 1, \qquad J_1 = a_0 + a_1 + a_2$$

(see (3.21)). By Corollary 3.1, we have $p_1^{\mathrm{conj}} = \min(2K, p_1^{x_1}) \leq 2K$ and, therefore, $t_1^{\mathrm{conj}} \leq 4K/\sqrt{r} = T$. Consequently, $(\widetilde{t}_1 + \varepsilon) - (\widetilde{t}_0 - \varepsilon) = T + 2\varepsilon > t_1^{\mathrm{conj}}$, and the interval $(\widetilde{t}_0 - \varepsilon, \widetilde{t}_1 + \varepsilon)$ contains a point t_1^{conj} conjugate to the instant $\widetilde{t}_0 - \varepsilon$. Thus, the arc $\widetilde{\Gamma}$ is not locally optimal, the more so is the arc Γ not locally optimal. $\square$

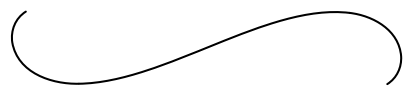

Fig. 5. Locally optimal elastica with 1 inflexion point

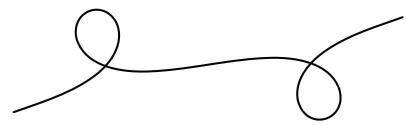

Fig. 6. Locally non-optimal elastica with 1 inflexion point

The mathematical notion of local optimality of an extremal trajectory $q(s) = (x_s, y_s, \theta_s)$ with respect to the functional of elastic energy corresponds to the *stability* of the corresponding elastica (x_s, y_s). Item (3) of Corollary 3.4 has a simple visual meaning: one cannot keep in hands an elastica having three inflexion points inside since such an elastica is unstable.

Remark. In the cases not considered in items (1)–(3) of Corollary 3.4, one can find examples both of locally optimal and non-optimal elasticas.

Let $k > k_0$. If $z = \operatorname{sn}^2 \tau = 1$ (i.e., the elastica is centered at its inflexion point), then, by Corollary 3.1, we have

$$p_1^{\mathrm{conj}} = \min(2K, p_1^{x_1}), \quad p_1^{x_1} \in (p_1^1, p_1^2) \subset (K, 4K).$$

For $p < K$, we obtain $p < p_1^{\mathrm{conj}}$, the corresponding elastica contains one inflexion point and is locally optimal (see Fig. 5). For $p_1^{x_1} < 2K$ (i.e., for $k \in (\bar{k}, 1)$, $\bar{k} \approx 0.998$) and $p \in (p_1^{x_1}, 2K)$, we obtain $p > p_1^{\mathrm{conj}} = p_1^{x_1}$, the corresponding elastica contains one inflexion point and is not locally optimal (see Fig. 6).

Let $k < k_0$ and $z = \operatorname{sn}^2 \tau = 0$ (the elastica is centered at its vertex). Then $p_1^{\mathrm{conj}} = p_1^1 \in (2K, 3K)$. If $p \in (K, 2K)$, then $p < p_1^1$, and then the corresponding elastica is locally optimal and contains 2 inflexion points (see Fig. 7).

Let $k > k_0$ and $z = \operatorname{sn}^2 \tau = 0$, then $p_1^{\mathrm{conj}} = p_1^1 \in (K, 2K)$. If $p > p_1^1$, then $p > p_1^{\mathrm{conj}}$, and then the corresponding elastica is not locally optimal and contains 2 inflexion points (see Fig. 8).

Corollary 3.1 provides the following description of an elastica centered at inflexion points or vertices and terminating at conjugate points.

Corollary 3.5. *Let $\lambda \in N_1$, and let $q(s) = \operatorname{Exp}_s(\lambda)$, $s \in [0, t]$, be the corresponding inflexional elastica.*

(1) *If the elastica $q(s)$ is centered at its vertex (i.e., $\operatorname{sn} \tau = 0$), then the terminal instant t is a conjugate time iff*

$$p = \frac{\sqrt{r}t}{2} \in \{p_n^1 \mid n \in \mathbb{N}\} \cup \{p_m^{x_2} \mid m \in \mathbb{N}\}.$$

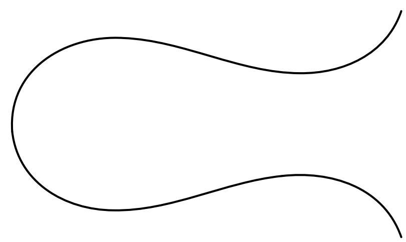

Fig. 7. Locally optimal elastica with 2 inflexion points

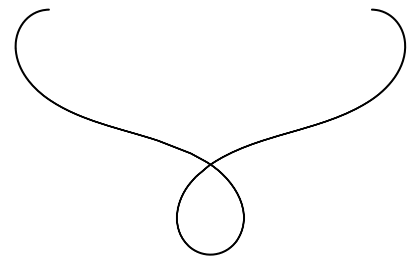

Fig. 8. Locally non-optimal elastica with 2 inflexion points

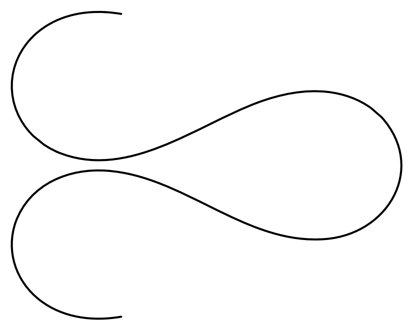

Fig. 9. Conjugate point, $\operatorname{sn}\tau = 0$, $p = p_1^{\mathrm{conj}}(k,\tau) = p_1^1(k)$

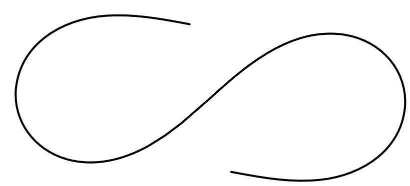

Fig. 10. Conjugate point, $\operatorname{cn}\tau = 0$, $p = p_1^{\mathrm{conj}}(k,\tau) = 2K$

(2) *If the elastica $q(s)$ is centered at its inflexion point (i.e., $\operatorname{cn}\tau = 0$), then the terminal instant t is a conjugate time iff*

$$p = \frac{\sqrt{r}t}{2} \in \{2Kn \mid n \in \mathbb{N}\} \cup \{p_m^{x_1} \mid m \in \mathbb{N}\}.$$

Figures 9 and 10 illustrate cases (1) and (2) of Corollary 3.5, respectively.

3.3. **The upper bound of the cut time.** On the basis of results about the local optimality obtained in this section, we can improve the statement on the upper bound of the time where elasticas lose their global optimality (i.e., on the cut time $t_{\mathrm{cut}}(\lambda)$; see [11, Theorem 12.1]). The argument uses the obvious inequality

$$t_{\mathrm{cut}}(\lambda) \le t_1^{\mathrm{conj}}(\lambda),$$

which holds since if a trajectory is not locally optimal, the more it is not globally optimal.

Theorem 3.3. *Let $\lambda \in N_1$. Then $t_{\mathrm{cut}}(\lambda) \le \mathbf{t}(\lambda)$.*

Proof. We must prove that the extremal trajectory $q(s) = \mathrm{Exp}_s(\lambda)$ is not optimal on any segment of the form $s \in [0, \mathbf{t}(\lambda) + \varepsilon]$, $\varepsilon > 0$. Compute the number $\tau = \sqrt{r}(2\varphi + \mathbf{t}(\lambda))/2$ for the covector $\lambda = (k, \varphi, r)$.

First, consider the case $k \in (0, k_0]$; then $\mathbf{t}(\lambda) = 4K/\sqrt{r}$. If $\operatorname{cn}\tau \operatorname{sn}\tau \ne 0$, then the inequality $t_{\mathrm{cut}}(\lambda) \le \mathbf{t}(\lambda)$ was proved in item (1) of [11, Theorem 12.1]. If $\operatorname{cn}\tau = 0$, then the instant $\mathbf{t}(\lambda)$ is a conjugate time by Corollary 3.5 and, therefore, the trajectory $q(s)$ is not locally optimal after this instant. Finally, if $\operatorname{sn}\tau = 0$, then the instant $\mathbf{t}(\lambda)$ is a Maxwell time by item (1.1) of [11, Theorem 11.1].

In the case $k \in (k_0, 1)$, we have $\mathbf{t}(\lambda) = 2p_1^1/\sqrt{r}$, and the argument is similar. If $\operatorname{cn}\tau \operatorname{sn}\tau \ne 0$, then the statement was proved in item (1) of [11, Theorem 12.1]. If $\operatorname{sn}\tau = 0$, then the instant $\mathbf{t}(\lambda)$ is a conjugate time by Corollary 3.5. And if $\operatorname{cn}\tau = 0$, then the instant $\mathbf{t}(\lambda)$ is a Maxwell time by item (1.2) of [11, Theorem 11.1]. □

4. Conjugate points on non-inflexional elasticas

In this section, we prove that inflexional elasticas ($\lambda \in N_2$), critical elasticas ($\lambda \in N_3$), and circles ($\lambda \in N_6$) do not contain conjugate points.

Let $\lambda \in N_2^+$. Similarly to Sec. 3, we first explicitly compute the Jacobian of the exponential mapping using the parametrization of extremals obtained in [11]:

$$\begin{aligned}
J &= \frac{\partial(x_t, y_t, \theta_t)}{\partial(\psi, k, r)} = \frac{1}{\sqrt{r}\cos(\theta_t/2)} \frac{\partial(x_t, y_t, \sin(\theta_t/2))}{\partial(\psi, k, \sqrt{r})} \\
&= -\frac{32}{(1-k^2)k^2 r^{3/2}\Delta^2} J_2, \qquad (4.1)
\end{aligned}$$

$$J_2 = c_2 z^2 + c_1 z + c_0, \quad z = \operatorname{sn}^2\tau \in [0,1], \qquad (4.2)$$

$$p = \sqrt{r}t/(2k), \quad \tau = \sqrt{r}(2\psi + t/k)/2, \quad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2\tau, \qquad (4.3)$$

$$c_2 = k^4 \operatorname{sn} p \operatorname{cn} p x_1, \qquad (4.4)$$

$$\begin{aligned}
x_1 &= 2\operatorname{cn} p \operatorname{sn} p \operatorname{E}^3(p) + (\operatorname{dn} p(3 - 6\operatorname{sn}^2 p) - (2-k^2)p \operatorname{cn} p \operatorname{sn} p)\operatorname{E}^2(p) \\
&\quad + (\operatorname{dn} p(k^2-2)p(1 - 2\operatorname{sn}^2 p) + \operatorname{cn} p \operatorname{sn} p(k^2(2p^2 - 1 + 6\operatorname{sn}^2 p) \\
&\quad - 2(2+p^2)))\operatorname{E}(p) + \operatorname{dn} p(2k^2 \operatorname{cn}^2 p \operatorname{sn}^2 p + (1-k^2)p^2(2\operatorname{sn}^2 p - 1)) \\
&\quad + p \operatorname{cn} p \operatorname{sn} p(2(2+p^2) - k^2(3 + (3-k^2)p^2 + (2-k^2)\operatorname{sn}^2 p)),
\end{aligned}$$

$$c_0 = -k f_2(p,k)\, x_2, \qquad (4.5)$$

$$x_2 = \operatorname{dn} p \operatorname{E}^2(p) - k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{E}(p) - (1-k^2)p^2 \operatorname{dn} p,$$

$$f_2(p,k) = 2(\operatorname{dn} p(2-k^2)p - 2\operatorname{E}(p)) + k^2 \operatorname{sn} p \operatorname{cn} p)/k,$$

$$c_2 + c_1 + c_0 = (1-k^2)c_0. \qquad (4.6)$$

4.1. **Preliminary lemmas.**

Lemma 4.1. *For any $p > 0$ and $k \in (0,1)$, we have $c_0 < 0$ and $c_0 + c_1 + c_2 < 0$.*

Proof. In view of decomposition (4.5) and Eq. (4.6), it suffices to show that

$$f_2(p,k) > 0, \quad x_2 > 0 \quad \forall p > 0 \quad \forall k \in (0,1). \tag{4.7}$$

We have

$$\left(\frac{f_2(p)}{\operatorname{dn} p}\right)' = k^4 \frac{\operatorname{cn}^2 p \operatorname{sn}^2 p}{\operatorname{dn}^2 p};$$

this identity means that $f_2(p)/\operatorname{dn} p$ increases with respect to the variable p. But $f_2(0) = 0$ and, therefore, $f_2(p) > 0$ for all $p > 0$ and $k \in (0,1)$.

Further, from the equalities

$$\left(\frac{x_2(p)}{\operatorname{dn} p \operatorname{E}(p)}\right)' = \frac{(1-k^2)(\operatorname{E}(p) - p \operatorname{dn}^2 p)^2}{\operatorname{dn}^2 p \operatorname{E}^2(p)},$$
$$\operatorname{E}(p) - p \operatorname{dn}^2 p = \frac{2}{3} k^2 p^3 + o(p^3) \not\equiv 0$$

it follows that $x_2(p)/(\operatorname{dn} p \operatorname{E}(p))$ increases with respect to p. Then the asymptotics

$$x_2(p) = \frac{4}{45}(1-k^2)p^6 + o(p^6) > 0, \quad p \to 0,$$

implies that $x_2 > 0$ for all $p > 0$ and $k \in (0,1)$.

Inequalities (4.7) are proved, and the statement of this lemma follows. □

Lemma 4.2. *For any $n \in \mathbb{N}$, $k \in (0,1)$, $z \in [0,1]$, we have $J_2(Kn, z, k) < 0$.*

Proof. Fix any n, k, and $p = Kn$ according to the assumption of this lemma. It follows from decomposition (4.4) that $c_2 = 0$. Thus, the function $J_2(z)$ becomes linear: $J_2(z) = c_1 z + c_0$, $z \in [0,1]$. By virtue of Lemma 4.1, this linear function is negative at the endpoints of the segment $z \in [0,1]$:

$$J_2(0) = c_0 < 0, \quad J_2(1) = c_1 + c_0 = c_2 + c_1 + c_0 < 0$$

and, therefore, it is also negative on the whole segment $[0,1]$. □

Lemma 4.3. *For any $p_1 > 0$ there exists $\widehat{k} = \widehat{k}(p_1) > 0$ such that for all $k \in (0, \widehat{k})$, $p \in (0, p_1)$, $z \in [0,1]$ we have $J_2(p,z,k) < 0$.*

Proof. In order to estimate the function J_2 for small k, we need the corresponding asymptotics as $k \to 0$:

$$c_0 = k^8 c_{00} + o(k^8), \quad c_1 = k^{10} c_{10} + o(k^{10}), \quad c_2 = k^{12} c_{20} + o(k^{12}), \tag{4.8}$$

$$c_{00} = -c_{10} = -\frac{1}{1024}(4p - \sin 4p)\, c_{01}(p), \tag{4.9}$$

$$c_{01} = 4p^2 - 1 + \cos 4p + p \sin 4p,$$

$$c_{20} = \frac{1}{8192} \cos p \sin p \, c_{21}(p),$$

$$\begin{aligned} c_{21} = &-3\cos 2p - 48p^2 \cos 2p + 3\cos 6p \\ &+ 42p \sin 2p - 64p^3 \sin 2p + 2p \sin 6p, \end{aligned}$$

and the asymptotics as $(p, k) \to (0, 0)$:

$$c_0 = -\frac{4}{135} k^8 p^9 + o(k^8 p^9), \quad c_2 = \frac{4}{4725} k^{12} p^{11} + o(k^{12} p^{11}); \tag{4.10}$$

all these asymptotic expansions are obtained via Taylor expansions of the Jacobi functions (see [10]).

(1) The equalities

$$\left(\frac{c_{01}}{p}\right)' = \frac{2(\sin 2p - 2p \cos 2p)^2}{p^2}, \quad c_{01} = \frac{128}{45} p^6 + o(p^6)$$

imply that $c_{01}(p) > 0$ for $p > 0$, whence, in view of decomposition (4.9), we obtain that $c_{00}(p) < 0$ for all $p > 0$.

Fix an arbitrary number $p_1 > 0$.

(2) Choose any $p_0 \in (0, p_1)$. We show that there exists $k_{01} = k_{01}(p_0, p_1) \in (0, 1)$ such that

$$J_2(p, z, k) < 0 \quad \forall p \in [p_0, p_1] \quad \forall z \in [0, 1] \quad \forall k \in (0, k_{01}). \tag{4.11}$$

Taking into account Eqs. (4.8), we obtain a Taylor expansion as $k \to 0$:

$$J_2(p, z, k) = k^8 c_{00}(p) + \frac{k^{10}}{10!} \frac{\partial^{10} J_2}{\partial k^{10}}(p, z, \widetilde{k}),$$

$$p \in [p_0, p_1], \ z \in [0, 1], \ \widetilde{k} \in (0, k).$$

By the continuity of the corresponding functions, we conclude that

$$\begin{aligned} &c_{00}(p) < -m, && m = m(p_0, p_1) > 0, \\ &\frac{1}{10!} \frac{\partial^{10} J_2}{\partial k^{10}}(p, z, \widetilde{k}) < m_1, && m_1 = m_1(p_0, p_1) > 0, \end{aligned}$$

whence $J_2 < k^8(-m + k^2 m_1) < 0$ for $k^2 < k_{01}^2 = m/m_1 > 0$. Inequality (4.11) follows.

(3) From asymptotics (4.10) and Eq. (4.6) we conclude that

$$J_2 = -\frac{4}{135} k^8 p^9 + o(k^8 p^9), \quad (p, k) \to 0.$$

Thus,

$$\exists p_0' > 0 \ \exists k_0' > 0 \ \forall p \in (0, p_0'] \ \forall k \in (0, k_0') \ \forall z \in [0,1] \quad J_2(p,z,k) < 0.$$

(4) Take $p_0' \in (0, p_1)$ and $k_0' \in (0,1)$ according to item (3) of this proof. Find $k_{01} = k_{01}(p_0', p_1)$ according to item (2). We set $\widehat{k}(p_1) = \min(k_0', k_{01}) > 0$. Then for any $k \in (0, \widehat{k}(p_1))$, we obtain the following: if $p \in (0, p_0]$, then $J_2 < 0$ by item (3), and if $p \in [p_0, p_1]$, then $J_2 < 0$ by item (2). Therefore, the number $\widehat{k}(p_1)$ satisfies conditions of this lemma. $\square$

4.2. **Absence of conjugate points on non-inflexional elasticas.**

Theorem 4.1. *If $\lambda \in N_2$, then the normal extremal trajectory $q(t) = \mathrm{Exp}_t(\lambda)$ does not contain conjugate points for $t > 0$.*

Proof. In view of the symmetry $i : N_2^+ \to N_2^-$ (see [11]), it suffices to consider the case $\lambda \in N_2^+$.

Denote $\lambda^1 = \lambda$. Fix any $n \in \mathbb{N}$ and prove that the trajectory

$$q^1(t) = \mathrm{Exp}_t(\lambda^1), \quad \lambda^1 = (\varphi, k^1, r) \in N_2^+,$$

does not contain conjugate points $t \in (0, t_1^1]$, $t_1^1 = 2k^1 K(k^1) n / \sqrt{r}$.

Consider the family of extremal trajectories

$$\begin{aligned} &\gamma^s = \{q^s(t) = \mathrm{Exp}_t(\lambda^s) \mid t \in [0, t_1^s]\}, \\ &\lambda^s = (\varphi, k^s, r) \in N_2^+, \quad t_1^s = 2k^s K(k^s) n / \sqrt{r}, \quad s \in [0,1], \end{aligned}$$

where the covector $\lambda^1 = (\varphi, k^1, r)$ is equal to λ given in the statement of this theorem and the covector $\lambda^0 = (\varphi, k^0, r)$ will be chosen below so that the parameter k^0 is sufficiently small.

According to Lemma 4.3, we choose a number $\widehat{k}(p^1) \in (0,1)$ corresponding to the number $p^1 = K(k^1)n$. We choose any $k^0 \in (0, \widehat{k}(p^1))$ and set $\lambda^0 = (\varphi, k^0, r) \in N_2^+$.

By Lemma 4.3, for any $p \in (0, p^1]$ and any $z \in [0,1]$, we have $J_2(p, z, k^0) < 0$. By Lemma 4.2, for any $z \in [0,1]$ and any $k \in [k_0, k_1]$, we have $J_2(K(k)n, z, k) < 0$.

Taking into account Eq. (4.1) and relations (4.2) and (4.3), we conclude that the trajectory γ^0 does not have conjugate points on the segment $t \in (0, t_1^0]$, and for any trajectory γ^s, $s \in [0,1]$, the endpoint $t = t_1^s$ is not conjugate. Now the statement of this theorem follows from Corollary 2.2. $\square$

4.3. **Absence of conjugate points for special cases.** The absence of conjugate points on extremals $\lambda_t \in N_2$ implies a similar fact for $\lambda_t \in N_3 \cup N_6$.

Theorem 4.2. *If $\lambda \in N_3 \cup N_6$, then the extremal trajectory $q(t) = \mathrm{Exp}_t(\lambda)$ does not contain conjugate points for $t > 0$.*

Proof. Let $\lambda \in N_3 \cup N_6$. Since the set $N_3 \cup N_6$ belongs to the boundary of the domain N_2, one can construct a continuous curve $\lambda^s : [0,1] \to N$ such that $\lambda^s \in N_2$ for $s \in [0,1)$ and $\lambda^1 = \lambda$.

Consider the family of extremal trajectories $q^s(t) = \operatorname{Exp}_t(\lambda^s)$, $t > 0$, $s \in [0,1]$. It follows from Theorem 4.1 that for $s \in [0,1)$ the trajectory $q^s(t)$ does not contain conjugate points $t > 0$. Then we conclude from Corollary 2.3 that the trajectory $q^1(t) = \operatorname{Exp}_t(\lambda)$ does not contain conjugate points for $t > 0$. □

5. Final remarks

Here we sum up our results of this work and [11] on cut points and conjugate points in the Euler elastic problem.

Given an extremal trajectory $q(t) = \operatorname{Exp}_t(\lambda)$ corresponding to a covector $\lambda \in T^*_{q_0}M = N = \bigcup_{i=1}^{7} N_i$, we obtained the following bounds on the cut time $t_{\mathrm{cut}}(\lambda)$ and the first conjugate time $t_1^{\mathrm{conj}}(\lambda)$ along this trajectory.

Theorem 5.1. (1) *For any $\lambda \in N$, we have $t_{\mathrm{cut}}(\lambda) \le \mathbf{t}(\lambda)$.*

(2) *If $\lambda \in N_2 \cup N_3 \cup N_6$, then $t_1^{\mathrm{conj}}(\lambda) = +\infty$.*

(3) *If $\lambda \in N_1$, then*:

- (3.1) $t_1^{\mathrm{conj}}(\lambda)$ *belongs to the segment bounded by* $\dfrac{4K}{\sqrt{r}}$ *and* $\dfrac{2p_1^1}{\sqrt{r}}$;
- (3.2) $t_1^{\mathrm{conj}}(\lambda) \ge \mathbf{t}(\lambda)$;
- (3.3) $t_1^{\mathrm{conj}}(\lambda) \in \left(\dfrac{T}{2}, \dfrac{3T}{2}\right)$;
- (3.4) *if the corresponding elastica does not contain inflection points, then it is locally optimal*;
- (3.5) *if the corresponding elastica contains at least three inflection points in its interior, then it is not locally optimal.*

Recall that the function $\mathbf{t}(\lambda)$ is defined as follows [11]:

$$\mathbf{t} : N \to (0, +\infty], \quad \lambda \mapsto \mathbf{t}(\lambda),$$

$$\lambda \in N_1 \;\Rightarrow\; \mathbf{t} = \frac{2}{\sqrt{r}} p_1(k),$$

$$p_1(k) = \min(2K(k), p_1^1(k)) = \begin{cases} 2K(k), & k \in (0, k_0], \\ p_1^1(k), & k \in [k_0, 1), \end{cases} \tag{5.1}$$

$$\lambda \in N_2 \;\Rightarrow\; \mathbf{t} = \frac{2k}{\sqrt{r}} p_1(k), \quad p_1(k) = K(k),$$

$$\lambda \in N_6 \;\Rightarrow\; \mathbf{t} = \frac{2\pi}{|c|},$$

$$\lambda \in N_3 \cup N_4 \cup N_5 \cup N_7 \;\Rightarrow\; \mathbf{t} = +\infty,$$

$K(k)$ is the complete elliptic integral of the first kind, $p_1^1(k) \in (K, 3K)$ is the first root of an equation in the Jacobi functions described in [11, Proposition 11.6], and T is the period of oscillation of the pendulum that parametrizes the vertical subsystem (2.13) of the normal Hamiltonian system.

Theorem 5.1 is a compilation of the following results: [11, Theorem 12.1], Theorems 3.3, 4.1, 4.2, and 3.2, and Corollaries 3.2, 3.3, and 3.4 of this work.

Note that the absence of conjugate points on elastic arcs without inflexion points (items (2) and (3.4) of Theorem 5.1) was known already to Max Born [5]; all other results are new.

On the basis of this information about conjugate points and description of Maxwell points obtained in [11], one can study the global structure of the exponential mapping in the Euler elastic problem: describe the domains where the exponential mapping is diffeomorphic and find a precise characterization of cut points. This will be the subject of our forthcoming work [12].

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