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CLASSIFICATION OF CONTROLLABLE SYSTEMS ON LOW-DIMENSIONAL SOLVABLE LIE GROUPS

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ABSTRACT. Right-invariant control systems on simply connected solvable Lie groups are studied. A complete and explicit description of controllable single-input right-invariant systems on such Lie groups up to dimension 6 is obtained.

0. Introduction

The aim of this paper is to study controllability of single-input right-invariant control systems on low-dimensional simply connected solvable Lie groups. It is an immediate continuation of our previous papers 6, 7, in which controllability of such systems on solvable Lie groups and, more generally, Lie groups different from their derived subgroups was studied. We use definitions and results of those papers (especially of the second one) throughout this paper.

A nice introduction to the subject of right-invariant systems on Lie groups is given in the recent book by V. Jurdjevic 3. One may also consult survey 8, which is an attempt to cover all results published so far on this subject (the majority of results of the present paper were announced in this survey).

0.1. Description of the problem and results. Given a Lie algebra L, there is the "largest" connected Lie group G having Lie algebra L, the simply connected one. All other connected Lie groups with Lie algebra L are "smaller" than G in the sense that they are quotients G/C, where C is a discrete subgroup of the center of G. A right-invariant system $\Gamma \subset L$ may thus be considered on any of these groups, and the simply connected group G is the hardest to control among them. Hence, given a right-invariant system Γ on a Lie group (or a homogeneous space of a Lie group) H, it is

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natural first to study its controllability on the simply connected covering $\widetilde{H}$ of H. If Γ is controllable on $\widetilde{H}$, then it is obviously controllable on H (and on all its homogeneous spaces); in the opposite case, one should use particular geometric properties of H (e.g., the existence of periodic one-parameter subgroups) to verify controllability of Γ on H. It is obvious and remarkable that controllability conditions on a simply connected Lie group G should have a completely Lie-algebraic form: they are completely determined by the Lie algebra L and its subset Γ (see, e.g., 1, 5, 6, 7).

This motivates the following definition. Let L be a finite-dimensional real Lie algebra.

Definition 1. A right-invariant system $\Gamma \subset L$ is called *controllable* if it is controllable on the (unique) connected simply connected Lie group with Lie algebra L.

The next definition makes sense at least for solvable Lie algebras in small dimensions.

Definition 2. A Lie algebra L is called *controllable* if there exist elements $A, B \in L$ such that the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable.

Remark. We define controllability of a Lie algebra L in terms of affine lines

$$A + \mathbb{R}B = \big\{A + uB \mid u \in \mathbb{R}\big\} \subset L.$$

It is easy to show that this definition is equivalent to a similar one in terms of affine segments

$$\{(1-u)A + uB \mid u \in [0,\, 1]\} \subset L;$$

see Theorem 8.2.

In this paper, we show that for solvable low-dimensional Lie algebras L, the following takes place:

- existence of a controllable single-input system $\Gamma = A + \mathbb{R}B \subset L$, i.e., controllability of L is a strong restriction on L;
- if L is controllable, then almost all pairs $(A, B) \in L \times L$ give rise to controllable systems $\Gamma = A + \mathbb{R}B$;
- controllability of a system $\Gamma = A + \mathbb{R}B \subset L$ depends primarily on L but not on Γ.

Moreover, these results yield a complete description of controllability in low-dimensional solvable Lie algebras, which is the main result of this work.

Up to dimension 6, we describe all solvable Lie algebras L that are controllable, and give controllability tests for single-input systems $\Gamma = A + \mathbb{R}B \subset L$.

The general "bird's-eye view" of controllable low-dimensional solvable Lie algebras is as follows:

$\dim L = 1$: the (unique) Lie algebra is controllable;
$\dim L = 2$: the two Lie algebras are noncontrollable;
$\dim L = 3$: there is one family of controllable Lie algebras $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 4$: there is one family of controllable Lie algebras $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 5$: there are two families of controllable Lie algebras:
(1) $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
(2) $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 6$: there are six families and, in addition, two controllable Lie algebras:
(1) $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
(2) $L_{6,II}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$,
(3) $L_{6,III}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
(4) $L_{6,IV}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$,
(5) $L_{6,V}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
(6) $L_{6,VI}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
(7) $L_{6,VII}$,
(8) $L_{6,VIII}$.

All these Lie algebras L have codimension one derived subalgebras $L^{(1)}$, and the complex parameters λ and μ are eigenvalues of the operators $\operatorname{ad} x|_{L^{(1)}}$, $x \in L \setminus L^{(1)}$. The Lie algebras in distinct families are nonisomorphic. Inside each family, the Lie algebras are isomorphic iff the corresponding sets $\{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}$ (or $\{\lambda, \bar{\lambda}\}$) are homothetic in $\mathbb{C}$ (for the family $L_{6,I}(\lambda, \mu)$, the corresponding sets $\{\lambda, \bar{\lambda}\}$ and $\{\mu, \bar{\mu}\}$ should be homothetic with the same coefficient).

For all controllable low-dimensional solvable Lie algebras, we obtain the following general result.

Theorem 7.1. *Let L be a controllable solvable Lie algebra,* $\dim L \leq 6$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*
(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:
   (1) $B \notin L^{(1)}$;
   (2) $\operatorname{Lie}(A, B) = L$.
(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

0.2. Structure of the paper. We obtain the description of controllable solvable Lie algebras and controllability conditions for systems of the form $\Gamma = A + \mathbb{R}B$ for dimensions 1–6 in Secs. 1–6, respectively.

In Sec. 7, we summarize these results in the form valid for all dimensions from 1 to 6.

While Secs. 1–7 are devoted to controllability of the lines $\Gamma = A + \mathbb{R}B$, in Sec. 8, we study controllability of the segments $S = \{(1-u)A + uB \mid u \in [0, 1]\}$. We relate controllability of segments with controllability of lines and give a general controllability test for segments in solvable Lie algebras in terms of half-planes containing the angles generated as cones by the segments.

In Sec. 9, we suggest several final remarks that might be helpful for the further study of controllability of right-invariant systems.

Finally, in the Appendix, we collect and prove some necessary auxiliary propositions.

0.3. Known facts. The main tools in the subsequent study of controllability are the results of 6, 7.

In addition to them, we also apply the following nice description of controllable right-invariant systems on solvable Lie groups (in the simply connected case, this description provides a controllability test). This controllability condition is applicable to Lie groups with cocompact radical, i.e., for Lie groups G such that the quotient $G/\operatorname{Rad} G$ modulo the maximal solvable normal subgroup $\operatorname{Rad} G$ is compact. In particular, this result applies to solvable Lie groups, for which $G = \operatorname{Rad} G$.

Proposition 1 (J. D. Lawson 5). *Assume that $G/\operatorname{Rad} G$ is compact; let $\Gamma \subset L$ be a right-invariant system that satisfies the rank condition $\operatorname{Lie}(\Gamma) = L$. If Γ is not contained in any half-space of L with boundary being a subalgebra, then Γ is controllable on the connected Lie group G. The converse holds if G is simply connected.*

1. One-dimensional Lie algebra

The unique one-dimensional Lie algebra is Abelian and isomorphic to $\mathbb{R}$.

Theorem 1.1. *The one-dimensional Lie algebra $\mathbb{R}$ is controllable.*
A system $\Gamma = A + \mathbb{R}B \subset \mathbb{R}$ is controllable if and only if $B \neq 0$.

Proof. The statement of the theorem is obvious. □

2. Two-dimensional Lie algebras

There are two nonisomorphic two-dimensional Lie algebras: Abelian $\mathbb{R}^2$, and solvable non-Abelian $S_2 = \operatorname{span}(x, y)$, $[x, y] = y$.

Theorem 2.1. *Both two-dimensional Lie algebras $\mathbb{R}^2$ and S_2 are not controllable.*

Proof. Both Lie algebras $L = \mathbb{R}^2$ and S_2 are completely solvable, i.e., all operators $\operatorname{ad} x$, $x \in L$, have real spectra. By Theorem 2 of 6, a completely solvable Lie algebra L is not controllable if $\dim L \geq 1$. □

3. Three-dimensional Lie algebras

3.1. Construction of controllable Lie algebras.

Construction 3.1. The Lie algebra $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 1.

$$L_3(\lambda) = \operatorname{span}(x, y, z),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z)} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \lambda = a + bi.$$

In other words, the following commutation relations hold in the Lie algebra $L_3(\lambda)$:

$$[x, y] = ay - bz, \quad [x, z] = by + az.$$

All other brackets of the base elements x, y, and z either are determined from these ones by skew-symmetry: $[y, x] = -ay + bz$, $[z, x] = -by - az$,or are zero: $[y, z] = [z, y] = 0$. We use such descriptions of multiplication in low-dimensional solvable Lie algebras in what follows.

Remark. The Jacobi identity for the (only) triple of base elements (x, y, z) holds; thus, $L_3(\lambda)$ is a Lie algebra. A similar argument with the Jacobi identity for all triples of base elements shows that all controllable Lie algebras defined in Secs. 4–6 are also realizable.

The Lie algebra $L_3(\lambda)$ is schematically represented in Fig. 1 by the eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$ and realifications of the eigenvectors y, $z \in L_3(\lambda)$ of the operator $\operatorname{ad} x|_{\operatorname{span}(y,z)}$.

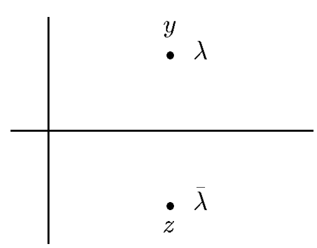

Fig. 1. $L_3(\lambda)$.

3.2. Controllability conditions.

Theorem 3.1. *Let $L = L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*
(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold:*
  (1) $B \notin L^{(1)}$;
  (2) $\operatorname{Lie}(A, B) = L$, *or*
  (2′) *the vectors A and B are linearly independent, or*
  (2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A) = L$.
(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Remark. In statement (b), we assert that controllability of the system Γ is equivalent to any one of the following (mutually equivalent) conditions: (1) & (2), or (1) & (2′), or (1) & (2″). We use such a convention in similar theorems for higher dimensions below.

Theorem 3.2. *A three-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

3.3. Proof of controllability conditions.

3.3.1. *Lie algebra $L_3(\lambda)$: Theorem* 3.1.

Proof. **Statement** (b), (1) & (2′). Sufficiency. We show that all hypotheses of Corollary 3 7 hold.

Conditions (1) and (2) obviously hold.

Condition (3). Consider the decomposition $B = B_x x + B_y y + B_z z$. By Lemma 10.2, we have

$$\operatorname{Sp}^{(1)} = \operatorname{Sp}\big(\operatorname{ad} B|_{L^{(1)}}\big) = B_x \cdot \operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = B_x \cdot \{\lambda, \bar{\lambda}\}.$$

The condition $B \notin L^{(1)}$ is equivalent to $B_x \neq 0$; thus, the spectrum $\operatorname{Sp}^{(1)}$ is simple.

Condition (4): $\operatorname{Sp}_r^{(2)} = \operatorname{Sp}_r^{(1)} = \emptyset$.

Condition (5), $A(a) \neq 0$ for all $a \in \operatorname{Sp}_c^{(1)}$, means that the vector A has a nonzero projection onto $L^{(1)}$ along the line $\mathbb{R}B$, i.e., A and B are linearly independent.

Condition (6): $\operatorname{Sp}_r^{(1)} = \emptyset$.

Now it follows from Corollary 3 in 7 that the system Γ is controllable.

Necessity follows from items (2) and (5) of Corollary 1 in 7.

Statement (b), (1) & (2): we prove that (2) $\Leftrightarrow$ (2′) under condition (1).

(2) $\Rightarrow$ (2′). If (2′) is violated, then (2) is also violated by Lemma 3.5 in 7.

(2) $\Leftarrow$ (2′). If (2) is violated, then Γ is not controllable by the rank condition; thus, (2′) is also violated.

Statement (b), (1) & (2″) follow from Lemma 10.4.

Statement (c) follows from statement (b), (1) & (2′).

Statement (a) follows from statement (c) and Lemma 10.5. □

3.3.2. *Controllable Lie algebras*: *Theorem* 3.2.

Proof. Necessity. Let L be a solvable three-dimensional Lie algebra, and let $\Gamma = A + \mathbb{R}B \subset L$ be a controllable system. By Theorem 1 in 7, $\dim L^{(1)} = 2$, $B \notin L^{(1)}$, and $\mathrm{Sp}_r^{(1)} = \mathrm{Sp}_r^{(2)}$. The derived subalgebra $L^{(1)}$ is nilpotent and two dimensional; thus, it is Abelian. Consequently, $L^{(2)} = \{0\}$, and hence $\mathrm{Sp}_r^{(1)} = \mathrm{Sp}_r^{(2)} = \emptyset$. Thus,

$$\mathrm{Sp}^{(1)} = \mathrm{Sp}\big(\mathrm{ad}\, B|_{L^{(1)}}\big) = \{\lambda, \bar{\lambda}\}, \quad \lambda = a + ib \in \mathbb{C} \setminus \mathbb{R}.$$

Then there exists a basis y, z of the plane $L^{(1)}$ such that

$$[B, y] = ay - bz \quad \text{and} \quad [B, z] = by + az.$$

Taking into account that $L^{(1)}$ is Abelian, we obtain that $L = L_3(\lambda)$; it remains to set $x = B$.

Sufficiency is an obvious consequence of Theorem 3.1 (c). □

3.4. Isomorphisms of controllable Lie algebras. We say that a set $S_1 \subset \mathbb{C}$ is *homothetic* to a set $S_2 \subset \mathbb{C}$ if $S_2 = k \cdot S_1$ for some number $k \in \mathbb{R} \setminus \{0\}$. We denote this by $S_1 \sim S_2$.

Theorem 3.3. *Lie algebras $L_3(\lambda_1)$ and $L_3(\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, are isomorphic if and only if $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.*

Proof. Necessity follows from Lemma 10.2.

Sufficiency. The following two cases are possible:

(1) $\lambda_2 = k\lambda_1$, $k \in \mathbb{R} \setminus \{0\}$;
(2) $\lambda_2 = k\bar{\lambda}_1$, $k \in \mathbb{R} \setminus \{0\}$.

In these cases, the required isomorphism $L_3(\lambda_2) \to L_3(\lambda_1)$ is defined on canonical bases $L_3(\lambda_i) = \mathrm{span}(x_i, y_i, z_i)$, $i = 1, 2$, given in Construction 3.1 as follows:

(1) $x_2 \mapsto kx_1$, $y_2 \mapsto y_1$, $z_2 \mapsto z_1$;
(2) $x_2 \mapsto kx_1$, $y_2 \mapsto z_1$, $z_2 \mapsto y_1$. □

4. Four-dimensional Lie algebras

4.1. Construction of controllable Lie algebras.

Construction 4.1. The Lie algebra $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 2.

$$L_4(\lambda) = \operatorname{span}(x, y, z, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,w)} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y, z] = w.$$

The arrows in the schematic representation of the Lie algebra $L_4(\lambda)$ in Fig. 2 mean that Lie bracket of the vectors y and z gives the vector w.

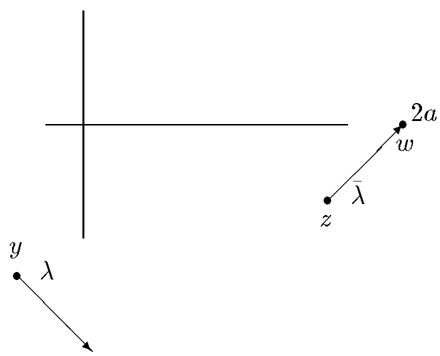

Fig. 2. $L_4(\lambda)$, $\operatorname{Re} \lambda = a$.

In the sequel, we consider the following decomposition for a vector $B \in L_4(\lambda)$:

$$B = B_x x + B_y y + B_z z + B_w w.$$

4.2. Controllability conditions.

Theorem 4.1. *Let $L = L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold:*

(1) $B \notin L^{(1)}$;

(2) $\operatorname{Lie}(A, B) = L$, *or*

(2′) $A(B_x\lambda) \neq 0$, *or*
(2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A, w) = L$.

(c) *Let* $B \in L \setminus L^{(1)}$. *Then the system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable for almost all* $A \in L$.

Theorem 4.2. *A four-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to* $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

4.3. Proof of controllability conditions.

4.3.1. *Lie algebra* $L_4(\lambda)$: *Theorem* 4.1.

Proof. First, we prove the theorem for the case $a = \operatorname{Re}\lambda \neq 0$.

Statement (b), (1) & (2′). Sufficiency follows from Corollary 3 in 7. Necessity follows from Corollary 1 in 7.

Statement (b), (1) & (2″): we show that (2′) ⇔ (2″) under condition (1). The line $I = \mathbb{R}w$ is an ideal in L. Consider the quotient Lie algebra

$$\tilde{L} = L/I = \operatorname{span}(\tilde{x}, \tilde{y}, \tilde{z}) \simeq L_3(\lambda). \tag{1}$$

(Here and below the sign of tilde denotes the passage to cosets.) Further, in view of Theorem 3.1, we obtain the chain of equivalent conditions

$$(2') \Leftrightarrow \tilde{A}(B_x\lambda) \neq 0 \Leftrightarrow \operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad}\tilde{B})\tilde{A}) = \tilde{L} \Leftrightarrow (2'').$$

The remainder of the proof: **statement** (b), (1) & (2), **statement** (c), and **statement** (a) follow as in Theorem 3.1.

Now we consider the case $a = \operatorname{Re}\lambda = 0$.

Statement (a). On the contrary, let $l \subset L$, $l \neq L^{(1)}$, be a codimension one subalgebra. Consider the quotient Lie algebra (1) and the image $\tilde{l} \subset \tilde{L}$ of the Lie algebra l. Obviously, $\dim\tilde{l}$ can be equal to 2 or 3.

Let $\dim\tilde{l} = 2$. By Theorem 3.1, we have $\tilde{l} = L_3^{(1)}(\lambda) = \operatorname{span}(\tilde{y}, \tilde{z})$. Then the Lie algebra l contains elements of the form

$$f^1 = y + f^1_w w, \quad f^1_w \in \mathbb{R},$$
$$f^2 = z + f^2_w w, \quad f^2_w \in \mathbb{R}.$$

Then $[f^1, f^2] = w \in l$; thus, $y, z \in l$. Therefore, $l = L^{(1)}$, a contradiction.

If $\dim\tilde{l} = 3$, then $\tilde{l} = \tilde{L}$, and we obtain $l = L$ by an argument similar to the previous one.

Statement (b), (1) & (2) follows from statement (a) and Proposition 1.

Statement (b), (1) & (2′). We prove that (2) ⇔ (2′) under condition (1).

(2) ⇒ (2′). If (2′) is violated, then (2) is violated by Lemma 3.5 in 7.

(2) $\Leftarrow$ (2′). Consider the quotient Lie algebra (1). In view of Theorem 3.1, we have the chain

$$(2') \Rightarrow \tilde{A}(B_x \lambda) \neq 0 \Rightarrow \mathrm{Lie}(\tilde{A}, \tilde{B}) = \tilde{L}.$$

Therefore, the Lie algebra $l = \mathrm{Lie}(A, B)$ contains elements of the form

$$\begin{aligned} f^1 &= y + f_w^1 w, & f_w^1 &\in \mathbb{R}; \\ f^2 &= z + f_w^2 w, & f_w^2 &\in \mathbb{R}; \\ f^3 &= x + f_w^3 w, & f_w^3 &\in \mathbb{R}. \end{aligned}$$

Then $[f^1, f^2] = w \in l$; consequently, $l = L$.

Statement (b), (1) & (2″) follows exactly as in the case $a \neq 0$.

Statement (c) follows from statement (b), (1) & (2′). □

4.3.2. *Controllable Lie algebras*: *Theorem* 4.2.

Proof. Necessity. Let L be a controllable solvable four-dimensional Lie algebra, and let $\Gamma = A + \mathbb{R}B \subset L$ be a controllable system. By Theorem 1 in 7, $\dim L^{(1)} = 3$, $B \notin L^{(1)}$, and $L_r^{(2)} = L_r^{(1)}$.

If $\mathrm{Sp}^{(1)} \subset \mathbb{R}$, then Lemma 10.3 yields $L_r^{(1)} \neq L_r^{(2)}$, which contradicts condition (3) of Theorem 1 in 7. Consequently, the operator $\mathrm{ad}\, B|_{L^{(1)}}$ has two complex and one real eigenvalue:

$$\mathrm{Sp}^{(1)} = \{a \pm bi,\ c\}, \quad a, b, c \in \mathbb{R}, \quad b \neq 0.$$

We can choose vectors $y, z, w \in L^{(1)}$ in such a way that

$$L^{(1)} = \mathrm{span}(y, z, w); \tag{2}$$

$$\mathrm{ad}\, B|_{\mathrm{span}(y,z,w)} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{3}$$

We have

$$\mathbb{R}w = L^{(1)}(c) = L_r^{(1)} = L_r^{(2)}. \tag{4}$$

The dimension of the space $L^{(2)}/L_r^{(2)}$ is even; thus, $\dim L^{(2)}/L_r^{(2)} = 2$ or 0. If $\dim L^{(2)}/L_r^{(2)} = 2$, then $\dim L^{(2)} = 3$, i.e., $L^{(2)} = L^{(1)}$, which contradicts to the nilpotency of $L^{(1)}$. Consequently, $L^{(2)} = L_r^{(2)}$. Taking into account (4), we obtain

$$\mathbb{R}w = L^{(2)}.$$

Thus, all the brackets $[y, w]$, $[z, w]$, and $[y, z]$ should have the form kw, $k \in \mathbb{R}$, and for at least one bracket, the coefficient k should be nonzero. The relation $[y, w] = kw$, $k \neq 0$, is impossible, since the operator $\mathrm{ad}\, y|_{L^{(1)}}$

is nilpotent (which follows from solvability of the Lie algebra L). Similarly, the relation $[z,w] = kw$, $k \neq 0$, is also impossible. Thus,

$$[y,w] = [z,w] = 0 \tag{5}$$

and $[y,z] = kw$, $k \neq 0$. We denote the vector kw by w and obtain

$$[y,z] = w. \tag{6}$$

Further, $y, z \in L^{(1)}(a+bi)$ and $w \in L^{(1)}(c)$, thus, $w = [y,z] \in L^{(1)}(2a) = L^{(1)}(c)$; consequently,

$$c = 2a. \tag{7}$$

Now we set $x = B$, take into account (2)–(7), and see that $L = L_4(\lambda)$, $\lambda = a + bi$. Necessity is proved.

Sufficiency follows from Theorem 4.1, statement (c). □

4.4. Isomorphisms of controllable Lie algebras.

Theorem 4.3. *Lie algebras $L_4(\lambda_1)$ and $L_4(\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, are isomorphic if and only if $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.*

Proof. Necessity. By Lemma 10.2, $\{\lambda_1, \bar{\lambda}_1, 2\operatorname{Re}\lambda_1\} \sim \{\lambda_2, \bar{\lambda}_2, 2\operatorname{Re}\lambda_2\}$; thus, $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.

Sufficiency is proved exactly as in Theorem 3.3. □

5. Five-dimensional Lie algebras

5.1. Construction of controllable Lie algebras.

Construction 5.1. The Lie algebra $L_{5,I}(\lambda,\mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 3.

$$L_{5,I}(\lambda,\mu) = \operatorname{span}(x,y,z,u,v);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{pmatrix}, \quad \lambda = a + bi, \quad \mu = c + di.$$

Construction 5.2. The Lie algebra $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 4.

$$L_{5,II}(\lambda) = \operatorname{span}(x,y,z,u,v);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 1 & 0 & a & b \\ 0 & 1 & -b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The circles around the eigenvalues λ, $\bar{\lambda}$ in Fig. 4 mean that they have algebraic multiplicity two. (Note that their geometric multiplicity is one.)

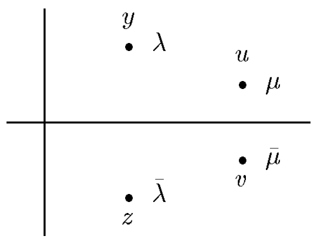

Fig. 3. $L_{5,I}(\lambda, \mu)$.

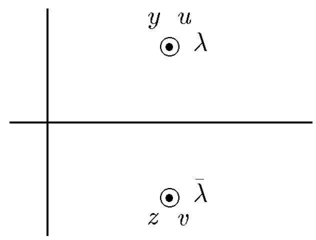

Fig. 4. $L_{5,II}(\lambda)$.

For an element B of the Lie algebras $L_{5,I}(\lambda, \mu)$ or $L_{5,II}(\lambda)$, we consider the following decomposition with respect to the base elements:

$$B = B_x x + B_y y + B_z z + B_u u + B_v v.$$

5.2. Controllability conditions.

Theorem 5.1. *Let $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:

(1) $B \notin L^{(1)}$;

(2) $\mathrm{Lie}(A, B) = L$, *or*

(2′) $A(B_x\lambda) \neq 0$ *and* $A(B_x\mu) \neq 0$, *or*

(2″) $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, (\mathrm{ad}\, B)^2 A, (\mathrm{ad}\, B)^3 A) = L$.

(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Theorem 5.2. *Let $L = L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:

(1) $B \notin L^{(1)}$;

(2) $\mathrm{Lie}(A, B) = L$, *or*

(2′) $\operatorname{top}(A, B_x\lambda) \neq 0$, *or*
(2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A, (\operatorname{ad} B)^2 A, (\operatorname{ad} B)^3 A) = L$.

(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Remark. The notation $\operatorname{top}(A, B_x\lambda) \neq 0$ in Theorem 5.2 (and in Theorem 6.5 below) means that the vector A has a nonzero component in the higher order root space of the operator $\operatorname{ad}_c B|_{L_c^{(1)}}$ corresponding to its eigenvalue $B_x\lambda$, see Definition 2 in 7.

Theorem 5.3. *A five-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, or $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

5.3. Proof of controllability conditions.

5.3.1. *Lie algebra $L_{5,I}(\lambda, \mu)$: Theorem* 5.1.

Proof. **Statement** (b), (1) & (2′). The Lie algebra $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, is meta-Abelian. A controllability test for simply connected meta-Abelian Lie groups is provided by Theorem 3 in 7. By this theorem, controllability of a system $\Gamma = A + \mathbb{R}B \subset L$ is equivalent to the following conditions:

(1) $B \notin L^{(1)}$;
(2) $\operatorname{top}(A, a) \neq 0$ for all $a \in \operatorname{Sp}_c^{(1)}$;

the other three conditions of Theorem 3 in 7 are satisfied for $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$.

We have

$$\operatorname{Sp}_c^{(1)} = \operatorname{Sp}^{(1)} = B_x \cdot \{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}.$$

Each of the conditions $\operatorname{top}(A, B_x\lambda) \neq 0$ and $\operatorname{top}(A, B_x\bar{\lambda}) \neq 0$ is equivalent to the inequality $A(B_x\lambda) \neq 0$; similarly, each of the conditions $\operatorname{top}(A, B_x\mu) \neq 0$ and $\operatorname{top}(A, B_x\bar{\mu}) \neq 0$ is equivalent to the relation $A(B_x\mu) \neq 0$.

The remainder of the proof: **statement** (b), (1) & (2) and (1) & (2″), **statement** (c), and **statement** (a) follow as in Theorem 3.1. □

5.3.2. *Lie algebra $L_{5,II}(\lambda)$: Theorem* 5.2.

Proof. The proof of this theorem is completely similar to that of Theorem 5.1. □

5.3.3. *Controllable Lie algebras*: *Theorem* 5.3.

Proof. Necessity. Let $\Gamma = A + \mathbb{R}B \subset L$ be a controllable system in a solvable five-dimensional Lie algebra L.

Theorem 1 in 7 implies the following:

$$\begin{aligned} &\dim L^{(1)} = 4; \\ &B \notin L^{(1)}; \\ &L_r^{(1)} = L_r^{(2)}. \end{aligned} \tag{8}$$

The operator $\operatorname{ad} B|_{L^{(1)}}$ has 4 eigenvalues. Now we consider their possible location in the complex plane and show that the controllability assumption leads necessarily to the cases indicated in this theorem: $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, or $L = L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(a) Four real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. In this case,

$$\operatorname{Sp}^{(1)} = \operatorname{Sp}(\operatorname{ad} B|_{L^{(1)}}) \subset \mathbb{R};$$

this is impossible, since Lemma 10.3 gives $L_r^{(1)} \neq L_r^{(2)}$, a contradiction to (8).

(b) Two complex eigenvalues and two real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. Now let

$$\operatorname{Sp}^{(1)} = \{\lambda, \bar{\lambda}, c, d\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad c, d \in \mathbb{R}.$$

We have the following decomposition of the derived subalgebra into invariant subspaces of the operator $\operatorname{ad} B$:

$$L^{(1)} = L^{(1)}(\lambda) + L^{(1)}(c) + L^{(1)}(d). \tag{9}$$

If $c \neq d$, then the vector spaces $L^{(1)}(c)$ and $L^{(1)}(d)$ are one dimensional and form a direct sum; if $c = d$, then the space $L^{(1)}(c) = L^{(1)}(d)$ is two dimensional. In both cases, the space

$$l = L^{(1)}(c) + L^{(1)}(d)$$

is two dimensional.

We commute relation (9) with itself and obtain

$$\begin{aligned} L^{(2)} = &[L^{(1)}(\lambda), L^{(1)}(\lambda)] + [L^{(1)}(\lambda), L^{(1)}(c)] + \\ &+ [L^{(1)}(\lambda), L^{(1)}(d)] + [L^{(1)}(c), L^{(1)}(d)]. \end{aligned} \tag{10}$$

The space l is a Lie algebra, since

$$[L^{(1)}(c), L^{(1)}(d)] \subset L^{(1)}(c+d) \subset L_r^{(1)} = l.$$

The Lie algebra l is nilpotent (as a subalgebra of the nilpotent Lie algebra $L^{(1)}$) and two dimensional; therefore, it is Abelian. Then for the summands of (10) we have

$$\begin{aligned}
&[L^{(1)}(\lambda), L^{(1)}(\lambda)] \subset L^{(1)}(2\operatorname{Re}\lambda) \cap L_r^{(2)};\\
&[L^{(1)}(\lambda), L^{(1)}(c)] \subset L^{(1)}(\lambda + c) \cap L_c^{(2)};\\
&[L^{(1)}(\lambda), L^{(1)}(d)] \subset L^{(1)}(\lambda + d) \cap L_c^{(2)};\\
&[L^{(1)}(c), L^{(1)}(d)] = 0.
\end{aligned}$$

Therefore, all the spaces $[L^{(1)}(\lambda), L^{(1)}(c)]$, $[L^{(1)}(\lambda), L^{(1)}(d)]$, and $[L^{(1)}(c), L^{(1)}(c)]$ have zero intersection with $L_r^{(2)}$, and $L_r^{(2)} = [L^{(1)}(\lambda), L^{(1)}(\lambda)]$; thus

$$\dim L_r^{(2)} = \dim[L^{(1)}(\lambda), L^{(1)}(\lambda)] \leq 1,$$

which contradicts to $L_r^{(2)} = L_r^{(1)} = l$ and $\dim l = 2$. Case (b) is impossible.

(c) Four distinct complex eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. Now let

$$\mathrm{Sp}^{(1)} = \{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda \neq \mu, \bar{\mu}.$$

Denote $\lambda = a + bi$ and $\mu = c + di$, $b, d \neq 0$.

Choose a basis y, z, u, v in $L^{(1)}$ in which the adjoint operator has the matrix

$$\operatorname{ad} B|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & b & 0 & 0\\ -b & a & 0 & 0\\ 0 & 0 & c & d\\ 0 & 0 & -d & c \end{pmatrix}.$$

Now it remains to prove that the Lie algebra $L^{(1)}$ is Abelian and to set $x = B$: then $L = L_{5,I}(\lambda, \mu)$.

In order to prove that $L^{(1)}$ is Abelian, we consider its complexification

$$L_c^{(1)} = L_c^{(1)}(\lambda) \oplus L_c^{(1)}(\bar{\lambda}) \oplus L_c^{(1)}(\mu) \oplus L_c^{(1)}(\bar{\mu})$$

and the corresponding basis

$$\begin{aligned}
&L_c^{(1)} = \operatorname{span}(e_\lambda, e_{\bar{\lambda}}, e_\mu, e_{\bar{\mu}}),\\
&L_c^{(1)}(\lambda) = \mathbb{C}e_\lambda,\ L_c^{(1)}(\bar{\lambda}) = \mathbb{C}e_{\bar{\lambda}},\ L_c^{(1)}(\mu) = \mathbb{C}e_\mu,\ L_c^{(1)}(\bar{\mu}) = \mathbb{C}e_{\bar{\mu}},\\
&\overline{e_\lambda} = e_{\bar{\lambda}}, \quad \overline{e_\mu} = e_{\bar{\mu}}.
\end{aligned}$$

We show that $L_c^{(1)}$ is Abelian. We have

$$\begin{aligned}
&[e_\lambda, e_{\bar{\lambda}}] \in L_c^{(1)}(\lambda + \bar{\lambda}) = L_c^{(1)}(2\operatorname{Re}\lambda) = \{0\};\\
&[e_\mu, e_{\bar{\mu}}] \in L_c^{(1)}(\mu + \bar{\mu}) = L_c^{(1)}(2\operatorname{Re}\mu) = \{0\};
\end{aligned}$$

consequently,

$$[e_\lambda, e_{\bar{\lambda}}] = [e_\mu, e_{\bar{\mu}}] = 0.$$

Further,

$$[e_\lambda, e_\mu] \in L_c^{(1)}(\lambda+\mu);$$
$$[e_\lambda, e_{\bar\mu}] \in L_c^{(1)}(\lambda+\bar\mu).$$

The eigenspace $L_c^{(1)}(\lambda+\mu) \neq \{0\}$ iff $\lambda+\mu$ equals one of the eigenvalues λ, $\bar\lambda$, μ, and $\bar\mu$. But $\lambda+\mu \neq \lambda, \mu$, since $\mu, \lambda \neq 0$, respectively. Further,

$$\lambda+\mu = \bar\lambda \;\Leftrightarrow\; \mu = \bar\lambda - \lambda = -2bi;$$
$$\lambda+\mu = \bar\mu \;\Leftrightarrow\; \lambda = \bar\mu - \mu = -2di.$$

Consequently,

$$L_c^{(1)}(\lambda+\mu) \neq \{0\} \;\Leftrightarrow\; \text{ either } \mu = -2bi \text{ or } \lambda = -2di.$$

Similarly,

$$L_c^{(1)}(\lambda+\bar\mu) \neq \{0\} \;\Leftrightarrow\; \text{ either } \mu = 2bi \text{ or } \lambda = 2di.$$

Consider the case $\mu = -2bi$ (the other three cases, $\lambda = -2di$, $\mu = 2bi$, and $\lambda = 2di$, are considered similarly). We have

$$[e_\lambda, e_\mu] \in L_c^{(2)}(\lambda+\mu) = L_c^{(1)}(\bar\lambda) = \mathbb{C}e_{\bar\lambda};$$
$$[e_\lambda, e_{\bar\mu}] \in L_c^{(2)}(\lambda+\bar\mu) = \{0\};$$
$$[e_{\bar\lambda}, e_\mu] \in L_c^{(2)}(\bar\lambda+\mu) = \{0\};$$
$$[e_{\bar\lambda}, e_{\bar\mu}] \in L_c^{(2)}(\bar\lambda+\bar\mu) = L_c^{(1)}(\lambda) = \mathbb{C}e_\lambda;$$

hence,

$$[e_\lambda, e_\mu] = ke_{\bar\lambda}, \quad k \in \mathbb{C},$$
$$[e_{\bar\lambda}, e_{\bar\mu}] = \bar k e_\lambda.$$

Suppose that $k \neq 0$. Then

$$[-(1/k)e_\mu, e_\lambda] = e_{\bar\lambda};$$
$$[-(1/\bar k)e_{\bar\mu}, e_{\bar\lambda}] = e_\lambda;$$

thus,

$$[-(1/k)e_\mu - (1/\bar k)e_{\bar\mu}, e_\lambda + e_{\bar\lambda}] = e_\lambda + e_{\bar\lambda},$$

i.e., $e_\lambda + e_{\bar\lambda}$ is an eigenvector of the operator $\operatorname{ad}(-(1/k)e_\mu - (1/\bar k)e_{\bar\mu})|_{L^{(1)}}$ with the eigenvalue 1. But this operator is nilpotent, since L is solvable. The contradiction shows that $k = 0$. The Lie algebra $L_c^{(1)}$ is Abelian; thus, $L^{(1)}$ is Abelian as well.

We set $x = B$ and obtain $L = L_{5,I}(\lambda, \mu)$ in case (c).

(d) Two complex eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. Finally, let

$$\mathrm{Sp}^{(1)} = \{\lambda, \bar\lambda\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Choose a basis

$$\{e^1_\lambda, e^2_\lambda, e^1_{\bar\lambda}, e^2_{\bar\lambda}\} \tag{11}$$

in $L_c^{(1)}$ such that

$$\operatorname{ad}_c B|_{\operatorname{span}(e^1_\lambda, e^2_\lambda, e^1_{\bar\lambda}, e^2_{\bar\lambda})} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \varepsilon & \lambda & 0 & 0 \\ 0 & 0 & \bar\lambda & 0 \\ 0 & 0 & \varepsilon & \bar\lambda \end{pmatrix}, \tag{12}$$

$$\varepsilon = 0 \text{ or } 1,$$
$$\overline{e^1_\lambda} = e^1_{\bar\lambda}, \quad \overline{e^2_\lambda} = e^2_{\bar\lambda}. \tag{13}$$

In order to show that $L_c^{(1)}$ is Abelian, we prove that all brackets for the base elements (11) are zero. First,

$$[e^1_\lambda, e^2_\lambda] \in L_c^{(1)}(2\lambda) = \{0\} \quad \Rightarrow \quad [e^1_\lambda, e^2_\lambda] = 0.$$

Taking into account (13), we obtain

$$[e^1_{\bar\lambda}, e^2_{\bar\lambda}] = 0.$$

Similarly,

$$[e^1_\lambda, e^1_{\bar\lambda}] = [e^2_\lambda, e^2_{\bar\lambda}] = 0.$$

Thus, the Lie algebra $L_c^{(1)}$ is Abelian, as well as $L^{(1)}$.

(d.1) Consider first the case where the operator $\operatorname{ad} B|_{L^{(1)}}$ is diagonalizable over $\mathbb{C}$, i.e., $\varepsilon = 0$ in (12). By the Abelian property of $L^{(1)}$, we have $L^{(2)} = \{0\}$; thus, $L^{(1)}/L^{(2)} = L^{(1)}$, and the quotient operator $\widetilde{\operatorname{ad} B}$: $L^{(1)}/L^{(2)} \to L^{(1)}/L^{(2)}$ has geometric multiplicity 2. That is, $\mathrm{j}(\lambda) = 2$ (see Definition 1 in 7). But this contradicts condition (6) of Theorem 1 in 7: $\mathrm{j}(a) \le 1$ for all $a \in \mathrm{Sp}^{(1)}$. This contradiction implies that case (d.1) is impossible.

(d.2) Therefore, the matrix of the operator $\operatorname{ad}_c B|_{L_c^{(1)}}$ should have Jordan blocks, i.e., $\varepsilon = 1$ in (12). We can find a real basis $\{y, z, u, v\}$ in $L^{(1)}$ that corresponds to the complex basis (11) in which the operator $\operatorname{ad} B|_{L^{(1)}}$ has the matrix

$$\operatorname{ad} B|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 1 & 0 & a & b \\ 0 & 1 & -b & a \end{pmatrix}.$$

Now we set $x = B$ and obtain $L = L_{5,II}(\lambda)$ in case (d.2). The proof of necessity in Theorem 5.3 is complete.

Sufficiency follows from Theorems 5.1, 5.2, (c). □

5.4. Isomorphisms of controllable Lie algebras.

Theorem 5.4. *Any two Lie algebras* $L_{5,I}(\lambda_1, \mu_1)$, $\lambda_1, \mu_1 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1 \neq \mu_1, \bar{\mu}_1$, *and* $L_{5,II}(\lambda_2)$, $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, *are nonisomorphic. All isomorphisms inside these classes are as follows*:

(1) $L_{5,I}(\lambda_1, \mu_1) \simeq L_{5,I}(\lambda_2, \mu_2)$, $\lambda_i, \mu_i \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_i \neq \mu_i, \bar{\mu}_i$, $i = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\, \mu_1, \bar{\mu}_1\} \sim \{\lambda_2, \bar{\lambda}_2\, \mu_2, \bar{\mu}_2\}$;

(2) $L_{5,II}(\lambda_1) \simeq L_{5,II}(\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.

Proof. Lie algebras $L_{5,I}(\lambda_1, \mu_1)$ and $L_{5,II}(\lambda_2)$ are nonisomorphic, since the corresponding sets $\{\lambda_1, \bar{\lambda}_1\, \mu_1, \bar{\mu}_1\}$ and $\{\lambda_2, \bar{\lambda}_2\}$ cannot be homothetic.

Statements (1) and (2) are proved exactly as in Theorem 4.3. □

6. Six-dimensional Lie algebras

6.1. Construction of controllable Lie algebras.

Construction 6.1. The Lie algebra $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 5.

$$L_{6,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v, w);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & -d & c & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix};$$

$$\lambda = a + bi, \ \mu = c + di;$$

$$[y, z] = w.$$

Construction 6.2. The Lie algebra $L_{6,II}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$; see Fig. 6.

$$L_{6,II}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v, w);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & a & d & 0 \\ 0 & 0 & -d & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix};$$

$$\lambda = a + bi, \ \mu = a + di;$$

$$[y, z] = w, \quad [u, v] = w.$$

Construction 6.3. The Lie algebra $L_{6,III}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 7.

$$L_{6,III}(\lambda) = \operatorname{span}(x, y, z, u, v, w);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & 3a & b & 0 \\ 0 & 0 & -b & 3a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix};$$

$$\lambda = a + bi;$$

$$[y, z] = w, \quad [w, y] = u, \quad [w, z] = v.$$

Construction 6.4. The Lie algebra $L_{6,IV}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$; see Fig. 8.

$$L_{6,IV}(\lambda) = \operatorname{span}(x, y, z, u, v, w);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & -a & b & 0 \\ 0 & 0 & -b & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda = a + bi;$$

$$[y, v] = -[z, u] = w.$$

Construction 6.5. The Lie algebra $L_{6,V}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 9.

$$L_{6,V}(\lambda) = \operatorname{span}(x, y, z, u, v, w);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 1 & 0 & a & b & 0 \\ 0 & 1 & -b & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix};$$

$$\lambda = a + bi;$$

$$[y, z] = w.$$

Construction 6.6. The Lie algebra $L_{6,VI}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 9.

$$L_{6,VI}(\lambda) = \operatorname{span}(x, y, z, u, v, w);$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 1 & 0 & a & b & 0 \\ 0 & 1 & -b & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix};$$
$$\lambda = a + bi;$$
$$[y, u] = [z, v] = w.$$

Construction 6.7. The Lie algebra $L_{6,VII}$; see Fig. 10.

$$L_{6,VII} = \operatorname{span}(x, y, z, u, v, w);$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$
$$[y, z] = w, \quad [w, y] = -v, \quad [w, z] = u.$$

Construction 6.8. The Lie algebra $L_{6,VIII}$; see Fig. 10.

$$L_{6,VIII} = \operatorname{span}(x, y, z, u, v, w);$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$
$$[y, z] = w, \quad [w, y] = v, \quad [w, z] = -u.$$

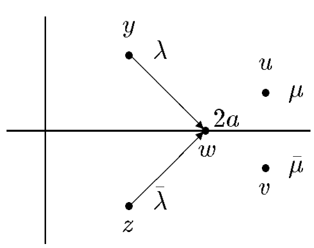

Fig. 5. $L_{6,I}(\lambda, \mu)$, $\operatorname{Re} \lambda = a$.

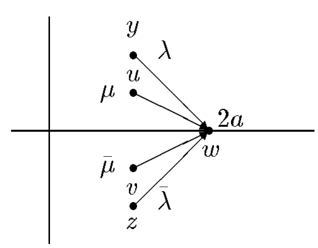

Fig. 6. $L_{6,II}(\lambda, \mu)$, $\operatorname{Re} \lambda = \operatorname{Re} \mu = a$.

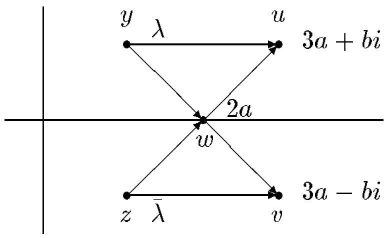

Fig. 7. $L_{6,III}(\lambda)$, $\lambda = a + bi$.

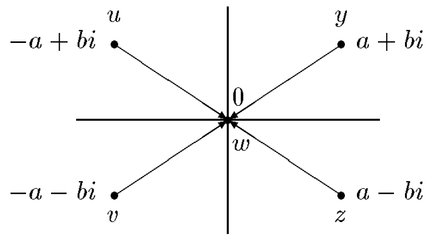

Fig. 8. $L_{6,IV}(\lambda)$, $\lambda = a + bi$.

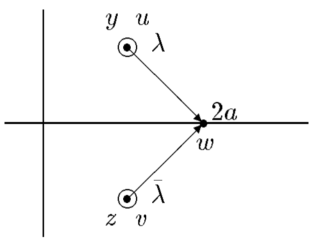

Fig. 9. $L_{6,V}(\lambda)$, $L_{6,VI}(\lambda)$, $\operatorname{Re}\lambda = a$.

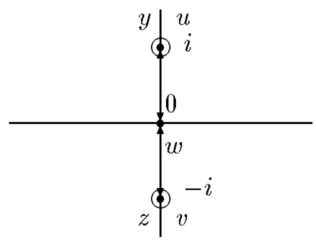

Fig. 10. $L_{6,VII}$, $L_{6,VIII}$.

In the sequel, we consider the following decomposition for a vector B in any of the Lie algebras $L_{6,I}$–$L_{6,VIII}$:

$$B = B_x x + B_y y + B_z z + B_u u + B_v v + B_w w.$$

6.2. Controllability conditions.

Theorem 6.1. *Let $L = L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold:*

  (1) $B \notin L^{(1)}$;

  (2) $\operatorname{Lie}(A, B) = L$, *or*

  (2′) $A(B_x \lambda) \neq 0$ *and* $A(B_x \mu) \neq 0$, *or*

  (2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A, (\operatorname{ad} B)^2 A, (\operatorname{ad} B)^3 A, w) = L$.

(c) *Let* $B \in L \setminus L^{(1)}$. *Then the system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable for almost all* $A \in L$.

Theorem 6.2. *Let* $L = L_{6,II}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in* L *is its derived subalgebra* $L^{(1)}$.

(b) *Let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:
  (1) $B \notin L^{(1)}$;
  (2) $\operatorname{Lie}(A, B) = L$, *or*
  (2′) $A(B_x\lambda) \neq 0$ *and* $A(B_x\mu) \neq 0$, *or*
  (2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A, (\operatorname{ad} B)^2 A, (\operatorname{ad} B)^3 A, w) = L$.

(c) *Let* $B \in L \setminus L^{(1)}$. *Then the system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable for almost all* $A \in L$.

Theorem 6.3. *Let* $L = L_{6,III}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in* L *is its derived subalgebra* $L^{(1)}$.

(b) *Let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:
  (1) $B \notin L^{(1)}$;
  (2) $\operatorname{Lie}(A, B) = L$, *or*
  (2′) $\operatorname{span}(B, A, (\operatorname{ad} B)A, w, u, v) = L$, *or*
  (2″) $A(B_x\lambda) \neq 0$ (*if* $\operatorname{Re} \lambda \neq 0$).

(c) *Let* $B \in L \setminus L^{(1)}$. *Then the system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable for almost all* $A \in L$.

Theorem 6.4. *Let* $L = L_{6,IV}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in* L *is its derived subalgebra* $L^{(1)}$.

(b) *Let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:
  (1) $B \notin L^{(1)}$;
  (2) $\operatorname{Lie}(A, B) = L$, *or*
  (2′) $A(B_x\lambda) \neq 0$ *and* $A(-B_x\lambda) \neq 0$, *or*
  (2″) $\operatorname{span}(B, A, (\operatorname{ad} B)A, (\operatorname{ad} B)^2 A, (\operatorname{ad} B)^3 A, w) = L$.

(c) *Let* $B \in L \setminus L^{(1)}$. *Then the system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable for almost all* $A \in L$.

Theorem 6.5. *Let*$L = L_{6,V}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *or* $L = L_{6,VI}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:
  (1) $B \notin L^{(1)}$;
  (2) $\mathrm{Lie}(A, B) = L$, *or*
  (2′) $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, (\mathrm{ad}\, B)^2 A, (\mathrm{ad}\, B)^3 A, w) = L$, *or*
  (2″) $\mathrm{top}(A, B_x \lambda) \neq 0$ (*if* $\mathrm{Re}\,\lambda \neq 0$).

(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Theorem 6.6. *Let $L = L_{6,VII}$ or $L = L_{6,VIII}$. Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*

(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:
  (1) $B \notin L^{(1)}$;
  (2) $\mathrm{Lie}(A, B) = L$, *or*
  (2′) $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, w, u, v) = L$.

(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Theorem 6.7. *A six-dimensional solvable Lie algebra L is controllable if and only if it is isomorphic to one of the following Lie algebras*:

(1) $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$;
(2) $L_{6,II}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\mathrm{Re}\,\lambda = \mathrm{Re}\,\mu$, $\lambda \neq \mu, \bar{\mu}$;
(3) $L_{6,III}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(4) $L_{6,IV}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$;
(5) $L_{6,V}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(6) $L_{6,VI}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(7) $L_{6,VII}$;
(8) $L_{6,VIII}$.

6.3. Proof of controllability conditions.

6.3.1. *Lie algebra $L_{6,I}(\lambda, \mu)$: Theorem* 6.1.

Proof. **Statement** (b), (1) & (2′). **Necessity**. By Lemma 10.2, we have $\mathrm{Sp}^{(1)} = B_x \cdot \{\lambda, \bar{\lambda}, \mu, \bar{\mu}, 2a\}$, $\mathrm{Sp}^{(2)} = B_x \cdot \{2a\}$, and the statement follows directly from items (2) and (5) of Corollary 1 in 7.

Sufficiency. If $\mathrm{Re}\,\lambda \neq 0$, then the operator $\mathrm{ad}\, B|_{L^{(1)}}$ has no N-pairs of real eigenvalues (see Definition 3 in 7). Then controllability of Γ follows from Theorem 2 in 7. Indeed, conditions (1) and (2) of this theorem are obviously satisfied. Condition (3):

$$L_r^{(1)} = L^{(1)}(B_x \cdot 2a) = \mathbb{R}w, \quad \text{and} \quad L_r^{(2)} = L^{(2)}(B_x \cdot 2a) = \mathbb{R}w.$$

Condition (4): the complex spectrum

$$\operatorname{Sp}_c^{(1)} = B_x \cdot \{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}$$

is simple; thus, all eigenspaces $L_c(a)$, $a \in \operatorname{Sp}_c^{(1)}$, are one-dimensional. Condition (5):

$$\operatorname{top}(A, B_x\lambda) = A(B_x\lambda) \quad \text{and} \quad \operatorname{top}(A, B_x\mu) = A(B_x\mu),$$

since both eigenvalues $B_x\lambda$ and $B_x\mu$ are simple. Condition (6): the only pair of real (coinciding) eigenvalues, $B_x \cdot 2a = B_x \cdot 2a$, is not an N-pair. All hypotheses (1)–(6) of Theorem 2 in 7 are satisfied; thus, the system Γ is controllable.

Consider the case $\operatorname{Re}\lambda = a = 0$. Now we show that the following assertions hold:

(1) $\operatorname{Lie}(A, B) = L$;
(2) there does not exist a codimension one subalgebra of L that contains the element B.

Then the system Γ is controllable by Proposition 1.

Statement (1). We have

$$L = L_{6,I}(\lambda, \mu) = \operatorname{span}(B, y, z, u, v, w);$$

see Construction 6.1. The line $\mathbb{R}w$ is an ideal in L. Consider the quotient Lie algebra

$$\tilde{L} = L/\mathbb{R}w = \operatorname{span}(\tilde{B}, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}). \tag{14}$$

The derived subalgebra is

$$\tilde{L}^{(1)} = [\tilde{L}, \tilde{L}] = \operatorname{span}(\tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}),$$

and the operator ad has the matrix

$$\operatorname{ad}\tilde{B}|_{\tilde{L}^{(1)}} = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{pmatrix};$$

thus, the spectrum

$$\operatorname{Sp}(\operatorname{ad}\tilde{B}|_{\tilde{L}^{(1)}}) = B_x \cdot \{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}$$

is simple. Moreover,

$$A(B_x\lambda) \neq 0,\ A(B_x\mu) \neq 0 \quad \Leftrightarrow \quad \tilde{A}(B_x\lambda) \neq 0,\ \tilde{A}(B_x\mu) \neq 0. \qquad (15)$$

By Lemma 10.4, we have

$$\operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad}\tilde{B})\tilde{A}, (\operatorname{ad}\tilde{B})^2\tilde{A}, (\operatorname{ad}\tilde{B})^3\tilde{A}) = \tilde{L}. \qquad (16)$$

This means that the image of the space

$$l = \operatorname{span}(B, A, (\operatorname{ad}B)A, (\operatorname{ad}B)^2A, (\operatorname{ad}B)^3A) \subset L$$

under the canonical projection $L \to \tilde{L}$ is the whole quotient Lie algebra $\tilde{L}$. Thus, $\dim l = 5$; then the space l contains vectors of the form

$$y_1 = y + \alpha w \quad \text{and} \quad z_1 = z + \beta w, \qquad \alpha, \beta \in \mathbb{R}.$$

Then

$$w = [y, z] = [y_1, z_1] \in \operatorname{Lie}(l);$$

thus, $\operatorname{Lie}(l) = L$. Consequently,

$$\operatorname{Lie}(A, B) = \operatorname{Lie}(l) = L;$$

the proof of statement (1) is complete.

Statement (2). On the contrary, we suppose that there exists a subalgebra l such that

$$l \subset L, \quad \dim l = \dim L - 1 \quad \text{and} \quad B \in l. \qquad (17)$$

The set

$$\Pi = \{C \in L \mid C(B_x\lambda) = 0 \text{ or } C(B_x\mu) = 0\}$$

is a union of two 4-dimensional spaces, and it can not contain the 5-dimensional space l:

$$l \not\subset \Pi.$$

Therefore, there exists a vector $C \in l \setminus \Pi$ such that

$$C \in l, \quad C(B_x\lambda) \neq 0 \quad \text{and} \quad C(B_x\mu) \neq 0.$$

By statement (1),

$$\operatorname{Lie}(C, B) = L.$$

But $l \supset \operatorname{Lie}(C, B)$; thus,

$$l = L.$$

The contradiction to (17) proves statement (2).

Now statements (1), (2), and Proposition 1 imply controllability of the system $\Gamma = A + \mathbb{R}B$. This completes the proof of sufficiency.

Statement (b), (1) & (2). We prove that (2) $\Leftrightarrow$ (2′) under condition (1).

$(2) \Rightarrow (2')$. Let $\mathrm{Lie}(A, B) = L$, and let condition $(2')$ be violated. For definiteness, let $A(B_x\lambda) = 0$. Then $\mathrm{j}(B_x\lambda) = 1$ and $\mathrm{top}(A, B_x\lambda) = 0$. Lemma 3.5 in 7 yields $\mathrm{Lie}(A, B) \neq L$; a contradiction.

$(2) \Leftarrow (2')$. If $\mathrm{Lie}(A, B) \neq L$, then the system $\Gamma = A + \mathbb{R}B$ is not controllable by the rank condition, and condition $(2')$ is violated.

Statement (b), (1) & $(2'')$. We prove that $(2') \Leftrightarrow (2'')$ under condition (1). As above, consider the quotient Lie algebra (14). In view of (15), condition $(2')$ is equivalent to (16), which, in turn, is equivalent to $(2'')$.

Statement (c) easily follows from statement (b), (1) & $(2')$.

Statement (a) follows from statement (c) and Lemma 10.5.

Theorem 6.1 is proved. □

6.3.2. *Lie algebra* $L_{6,II}(\lambda, \mu)$: *Theorem* 6.2.

Proof. The proof is completely similar to that of Theorem 6.1. □

6.3.3. *Lie algebra* $L_{6,III}(\lambda)$: *Theorem* 6.3.

Proof. **Case 1:** $\mathrm{Re}\,\lambda \neq 0$.

Statement (b), (1) & $(2'')$. Necessity follows directly from items (2) and (5) of Corollary 1 in 7, since $\mathrm{Sp}^{(1)} = B_x \cdot \{\lambda, \bar{\lambda}, \mu, \bar{\mu}, 2a\}$ and $\mathrm{Sp}^{(2)} = B_x \cdot \{\mu, \bar{\mu}, 2a\}$; see Lemma 10.2.

Sufficiency. In the generic case $A(B_x\mu) \neq 0$, controllability follows directly from Theorem 2 in 7. Indeed, conditions (1), (2), (4)–(6) of this theorem are obviously satisfied. Consider condition (3). We have

$$\mathrm{Sp}_r^{(1)} = \mathrm{Sp}_r^{(2)} = \{B_x \cdot 2a\};$$

thus,

$$L_r^{(1)} = L^{(1)}(B_x \cdot 2a) \quad \text{and} \quad L_r^{(2)} = L^{(2)}(B_x \cdot 2a).$$

Further,

$$L^{(2)}(B_x \cdot 2a) \subset L^{(1)}(B_x \cdot 2a).$$

Moreover, the eigenvalue $B_x \cdot 2a$ is simple; thus,

$$\dim L^{(2)}(B_x \cdot 2a) = \dim L^{(1)}(B_x \cdot 2a) = 1.$$

Consequently,

$$L^{(2)}(B_x \cdot 2a) = L^{(1)}(B_x \cdot 2a)$$

and

$$L_r^{(2)} = L_r^{(1)}.$$

By Theorem 2 in 7, the system Γ is controllable.

In the case $A(B_x\mu) = 0$, we have to slightly modify the proof of this theorem. By Lemma 4.2 in 7, we obtain

$$L^{(1)}(B_x\lambda) \subset \mathrm{LS}(\Gamma). \tag{18}$$

We can choose a basis in the space $L^{(1)}(B_x\lambda)$ of the form

$$y^1 = y + y^1_u u + y^1_v v + y^1_w w, \quad y^1_u, y^1_v, y^1_w \in \mathbb{R}, \tag{19}$$
$$z^1 = z + z^1_u u + z^1_v v + z^1_w w, \quad z^1_u, z^1_v, z^1_w \in \mathbb{R}. \tag{20}$$

Since

$$\operatorname{span}(y^1, z^1) = L^{(1)}(B_x\lambda) \subset \mathrm{LS}(\Gamma),$$

we have

$$\pm [y^1, z^1] = \pm w^1 = \pm(w + w^1_u u + w^1_v v) \in \mathrm{LS}(\Gamma). \tag{21}$$

Using a linear combination of (19), (20), and (21), we obtain

$$\begin{aligned} &y^2 = y + y^2_u u + y^2_v v, \quad y^2_u,\ y^2_v \in \mathbb{R};\\ &z^2 = z + z^2_u u + z^2_v v, \quad z^2_u,\ z^2_v \in \mathbb{R};\\ &\operatorname{span}(y^2, z^2) \subset \mathrm{LS}(\Gamma). \end{aligned}$$

Therefore,

$$\pm w = \pm[y^2, z^2] \in \mathrm{LS}(\Gamma).$$

Then

$$\begin{aligned} \pm[w, y^2] &= \pm[w, y] = \pm u \in \mathrm{LS}(\Gamma);\\ \pm[w, z^2] &= \pm[w, z] = \pm v \in \mathrm{LS}(\Gamma). \end{aligned}$$

Therefore, we have

$$L^{(1)} = \operatorname{span}(y, z, u, v, w) \subset \mathrm{LS}(\Gamma).$$

Since $B \notin L^{(1)}$, we have $\mathrm{LS}(\Gamma) = L$, and the system Γ is controllable (see 4 or controllability condition (15) in 7).

Statement (b), (1) & (2′). The space $I = \operatorname{span}(u, v)$ is an ideal in L. Consider the quotient Lie algebra

$$\tilde{L} = L/I = \operatorname{span}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}) \simeq L_4(\lambda).$$

In view of Theorem 4.1, we obtain the following chain:

$$(2'') \Leftrightarrow \tilde{A}(B_x\lambda) \neq 0 \Leftrightarrow \operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad}\tilde{B})\tilde{A}, \tilde{w}) = \tilde{L} \Leftrightarrow (2').$$

The remainder of the proof: **statement** (b), (1) & (2), **statement** (c), and **statement** (a), follow exactly as in the proof of Theorem 6.1.

Case 2: $\operatorname{Re}\lambda = 0$.

Statement (a). On the contrary, assume that the Lie algebra $L = L_{6,III}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, contains a codimension one subalgebra $l \neq L^{(1)}$.

The space $I = \operatorname{span}(u, v)$ is an ideal in L; therefore, we can consider the quotient Lie algebra

$$\tilde{L} = L/I = \operatorname{span}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}), \tag{22}$$

$$\operatorname{ad} \tilde{x}|_{\operatorname{span}(\tilde{y},\tilde{z},\tilde{w})} = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$[\tilde{y}, \tilde{z}] = \tilde{w}, \tag{24}$$

and the corresponding image

$$\tilde{l} \subset \tilde{L}$$

of the Lie algebra l. It is easy to see that $\dim \tilde{l}$ can be equal to 3 or 4.

(1) $\dim \tilde{l} = 4$. Then $\tilde{l} = \tilde{L}$; thus, the Lie algebra l contains elements of the form

$$f^1 = x + f_u^1 u + f_v^1 v, \quad f_u^1,\ f_v^1 \in \mathbb{R}; \tag{25}$$

$$f^2 = y + f_u^2 u + f_v^2 v, \quad f_u^2,\ f_v^2 \in \mathbb{R}; \tag{26}$$

$$f^3 = z + f_u^3 u + f_v^3 v, \quad f_u^3,\ f_v^3 \in \mathbb{R}; \tag{27}$$

$$f^4 = w + f_u^4 u + f_v^4 v, \quad f_u^4,\ f_v^4 \in \mathbb{R}. \tag{28}$$

We have $[f^2, f^3] = w \in l$. Further, $[f^2, w] = -u \in l$ and $[f^3, w] = -v \in l$. Since vectors (25)–(28) belong to l, we see that $x, y, z, w \in l$. We have $l = L$, a contradiction.

(2) $\dim \tilde{l} = 3$. In view of (22)–(24), the Lie algebra $\tilde{L}$ is isomorphic to the Lie algebra $L_4(bi)$, see Construction 4.1. Since $\tilde{l}$ is a codimension one subalgebra of $\tilde{L}$, Theorem 4.1 yields

$$\tilde{l} = \tilde{L}^{(1)} = \operatorname{span}(\tilde{y}, \tilde{z}, \tilde{w}).$$

Therefore, the Lie algebra l contains elements of the form

$$f^1 = y + f_u^1 u + f_v^1 v, \quad f_u^1,\ f_v^1 \in \mathbb{R};$$

$$f^2 = z + f_u^2 u + f_v^2 v, \quad f_u^2,\ f_v^2 \in \mathbb{R}.$$

Then $[f^1, f^2] = w \in l$, $[f^1, w] = -u \in l$, and $[f^2, w] = -v \in l$. Thus, $l = L^{(1)}$, a contradiction.

Statement (b), (1) & (2), follows from Proposition 1 and (a).

Statement (b), (1) & (2′). We prove that (2) $\Leftrightarrow$ (2′) under condition (1). Consider the quotient Lie algebra (22). In view of Theorem 4.1, it is easy to see that

$$(2) \Leftrightarrow \operatorname{Lie}(\tilde{A}, \tilde{B}) = \tilde{L} \Leftrightarrow \operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad} \tilde{B})\tilde{A}, \tilde{w}) = \tilde{L} \Leftrightarrow (2').$$

Statement (c) follows from statement (b), (1) & (2′). □

6.3.4. *Lie algebra* $L_{6,IV}(\lambda)$: *Theorem* 6.4.

Proof. **Statement** (b), (1) & (2′). Necessity follows from items (2) and (5) of Corollary 1 in 7, since

$$\mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)} = B_x \cdot \{\pm\lambda, \pm\bar{\lambda}\}.$$

Sufficiency follows exactly as in the proof of Theorem 6.1 for the case $\operatorname{Re}\lambda = 0$: one should just replace the Lie algebra $L_{6,I}(\lambda,\mu)$, the eigenvalue μ, and Construction 6.1, respectively, by the Lie algebra $L_{6,IV}(\lambda)$, the eigenvalue $-\lambda$, and Construction 6.4.

The remainder of the proof: **statement** (b), (1) & (2) and (1) & (2″), **statement** (c), and **statement** (a) follow exactly as in the proof of Theorem 6.1. □

6.3.5. *Lie algebras* $L_{6,V}(\lambda)$ *and* $L_{6,VI}(\lambda)$: *Theorem* 6.5.

Proof. **Case 1:** $\operatorname{Re}\lambda \neq 0$.

Statement (b), (1) & (2″). Necessity follows directly from items (2) and (7) of Theorem 1 in 7, since

$$\mathrm{Sp}^{(1)} = B_x \cdot \{\lambda, \bar{\lambda}, 2a\}, \quad \mathrm{Sp}^{(2)} = \{2B_x a\},$$
$$\mathrm{j}(B_x\lambda) = 1, \quad \mathrm{j}(2B_x a) = 0$$

(see Definition 1 in 7 and remarks after it).

Sufficiency follows from Corollary 2 in 7.

Statement (b), (1) & (2′). The line $I = \mathbb{R}w$ is an ideal in L. Consider the quotient Lie algebra

$$\tilde{L} = L/I = \mathrm{span}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}) \simeq L_{5,II}(\lambda).$$

Taking into account Theorem 5.2, we obtain the following chain:

$$(2'') \Leftrightarrow \mathrm{top}(\tilde{A}, B_x\lambda) \neq 0 \Leftrightarrow$$
$$\mathrm{span}(\tilde{B}, \tilde{A}, (\operatorname{ad}\tilde{B})\tilde{A}, (\operatorname{ad}\tilde{B})^2\tilde{A}, (\operatorname{ad}\tilde{B})^3\tilde{A}) = \tilde{L} \Leftrightarrow (2').$$

The remainder of the proof: **statement** (b), (1) & (2), **statement** (c), and **statement** (a) follow exactly as in the proof of Theorem 6.1.

Case 2: $\operatorname{Re}\lambda = 0$.

Statement (a). On the contrary, suppose that the Lie algebra L contains a codimension one subalgebra $l \neq L^{(1)}$.

The line $I = \mathbb{R}w$ is an ideal in L. Consider the quotient Lie algebra

$$\tilde{L} = L/I = \mathrm{span}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}), \tag{29}$$
$$\operatorname{ad}\tilde{x}|_{\mathrm{span}(\tilde{y},\tilde{z},\tilde{u},\tilde{v})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix};$$

thus, $\tilde{L} = L_{5,II}(i)$ (see Construction 5.2), and the corresponding image

$$\tilde{l} \subset \tilde{L}.$$

Obviously,

$$\dim \tilde{l} = \begin{cases} 4 & \text{if } w \in l, \\ 5 & \text{if } w \notin l. \end{cases}$$

(a) Let $\dim \tilde{l} = 4$. Then $\tilde{l}$ is a codimension one subalgebra in $\tilde{L}$, and by Theorem 5.2, $\tilde{l} = L^{(1)}{}_{5,II}(i) = \operatorname{span}(\tilde{y}, \tilde{z}, \tilde{u}, \tilde{v})$. Since $w \in l$, we obtain $l = L^{(1)}$, a contradiction.

(b) Let $\dim \tilde{l} = 5$. Then $\tilde{l} = \tilde{L} \ni \tilde{y}, \tilde{z}, \tilde{u}$; thus, the Lie algebra l contains elements of the form

$$\begin{aligned} f^1 &= y + f^1_w w, \quad f^1_w \in \mathbb{R}; \\ f^2 &= z + f^2_w w, \quad f^2_w \in \mathbb{R}; \\ f^3 &= u + f^1_w w, \quad f^3_w \in \mathbb{R}. \end{aligned}$$

Consequently, $[f^1, f^2] = w \in l$. Thus, $l = L$, a contradiction.

Statement (b), (1) & (2), follows from Proposition 1 and (a).

Statement (b), (1) & (2′). We prove that (2) ⇔ (2′) under condition (1). As above, consider the quotient Lie algebra (29).

(2) ⇒ (2′). If $\operatorname{Lie}(A, B) = L$, then $\operatorname{Lie}(\tilde{A}, \tilde{B}) = \tilde{L}$. By Theorem 5.2, we have $\operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad} \tilde{B})\tilde{A}, (\operatorname{ad}^2 \tilde{B})\tilde{A}, (\operatorname{ad}^3 \tilde{B})\tilde{A}) = \tilde{L}$, which is equivalent to (2′).

(2) ⇐ (2′). Condition (2′) implies $\operatorname{Lie}(\tilde{A}, \tilde{B}) = \tilde{L}$. Then we show that $\operatorname{Lie}(A, B) = L$ exactly as in item (b) above.

Statement (c) follows from statement (b), (1) & (2′). □

6.3.6. *Lie algebras* $L_{6,VII}$ *and* $L_{6,VIII}$: *Theorem* 6.6.

Proof. **Statement** (a). Let $l \subset L$ be a codimension one subalgebra, $l \neq L^{(1)}$. The space $I = \operatorname{span}(u, v)$ is an ideal in L; thus, we can consider the quotient Lie algebra

$$\begin{aligned} &\tilde{L} = L/I = \operatorname{span}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}), \qquad (30) \\ &\operatorname{ad} \tilde{x}|_{\operatorname{span}(\tilde{y}, \tilde{z}, \tilde{w})} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &[\tilde{y}, \tilde{z}] = \tilde{w}, \end{aligned}$$

and the corresponding image $\tilde{l} \subset \tilde{L}$ of the Lie algebra l. Obviously, $\dim \tilde{l}$ can be equal to 3 or 4.

(1) Let $\dim \tilde{l} = 3$. Then $\tilde{l}$ is a codimension one subalgebra of the Lie algebra $\tilde{L}$, which is isomorphic to the Lie algebra $L_4(i)$; see Construction 4.1. By

Theorem 4.1, $\tilde{l}$ coincides with the derived subalgebra $\tilde{L}^{(1)} = \operatorname{span}(\tilde{y}, \tilde{z}, \tilde{w})$. Thus, the Lie algebra l contains vectors of the form

$$f^1 = y + f_u^1 u + f_v^1 v, \qquad f_u^1,\ f_v^1 \in \mathbb{R}, \tag{31}$$

$$f^2 = z + f_u^2 u + f_v^2 v, \qquad f_u^2,\ f_v^2 \in \mathbb{R}, \tag{32}$$

$$f^3 = w + f_u^3 u + f_v^3 v, \qquad f_u^3,\ f_v^3 \in \mathbb{R}. \tag{33}$$

The condition $\dim \tilde{l} = 3$ implies $\operatorname{span}(u, v) \subset l$, which, together with (31)–(33), yields $y, z, w \in l$. Thus, $l = L^{(1)}$, a contradiction.

(2) Let $\dim \tilde{l} = 4$; then $l + \operatorname{span}(u, v) = L$. Consequently, the Lie algebra l contains elements of the form

$$f^1 = y + f_u^1 u + f_v^1 v, \qquad f_u^1,\ f_v^1 \in \mathbb{R};$$

$$f^2 = z + f_u^2 u + f_v^2 v, \qquad f_u^2,\ f_v^2 \in \mathbb{R}.$$

Then $[f^1, f^2] = w \in l$; thus, the elements $[f^1, w] = \pm v$ and $[f^2, w] = \mp u$ also belong to the Lie algebra l. Therefore, we obtain $l = L$, a contradiction.

Statement (b), (1) & (2), follows from Proposition 1 and (a).

Statement (b), (1) & (2′). We prove that (2) $\Leftrightarrow$ (2′) under condition (1). Consider the quotient Lie algebra (30).

(2) $\Rightarrow$ (2′). Let $\operatorname{Lie}(A, B) = L$; then $\operatorname{Lie}(\tilde{A}, \tilde{B}) = \tilde{L}$. By Theorem 4.1, we obtain $\operatorname{span}(\tilde{B}, \tilde{A}, (\operatorname{ad} \tilde{B})\tilde{A}, \tilde{w}) = \tilde{L}$, which is equivalent to hypothesis (2′) of this theorem.

(2) $\Leftarrow$ (2′). Conversely, hypothesis (2′) implies $\operatorname{Lie}(\tilde{A}, \tilde{B}) = \tilde{L}$. Therefore, the Lie algebra $\operatorname{Lie}(A, B)$ contains elements of the form

$$f^1 = y + f_u^1 u + f_v^1 v, \qquad f_u^1,\ f_v^1 \in \mathbb{R};$$

$$f^2 = z + f_u^2 u + f_v^2 v, \qquad f_u^2,\ f_v^2 \in \mathbb{R};$$

$$f^3 = x + f_u^3 u + f_v^3 v, \qquad f_u^3,\ f_v^3 \in \mathbb{R}.$$

Then the elements $[f^1, f^2] = w$, $[f^1, w] = \pm v$, and $[f^2, w] = \mp u$ are in $\operatorname{Lie}(A, B)$; consequently, $\operatorname{Lie}(A, B) = L$.

Statement (c) follows from statement (b), (1) & (2′). □

6.3.7. *Controllable Lie algebras*: *Theorem* 6.7.

Proof. **Necessity**. Let a six-dimensional solvable Lie algebra L be controllable, i.e., there exist $A, B \in L$ such that the system $\Gamma = A + \mathbb{R}B$ is controllable. Then Theorem 1 in 7 implies the following:

$$\dim L^{(1)} = 5;$$

$$B \notin L^{(1)};$$

$$L_r^{(1)} = L_r^{(2)}. \tag{34}$$

Now we consider step by step all possibilities for location of the spectrum $\operatorname{Sp}^{(1)} = \operatorname{Sp}(\operatorname{ad} B|_{L^{(1)}})$ in the complex plane.

(a) Five real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. The operator $\operatorname{ad} B|_{L^{(1)}}$ cannot have real spectrum, since if $\operatorname{Sp}^{(1)} \subset \mathbb{R}$ then $L_r^{(1)} \neq L_r^{(2)}$ by Lemma 10.3, a contradiction with (34).

(b) Three real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$. Suppose that

$$\operatorname{Sp}^{(1)} = \{\lambda, \bar{\lambda}, c, d, e\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad c, d, e \in \mathbb{R}$$

(some or all of the numbers c, d, e may coincide one with other). Then we have the decomposition

$$L^{(1)} = L^{(1)}(\lambda) \oplus L_r^{(1)}, \tag{35}$$

where $\dim L^{(1)}(\lambda) = 2$ and $L_r^{(1)} = L^{(1)}(c) + L^{(1)}(d) + L^{(1)}(e)$,

$$\dim L_r^{(1)} = 3. \tag{36}$$

Thus,

$$L^{(2)} = [L^{(1)}(\lambda), L^{(1)}(\lambda)] + [L^{(1)}(\lambda), L_r^{(1)}] + [L_r^{(1)}, L_r^{(1)}].$$

Taking into account Lemma 10.1 and decomposition (35), we obtain

$$[L^{(1)}(\lambda), L_r^{(1)}] \subset L^{(2)}(\lambda) \subset L_c^{(2)},$$
$$[L^{(1)}(\lambda), L^{(1)}(\lambda)] \subset L^{(2)}(2 \operatorname{Re} \lambda) \subset L_r^{(2)}.$$

Consequently,

$$L_r^{(2)} = [L^{(1)}(\lambda), L^{(1)}(\lambda)] + [L_r^{(1)}, L_r^{(1)}].$$

Now we estimate the dimensions of the summands in the right-hand side.

The space $L^{(1)}(\lambda)$ is two-dimensional; thus, $\dim[L^{(1)}(\lambda), L^{(1)}(\lambda)] \leq 1$.

The space $L_r^{(1)}$ is three-dimensional, and by Lemma 10.1

$$[L_r^{(1)}, L_r^{(1)}] \subset L_r^{(1)},$$

i.e., $L_r^{(1)}$ is a Lie algebra. But L is solvable, hence $L_r^{(1)} \subset L^{(1)}$ is nilpotent. Thus, $L_r^{(1)}$ is a three-dimensional nilpotent Lie algebra. Consequently, $L_r^{(1)}$ is either Abelian, or a (unique) three-dimensional nilpotent non-Abelian Lie algebra. In both cases, $\dim[L_r^{(1)}, L_r^{(1)}] \leq 1$.

Therefore, $\dim L_r^{(2)} \leq \dim[L^{(1)}(\lambda), L^{(1)}(\lambda)] + \dim[L_r^{(1)}, L_r^{(1)}] \leq 2$, which contradicts equalities (34) and (36). Therefore, case (b) is impossible.

(c) One real eigenvalue of the operator $\operatorname{ad} B|_{L^{(1)}}$. Since both cases (a) and (b) are impossible, we should have

$$\operatorname{Sp}^{(1)} = \{\lambda, \bar{\lambda}, \mu, \bar{\mu}, e\}, \quad \lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}, \quad \mu = c + di \in \mathbb{C} \setminus \mathbb{R}, \quad e \in \mathbb{R}.$$

We can assume that $b, d > 0$. Replacing, if necessary, B by $-B$, we obtain $e \geq 0$.

(c.1) Let $\lambda \neq \mu$. Then the operator $\mathrm{ad}_c B|_{L_c^{(1)}}$ has a simple spectrum, and there is the following decomposition into one-dimensional eigenspaces:

$$L_c^{(1)} = L_c^{(1)}(\lambda) \oplus L_c^{(1)}(\bar{\lambda}) \oplus L_c^{(1)}(\mu) \oplus L_c^{(1)}(\bar{\mu}) \oplus L_c^{(1)}(e).$$

Choose the corresponding eigenvectors

$$\mathbb{C}f_\lambda = L_c^{(1)}(\lambda), \quad \mathbb{C}f_{\bar{\lambda}} = L_c^{(1)}(\bar{\lambda}), \quad \mathbb{C}f_\mu = L_c^{(1)}(\mu), \quad \mathbb{C}f_{\bar{\mu}} = L_c^{(1)}(\bar{\mu}), \quad \text{and} \quad \mathbb{C}f_e = L_c^{(1)}(e),$$

so that their complex conjugates in L_c satisfy

$$\overline{f_\lambda} = f_{\bar{\lambda}}, \quad \overline{f_\mu} = f_{\bar{\mu}}, \quad \text{and} \quad \overline{f_e} = f_e.$$

By Lemma 10.1,

$$[f_\lambda, f_\mu] \in L_c^{(1)}(\lambda + \mu) = L_c^{(1)}(a + c + (b + d)i) = \{0\},$$

hence

$$[f_\lambda, f_\mu] = 0. \tag{37}$$

We take complex conjugate and obtain

$$[f_{\bar{\lambda}}, f_{\bar{\mu}}] = 0. \tag{38}$$

(c.1.1) Let $b = \mathrm{Im}\,\lambda \neq \mathrm{Im}\,\mu = d$. We show that $[f_\lambda, f_e] = 0$. On the contrary, we suppose that $[f_\lambda, f_e] \neq 0$. By Lemma 10.1,

$$[f_\lambda, f_e] \in L_c^{(1)}(\lambda + e) = L_c^{(1)}(a + e + bi).$$

Since $[f_\lambda, f_e] \neq 0$, we have $a + e + bi \in \mathrm{Sp}^{(1)}$. It is obvious that $a + e + bi$ cannot equal any one of the eigenvalues e, $\bar{\lambda} = a - bi$, and $\bar{\mu} = c - di$ (recall that $b, d > 0$). Further, $a + e + bi \neq \mu = c + di$, since $b \neq d$. Consequently, $a+e+bi = lam = a+bi$ and $[f_\lambda, f_e] \in L_c^{(1)}(\lambda)$, i.e., $[f_\lambda, f_e] = kf_\lambda$, $k \in \mathbb{C}\setminus\{0\}$. But this contradicts the nilpotency of the operator $\mathrm{ad}\, f_e : L_c^{(1)} \to L_c^{(1)}$ (L is solvable; hence $L^{(1)}$ is nilpotent and $\mathrm{ad}\, f_e|_{L_c^{(1)}}$ is nilpotent). This contradiction shows that

$$[f_\lambda, f_e] = 0. \tag{39}$$

We prove similarly that

$$[f_{\bar{\lambda}}, f_e] = [f_\mu, f_e] = [f_{\bar{\mu}}, f_e] = 0. \tag{40}$$

Now we show that $[f_\lambda, f_{\bar{\mu}}] = 0$. On the contrary, we suppose that $[f_\lambda, f_{\bar{\mu}}] \neq 0$. Then

$$[f_\lambda, f_{\bar{\mu}}] \in L_c^{(1)}(\lambda + \bar{\mu}) = L_c^{(1)}(a + c + (b - d)i).$$

It is easy to see that $\lambda + \bar\mu = a + c + (b-d)i$ cannot equal any one of the eigenvalues e (since $b \neq d$), $\lambda = a + bi$ (since $c - di = \bar\mu \neq 0$), and $\bar\mu = c - di$ (since $a + bi = \lambda \neq 0$). There are only two mutually exclusive possibilities:

$$a + c + (b-d)i = \bar\lambda = a - bi \quad \Leftrightarrow \quad c = 0,\ d = 2b$$

or

$$a + c + (b-d)i = \mu = c + di \quad \Leftrightarrow \quad a = 0,\ b = 2d.$$

Let $a = 0$ and $b = 2d$ (the case $c = 0$, $d = 2b$ is considered similarly). We have

$$[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(a + c + (b-d)i) = L_c^{(1)}(c + di) = L_c^{(1)}(\mu);$$

thus,

$$[f_\lambda, f_{\bar\mu}] = k f_\mu, \quad k \in \mathbb{C} \setminus \{0\}.$$

We take complex conjugate and obtain

$$[f_{\bar\lambda}, f_\mu] = \bar k f_{\bar\mu}.$$

Then, in view of (37) and (38), we have

$$[(1/k)f_\lambda + (1/\bar k)f_{\bar\lambda}, f_\mu + f_{\bar\mu}] = f_\mu + f_{\bar\mu}.$$

This contradicts the nilpotency of the operator $\operatorname{ad}((1/k)f_\lambda + (1/\bar k)f_{\bar\lambda})|_{L_c^{(1)}}$. Consequently,

$$[f_\lambda, f_{\bar\mu}] = 0 \tag{41}$$

and

$$[f_{\bar\lambda}, f_\mu] = 0. \tag{42}$$

Now we consider the remaining brackets

$$[f_\lambda, f_{\bar\lambda}] \in L_c^{(1)}(2a) \quad \text{and} \quad [f_\mu, f_{\bar\mu}] \in L_c^{(1)}(2c).$$

If $2a \neq e$ and $2c \neq e$, then $[f_\lambda, f_{\bar\lambda}] = [f_\mu, f_{\bar\mu}] = 0$, and the derived subalgebra $L_c^{(1)}$ is Abelian (see (37)–(42)). Then $L^{(1)}$ is Abelian and $L^{(2)} = \{0\}$, which contradicts condition (34), since $L_r^{(1)} = L^{(1)}(e)$.

(c.1.1.1) Let $2a = e$ and $2c \neq e$. Then

$$[f_\mu, f_{\bar\mu}] = 0; \tag{43}$$

$$[f_\lambda, f_{\bar\lambda}] = k f_e, \quad k \in \mathbb{C} \setminus \{0\}, \tag{44}$$

where $k \neq 0$, since $L_r^{(2)} = L_r^{(1)} = L^{(1)}(e)$. Therefore, the only nonzero bracket in $L_c^{(1)}$ (see (37)–(44)) is bracket (44). We take complex conjugate of this relation and obtain $[f_{\bar\lambda}, f_\lambda] = \bar k f_e$; thus, $\bar k = -k$, i.e., $k = il$, $l \in \mathbb{R} \setminus \{0\}$.

Now we return from $L_c^{(1)}$ to $L^{(1)}$. Denote

$$x = B; \tag{45}$$

$$y = (f_\lambda + f_{\bar\lambda})/2, \quad z = (f_\lambda - f_{\bar\lambda})/(2i); \tag{46}$$

$$u = (f_\mu + f_{\bar\mu})/2, \quad v = (f_\mu - f_{\bar\mu})/(2i); \tag{47}$$

$$w = -(l/2) f_e. \tag{48}$$

Now an immediate verification of multiplication rules in the Lie algebra $L = \operatorname{span}(x, y, z, u, v, w)$ shows that $L = L_{6,I}(\lambda, \mu)$.

(c.1.1.2) Let $2a \neq e$ and $2c = e$. This case is completely similar to case (c.1.1.1); one should just switch λ and μ. Thus, $L = L_{6,I}(\mu, \lambda)$.

(c.1.1.3) Let $2a = 2c = e$. Then

$$[f_\lambda, f_{\bar\lambda}] = k f_e, \quad [f_\mu, f_{\bar\mu}] = l f_e, \quad k,\ l \in \mathbb{C}.$$

If $k \neq 0$ and $l = 0$, then $L = L_{6,I}(\lambda, \mu)$.

If $k = 0$ and $l \neq 0$, then $L = L_{6,I}(\mu, \lambda)$. If $k \neq 0$ and $l \neq 0$, then $L = L_{6,II}(\lambda, \mu)$.

(c.1.2) Let $b = \operatorname{Im} \lambda = \operatorname{Im} \mu = d$. Consider the bracket

$$[f_\lambda, f_e] \in L_c^{(1)}(\lambda + e) = L_c^{(1)}(a + e + bi). \tag{49}$$

It is obvious that $a+e+bi \neq e, a-bi, c-di$. There can be only the following two mutually exclusive possibilities:

$$a + e + bi = c + di \ \Leftrightarrow\ a + e = c$$

or

$$a + e + bi = a + bi \ \Leftrightarrow\ e = 0.$$

(c.1.2.1) Let $a + e = c$ and let $e \neq 0$; thus, $e > 0$. We have

$$[f_\lambda, f_e] \in L_c^{(1)}(a + e + bi) = L_c^{(1)}(c + di);$$

consequently,

$$[f_\lambda, f_e] = k f_\mu, \quad k \in \mathbb{C}; \tag{50}$$

$$[f_{\bar\lambda}, f_e] = \bar k f_{\bar\mu}. \tag{51}$$

Now consider the bracket

$$[f_\mu, f_e] \in L_c^{(1)}(\mu + e) = L_c^{(1)}(c + e + bi) = L_c^{(1)}(a + 2e + bi).$$

It is obvious that the number $c + e + bi = a + 2e + bi$ is not an eigenvalue of $\operatorname{ad} B|_{L^{(1)}}$, since it is not equal to any of the numbers e, $a \pm bi$, and $c \pm bi$. Consequently,

$$[f_\mu, f_e] = [f_{\bar\mu}, f_e] = 0.$$

(c.1.2.1.1) Let $e = 2a \neq 2c$. Then

$$[f_\mu, f_{\bar\mu}] \in L_c^{(1)}(2c) = \{0\};$$

thus,

$$[f_\mu, f_{\bar\mu}] = 0.$$

Further,

$$[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(a+c),$$

but $a + c \neq e = 2a$; hence $L_c^{(1)}(a+c) = \{0\}$ and

$$[f_\lambda, f_{\bar\mu}] = [f_{\bar\lambda}, f_\mu] = 0.$$

Finally,

$$[f_\lambda, f_{\bar\lambda}] = l f_e, \quad l \in \mathbb{C} \setminus \{0\}, \tag{52}$$

where $l \neq 0$, since otherwise $\mathbb{R} f_e = L_r^{(1)} \neq L_r^{(2)} = \{0\}$. We take complex conjugate of (52) and obtain

$$\operatorname{Re} l = 0.$$

Therefore, the only nonzero brackets in $L_c^{(1)}$ are (52) and, probably, (50) and (51).

We pass to the real basis in L corresponding to the basis B, f_λ, $f_{\bar\lambda}$, f_μ, $f_{\bar\mu}$, f_e in L_c and observe that if $k \neq 0$ in (50) and (51), then $L = L_{6,III}(\lambda)$. If $k = 0$ in (50) and (51), then $L = L_{6,I}(\lambda, \mu)$.

(c.1.2.1.2) Let $e = 2c \neq 2a$. Then

$$[f_\lambda, f_{\bar\lambda}] \in L_c^{(1)}(\lambda + \bar\lambda) = L_c^{(1)}(2a) = \{0\};$$

thus,

$$[f_\lambda, f_{\bar\lambda}] = 0.$$

Similarly,

$$[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(a+c) = \{0\},$$

since $a + c \neq e = 2c$, we have

$$[f_\lambda, f_{\bar\mu}] = 0.$$

But

$$[f_\mu, f_{\bar\mu}] \in L_c^{(1)}(\mu + \bar\mu) = L_c^{(1)}(2c) = L_c^{(1)}(e);$$

thus,

$$[f_\mu, f_{\bar\mu}] = l f_e, \quad l \in \mathbb{C} \setminus \{0\}, \tag{53}$$

where $l \neq 0$, since otherwise condition (34) is violated as in the previous item. Jacobi identity for the triple $f_\mu, f_{\bar\mu}, f_\lambda$ yields $kl = 0$. But $l \neq 0$; consequently, $k = 0$ in (50) and (51).

Therefore, the only nonzero bracket in $L_c^{(1)}$ is (53). Therefore, $L = L_{6,I}(\mu, \lambda)$.

(c.1.2.2) Let $e = 0$ and $a + e \neq c$. In view of (49), we have $[f_\lambda, f_e] \in L_c^{(1)}(a+bi)$; thus, $[f_\lambda, f_e] = k f_\lambda$. The operator $\operatorname{ad} f_e|_{L_c^{(1)}}$ is nilpotent; therefore, $k = 0$, i.e.,

$$[f_\lambda, f_e] = [f_{\bar\lambda}, f_e] = 0.$$

Similarly

$$[f_\mu, f_e] = [f_{\bar\mu}, f_e] = 0.$$

Now consider the bracket

$$[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(a+c).$$

(c.1.2.2.1) Let $a + c = 0$. Then

$$[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(e);$$

consequently,

$$[f_\lambda, f_{\bar\mu}] = k f_e, \quad k \in \mathbb{C}; \tag{54}$$

$$[f_{\bar\lambda}, f_\mu] = \bar k f_e. \tag{55}$$

Further, $[f_\lambda, f_{\bar\lambda}] \in L^{(1)}(2a) = \{0\}$ ($a \neq 0$, since otherwise $c = 0$ and $\lambda = \mu$, which contradicts condition (c.1)). Consequently,

$$[f_\lambda, f_{\bar\lambda}] = 0.$$

Similarly,

$$[f_\mu, f_{\bar\mu}] = 0.$$

Therefore, the only possibly nonzero brackets in $L_c^{(1)}$ are given by (54) and (55). If $k = 0$ in these relations, then $L_c^{(1)}$ is Abelian, which contradicts condition (34). Consequently, $k \neq 0$. We pass to the real basis in L and obtain $L = L_{6,IV}(\lambda)$.

(c.1.2.2.2) Let $a + c \neq 0$. Then $[f_\lambda, f_{\bar\mu}] \in L_c^{(1)}(a+c) = \{0\}$, thus

$$[f_\lambda, f_{\bar\mu}] = [f_{\bar\lambda}, f_\mu] = 0.$$

Now consider the bracket

$$[f_\lambda, f_{\bar\lambda}] \in L_c^{(1)}(2a).$$

(c.1.2.2.2.1) Let $a \neq 0$. Then

$$[f_\lambda, f_{\bar\lambda}] = 0.$$

Further, $[f_\mu, f_{\bar\mu}] \in L_c^{(1)}(2c)$. If $c \neq q0$, then $[f_\mu, f_{\bar\mu}] = 0$ and $L_c^{(1)}$ is Abelian, which contradicts condition (34). Consequently, $c = 0$, and

$$[f_\mu, f_{\bar\mu}] = k f_e, \quad k \in \mathbb{C} \setminus \{0\},$$

is the only nonzero bracket in $L_c^{(1)}$. We pass to the real basis in L and obtain $L = L_{6,I}(\mu, \lambda)$.

(c.1.2.2.2.2) Let $a = 0$. Then $[f_\lambda, f_{\bar\lambda}] \in L_c^{(1)}(2a) = L_c^{(1)}(e)$; thus,

$$[f_\lambda, f_{\bar\lambda}] = k f_e, \quad k \in \mathbb{C}. \tag{56}$$

By condition (c.1.2.2.2), we have $a + c \neq 0$; therefore, $c \neq 0$. Consequently, $[f_\mu, f_{\bar\mu}] \in L_c^{(1)}(2c) = \{0\}$ and

$$[f_\mu, f_{\bar\mu}] = 0.$$

The only possibly nonzero bracket in $L_c^{(1)}$ is given by (56). If $k = 0$ in this relation, then $L_c^{(1)}$ is Abelian, which is impossible in view of (34). Thus, $k \neq 0$ and $L = L_{6,I}(\lambda, \mu)$.

(c.2) Let $\lambda = \mu$. The operator $\operatorname{ad} B|_{L^{(1)}}$ has the multiple spectrum

$$\operatorname{Sp}^{(1)} = \{\lambda, \bar\lambda, e\},$$

where $\lambda = a + bi$, $b > 0$, is a double eigenvalue and $e \geq 0$ is a simple eigenvalue. We have the decomposition

$$L^{(1)} = L^{(1)}(\lambda) \oplus L^{(1)}(e) \quad \text{with} \quad \dim L^{(1)}(\lambda) = 4 \quad \text{and} \quad \dim L^{(1)}(e) = 1.$$

Thus,

$$L^{(2)} = [L^{(1)}(\lambda), L^{(1)}(\lambda)] + [L^{(1)}(\lambda), L^{(1)}(e)],$$

where

$$[L^{(1)}(\lambda), L^{(1)}(e)] \subset L^{(1)}(\lambda + e),$$
$$[L^{(1)}(\lambda), L^{(1)}(\lambda)] \subset L^{(1)}(2a).$$

Then the condition $L_r^{(1)} = L_r^{(2)} = L^{(1)}(e)$ implies $e = 2a$.

(c.2.1) Let $a \neq 0$.

(c.2.1.1) Let the operator $\operatorname{ad}_c B|_{L_c^{(1)}}$ be not diagonalizable, i.e., in some basis $f_\lambda, g_\lambda, f_{\bar\lambda}, g_{\bar\lambda}, f_e$ of the space $L_c^{(1)}$ for which

$$\overline{f_\lambda} = f_{\bar\lambda}, \quad \overline{g_\lambda} = g_{\bar\lambda}, \quad \text{and} \quad \overline{f_e} = f_e,$$

this operator has the matrix

$$\operatorname{ad}_c B|_{L_c^{(1)}} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \bar\lambda & 0 & 0 \\ 0 & 0 & 1 & \bar\lambda & 0 \\ 0 & 0 & 0 & 0 & e \end{pmatrix}.$$

We have the decomposition

$$L_c^{(1)} = L_c^{(1)}(\lambda) \oplus L_c^{(1)}(\bar\lambda) \oplus L_c^{(1)}(e),$$

where $L_c^{(1)}(\lambda) = \operatorname{span}(f_\lambda, g_\lambda)$, $L_c^{(1)}(\bar\lambda) = \operatorname{span}(f_{\bar\lambda}, g_{\bar\lambda})$, and $L_c^{(1)}(e) = \mathbb{C}f_e$.

Because $[f_\lambda, f_e] \in L_c^{(1)}(\lambda + e) = \{0\}$, we obtain

$$[f_\lambda, f_e] = 0.$$

Similarly,

$$[g_\lambda, f_e] = [f_{\bar\lambda}, f_e] = [g_{\bar\lambda}, f_e] = 0.$$

Further, all pairwise brackets for the vectors f_λ, g_λ, $f_{\bar\lambda}$, and $g_{\bar\lambda}$ are contained in $L_c^{(1)}(e)$; therefore,

$$\begin{aligned}
&[f_\lambda, g_\lambda] = \alpha f_e, \quad \alpha = c + di, \quad c, d \in \mathbb{R};\\
&[f_\lambda, g_{\bar\lambda}] = \beta f_e, \quad \beta = p + qi, \quad p, q \in \mathbb{R};\\
&[f_\lambda, f_{\bar\lambda}] = \gamma f_e, \quad \gamma \in \mathbb{C};\\
&[g_\lambda, g_{\bar\lambda}] = \delta f_e, \quad \delta \in \mathbb{C}.
\end{aligned}$$

We apply complex conjugation to the last two relations and obtain

$$\begin{aligned}
\gamma = -\bar\gamma \quad &\Rightarrow \quad \gamma = ik, \quad k \in \mathbb{R},\\
\delta = -\bar\delta \quad &\Rightarrow \quad \delta = il, \quad l \in \mathbb{R}.
\end{aligned}$$

Consider the following real basis in $L^{(1)}$:

$$\begin{aligned}
&y = (f_\lambda + f_{\bar\lambda})/2, \quad z = (f_\lambda - f_{\bar\lambda})/(2i),\\
&u = (g_\lambda + g_{\bar\lambda})/2, \quad v = (g_\lambda - g_{\bar\lambda})/(2i),\\
&w = f_e
\end{aligned}$$

and obtain the following multiplication table:

	y	z	u	v
y	0	$(k/2)w$	$((c+p)/2)w$	$((d-q)/2)w$
z	$(-k/2)w$	0	$((d+q)/2)w$	$((p-c)/2)w$
u	$(-(c+p)/2)w$	$(-(d+q)/2)w$	0	$(-l/2)w$
v	$(-(d-q)/2)w$	$(-(p-c)/2)w$	$(l/2)w$	0

The Jacobi identity implies

$$\begin{aligned}
(B, y, z) \quad &\Rightarrow \quad (d+q)/2 = (d-q)/2,\\
(B, y, z) \quad &\Rightarrow \quad b((p-c)/2 - (c+p)/2) = -l/2,\\
(B, z, u) \quad &\Rightarrow \quad b((c+p)/2 - (p-c)/2) = -l/2,\\
(B, z, v) \quad &\Rightarrow \quad (d+q)/2 = -(d-q)/2.
\end{aligned}$$

Thus, $d = q = l = c = 0$, i.e., the multiplication tabl e takes the form

	y	z	u	v
y	0	$(k/2)w$	$(p/2)w$	0
z	$(-k/2)w$	0	0	$(p/2)w$
u	$(-p/2)w$	0	0	0
v	0	$(-p/2)w$	0	0

Note that $k^2 + p^2 \neq 0$, since if $k = p = 0$, then the derived subalgebra $L^{(1)}$ is Abelian, a contradiction to (34). If $p = 0$, then $L = L_{6,V}(\lambda)$, and if $p \neq 0$, then $L = L_{6,VI}(\lambda)$.

(c.2.1.2) Let the operator $\mathrm{ad}_c B|_{L_c^{(1)}}$ be diagonalizable. Then there exists a basis $\{y, z, u, v, w\}$ of the space $L^{(1)}$ in which the operator $\mathrm{ad}\, B|_{L^{(1)}}$ has the matrix

$$\mathrm{ad}\, B|_{L^{(1)}} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & -b & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi.$$

In the same way as in item (c.2.1.1) we show that

$$[y, w] = [z, w] = [u, w] = [v, w] = 0.$$

Thus,

$$\begin{aligned} L^{(2)} &= [L^{(1)}, L^{(1)}] = [L^{(1)}(\lambda) \oplus L^{(1)}(e), L^{(1)}(\lambda) \oplus L^{(1)}(e)] = \\ &= [L^{(1)}(\lambda), L^{(1)}(\lambda)] \oplus [L^{(1)}(\lambda), L^{(1)}(e)] = [L^{(1)}(\lambda), L^{(1)}(\lambda)] = \\ &= L^{(1)}(e). \end{aligned}$$

Consequently, $L^{(1)}/L^{(2)} = L^{(1)}(\lambda)$, and the quotient operator

$$\widetilde{\mathrm{ad}\, B} : L^{(1)}/L^{(2)} \to L^{(1)}/L^{(2)}$$

has the matrix

$$\widetilde{\mathrm{ad}\, B} = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}.$$

The complexification of this operator $\widetilde{\mathrm{ad}_c\, B}$ has two linearly independent eigenvectors, i.e., $\mathrm{j}(\lambda) = 2$, see Definition 1 in 7. By Theorem 1 in 7, the system $\Gamma = A + \mathbb{R}B$ cannot be controllable; thus, case (c.2.1.2) is impossible.

(c.2.2) Let $a = 0$. Then $\mathrm{Sp}^{(1)} = \{\pm bi, 0\}$, $b \neq 0$. Both eigenvalues $\pm bi$ have algebraic multiplicity two and 0 is a simple eigenvalue.

In order to obtain $b = 1$, we replace the element B by the element B/b and denote it by x in the sequel. Thus, $\mathrm{Sp}^{(1)} = \mathrm{Sp}\,\mathrm{ad}\,x|_{L^{(1)}}) = \{\pm i, 0\}$.

(c.2.2.1) Let the both eigenvalues $\pm i$ have the geometric multiplicity one. In the complexification L_c, we can choose a Jordan basis of the operator $\mathrm{ad}_c\, x|_{L^{(1)}}$:

$$L_c = \mathrm{span}(x, e_1, e_2, f_1, f_2, g), \tag{57}$$

$$\overline{e}_1 = f_1, \quad \overline{e}_2 = f_2, \quad \overline{g} = g, \tag{58}$$

$$L_c^{(1)} = \mathrm{span}(e_1, e_2, f_1, f_2, g), \tag{59}$$

$$\mathrm{ad}_c\, x|_{\mathrm{span}(e_1,e_2,f_1,f_2,g)} = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{60}$$

Now we examine Lie brackets in the derived subalgebra $L_c^{(1)}$.

The vectors e_1 and e_2 belong to $L_c^{(1)}(i)$; thus, by Lemma 10.1,

$$[e_1, e_2] \in L_c^{(1)}(2i) = \{0\},$$

that is,

$$[e_1, e_2] = 0. \tag{61}$$

Similarly,

$$[f_1, f_2] = 0. \tag{62}$$

A similar argument using Lemma 10.1 yields

$$[e_1, f_1] = ag, \qquad a \in \mathbb{C}; \tag{63}$$

$$[e_2, f_2] = bg, \qquad b \in \mathbb{C}; \tag{64}$$

$$[e_1, f_2] = cg, \qquad c \in \mathbb{C}; \tag{65}$$

$$[e_2, f_1] = dg, \qquad d \in \mathbb{C}. \tag{66}$$

Further,

$$\overline{a}g = \overline{ag} = \overline{[e_1, f_1]} = [\overline{e}_1, \overline{f}_1] = [f_1, e_1] = -ag,$$

consequently,

$$a = -\overline{a} = i\alpha, \quad \alpha \in \mathbb{R}. \tag{67}$$

Similarly,

$$b = -\overline{b} = i\beta, \quad \alpha \in \mathbb{R}, \tag{68}$$

and

$$c = -\overline{d} = \gamma + i\delta, \quad \gamma,\ \delta \in \mathbb{R}. \tag{69}$$

In view of (67)–(69), the commutation relations (63)–(66) are rewritten as

$$[e_1, f_1] = i\alpha g, \quad \alpha \in \mathbb{R}; \tag{70}$$

$$[e_2, f_2] = i\beta g, \quad \beta \in \mathbb{R}; \tag{71}$$

$$[e_1, f_2] = (\gamma + i\delta)g, \quad \gamma,\ \delta \in \mathbb{R}; \tag{72}$$

$$[e_2, f_1] = (-\gamma + i\delta)g. \tag{73}$$

Now we consider Lie brackets with g. By Lemma 10.1, the operator $\operatorname{ad} g$ leaves the subspace $L_c^{(1)}(i) = \operatorname{span}(e_1, e_2)$ invariant. Moreover, since $L^{(1)}$ is nilpotent, the operator

$$\operatorname{ad} g : L_c^{(1)}(i) \to L_c^{(1)}(i) \tag{74}$$

is nilpotent.

(c.2.2.1.1) Let operator (74) be nonzero. Then it has a one-dimensional image.

(c.2.2.1.1.1) Let $\operatorname{Im}\left(\operatorname{ad} g|_{L_c^{(1)}(i)}\right) \neq \mathbb{C}e_2$. We choose a vector

$$e_1' \in L_c^{(1)}(i) = \operatorname{span}(e_1, e_2)$$

such that

$$e_1' = e_1 + ke_2, \quad k \in \mathbb{C}, \quad \text{and} \quad \operatorname{Im}\left(\operatorname{ad} g|_{L_c^{(1)}(i)}\right) = \mathbb{C}e_1'. \tag{75}$$

The operator $\operatorname{ad} x : L_c^{(1)}(i) \to L_c^{(1)}(i)$ has the same matrix

$$\begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix}$$

in the both bases $\{e_1, e_2\}$ and $\{e_1', e_2\}$. Therefore, we can denote e_1' by e_1, choose the corresponding vector $f_1 = \overline{e}_1$, and preserve the previous notation for the new basis $L_c = \operatorname{span}(x, e_1, e_2, f_1, f_2, g)$ with the additional property (see (75))

$$\operatorname{Im}\left(\operatorname{ad} g|_{L_c^{(1)}(i)}\right) = \mathbb{C}e_1.$$

Then, since the operator $\operatorname{ad} g : L_c^{(1)}(i) \to L_c^{(1)}(i)$ is nilpotent and nonzero, it has the matrix

$$\operatorname{ad} g|_{\operatorname{span}(e_1, e_2)} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \quad r \in \mathbb{C} \setminus \{0\}.$$

But the Jacobi identity for the triple (x, g, e_2) yields $r = 0$. The contradiction shows that case (c.2.2.1.1.1) is impossible.

(c.2.2.1.1.2) Let $\operatorname{Im}\left(\operatorname{ad} g|_{L_c^{(1)}(i)}\right) = \mathbb{C}e_2$. Then

$$\operatorname{ad} g|_{\operatorname{span}(e_1,e_2)} = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \quad q \in \mathbb{C} \setminus \{0\};$$
$$\operatorname{ad} g|_{\operatorname{span}(f_1,f_2)} = \begin{pmatrix} 0 & 0 \\ \overline{q} & 0 \end{pmatrix},$$

The Jacobi identity yields

$$\begin{aligned} (x, e_1, f_1) &\Rightarrow \quad \delta = 0, \\ (x, e_1, f_2) &\Rightarrow \quad \beta = 0, \\ (e_1, e_2, f_1) &\Rightarrow \quad \gamma = 0. \end{aligned}$$

Therefore, the preceding relations should be satisfied; if they hold, the Jacobi identity for all possible triples of base elements in L_c is satisfied. Finally, multiplication rules in the Lie algebra $L_c = \operatorname{span}(x, e_1, e_2, f_1, f_2, g)$ are determined by the following Lie brackets: (60) and

$$\begin{aligned} &[e_1, f_1] = i\alpha g, \quad \alpha \in \mathbb{R} \setminus \{0\}, \\ &\operatorname{ad} g|_{\operatorname{span}(e_1,e_2)} = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \quad q = q_1 + iq_2 \in \mathbb{C} \setminus \{0\}, \\ &\operatorname{ad} g|_{\operatorname{span}(f_1,f_2)} = \begin{pmatrix} 0 & 0 \\ \overline{q} & 0 \end{pmatrix}, \end{aligned} \tag{76}$$

all other brackets between the base elements are either zero or follow from the preceding ones by the skew-symmetry property (note that $\alpha \neq 0$ in (77), since $g \in L_c^{(1)}$).

Choose a new base vector

$$x' = x + \gamma g, \quad \gamma \in \mathbb{R}, \tag{77}$$

in L_c; the value of the parameter γ will be specified later. We have

$$\begin{aligned} &[x', e_1] = ie_1 + e_2 + q\gamma e_2 = ie_1 + (1 + q\gamma)e_2; \\ &[x', f_1] = -if_1 + f_2 + \overline{q}\gamma f_2 = -if_1 + (1 + \overline{q}\gamma)f_2. \end{aligned}$$

(c.2.2.1.1.2.1) Let $q \notin \mathbb{R}$. Choose the new base vectors

$$\begin{aligned} &e_1' = e_1, \quad e_2' = (1 + q\gamma)e_2, \\ &f_1' = f_1, \quad f_2' = (1 + \overline{q}\gamma)f_2, \\ &g' = -(\alpha/2)g \end{aligned}$$

in L_c and define the constants

$$\gamma = -\frac{q_1}{q_1^2 + q_2^2} \quad \text{and} \quad K = -\frac{\alpha}{2}\frac{q_1^2 + q_2^2}{q_2} \neq 0.$$

We have

$$[g', e_1'] = iKe_2';$$
$$[g', f_1'] = -iKf_2'.$$

Further, we divide the base vectors e_1', f_1', e_2', and f_2' by $\sqrt{|K|}$, the vector g' by $|K|$, denote the vectors obtained by e_1, f_1, e_2, f_2, and g, and come to the multip lication rules in L_c

$$[e_1, f_1] = -2ig;$$
$$[g, e_1] = \pm ie_2;$$
$$[g, f_1] = \mp if_2.$$

We pass to the real basis in L and obtain $L = L_{6,VII}$ or $L = L_{6,VIII}$.

(c.2.2.1.1.2.2) Let $q \in \mathbb{R}$.

We set $\gamma = -1/q$ in (77) and choose the following new base vectors in L_c:

$$e_1' = e_1, \quad e_2' = -\alpha(q/2)e_2,$$
$$f_1' = f_1, \quad f_2' = -\alpha(q/2)f_2,$$
$$g' = -(\alpha/2)g;$$

preserving the old notation for the new basis, we obtain the multiplication rules

$$[e_1, f_1] = -2ig,$$
$$[g, e_1] = e_2,$$
$$[g, f_1] = f_2.$$

In the corresponding real basis:

$$L = \operatorname{span}(x, y, z, u, v, w), \tag{78}$$
$$y = (e_1 + f_1)/2, \quad z = (e_1 - f_1)/(2i), \tag{79}$$
$$u = (e_2 + f_2)/2, \quad v = (e_2 - f_2)/(2i), \tag{80}$$
$$w = g, \tag{81}$$

we have $L = L_{6,III}(i)$.

(c.2.2.1.2) Now let operator (74) be zero:

$$\operatorname{ad} g|_{\operatorname{span}(e_1,e_2)} = 0; \tag{82}$$

thus,

$$\operatorname{ad} g|_{\operatorname{span}(f_1,f_2)} = 0. \tag{83}$$

Multiplication rules in L_c are determined by (60)–(62), (70)–(73), (82), and (83). The Jacobi identity implies

$$\begin{aligned}(x, e_1, f_1) &\Rightarrow \quad \delta = 0,\\ (x, e_1, f_2) &\Rightarrow \quad \beta = 0.\end{aligned}$$

If $\delta = \beta = 0$, then the Jacobi identity for all base elements in L_c holds.

Therefore, Lie brackets in L_c are determined by the following ones: (60) and

$$\begin{aligned}&[e_1, f_1] = i\alpha g, \quad \alpha \in \mathbb{R},\\ &[e_1, f_2] = -[e_2, f_1] = \gamma g, \quad \gamma \in \mathbb{R},\end{aligned}$$

where

$$\alpha^2 + \gamma^2 \neq 0,$$

since $g \in L_c^{(1)}$.

Now we proceed exactly as in item (c.2.2.1.1.2): we choose the real basis (79)–(81) and obtain

$$\begin{aligned}&[y, z] = kw, \quad k = -\alpha/2 \in \mathbb{R},\\ &[y, u] = [z, v] = lw, \quad l = \gamma/2 \in \mathbb{R},\\ &k^2 + l^2 \neq 0.\end{aligned}$$

If $l = 0$, then $L = L_{6,V}(i)$, and if $l \neq 0$, then $L = L_{6,VI}(i)$ (see Constructions 6.5, 6.6) in case (c.2.2.1.2).

(c.2.2.2) Let both eigenvalues $\pm i$ have algebraic multiplicity two. There exists a basis e_1, e_2, f_1, f_2, g of the derived subalgebra $L_c^{(1)}$ in which the operator ad has the diagonal matrix

$$\operatorname{ad} x|_{\operatorname{span}(e_1,e_2,f_1,f_2,g)} = \begin{pmatrix} i & 0 & 0 & 0 & 0\\ 0 & i & 0 & 0 & 0\\ 0 & 0 & -i & 0 & 0\\ 0 & 0 & 0 & -i & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (84)$$

In the same way as before, we obtain the Lie brackets

$$\begin{aligned}&[e_1, f_1] = i\alpha g, \quad \alpha \in \mathbb{R},\\ &[e_2, f_2] = i\beta g, \quad \beta \in \mathbb{R},\\ &[e_1, f_2] = (\gamma + i\delta) g, \quad \gamma,\ \delta \in \mathbb{R},\\ &[e_2, f_1] = (-\gamma + i\delta) g.\end{aligned}$$

The operator $\operatorname{ad} g : \operatorname{span}(e_1, e_2) \to \operatorname{span}(e_1, e_2)$ is nilpotent; moreover, it is nonzero, since otherwise $L_c^{(2)} = \mathbb{C} g$; hence $\mathrm{j}(\pm i) = 2$, a contradiction to

Theorem 1 in 7. Therefore, there exists a basis e_1, e_2 in $L_c^{(1)}(i)$ and the corresponding basis $f_1 = \overline{e}_1$, $f_2 = \overline{e}_2$ in $L_c^{(1)}(-i)$ in which

$$\operatorname{ad} g|_{\operatorname{span}(e_1,e_2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{ad} g|_{\operatorname{span}(f_1,f_2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \tag{85}$$

note that the matrix in (84) remains without changes in the new basis.

We write the Jacobi identity for the triples of elements and obtain

$$\begin{aligned} (g, e_2, f_1) &\Rightarrow \quad \alpha = 0; \\ (g, e_2, f_2) &\Rightarrow \quad \delta = 0; \\ (f_2, e_1, e_2) &\Rightarrow \quad \gamma = 0. \end{aligned}$$

If $\alpha = \delta = \gamma = 0$, then Jacobi identity holds for all possible triples of base elements of L_c.

Therefore, multiplication rules in L_c are determined by (84), (85), and

$$[e_2, f_2] = i\beta g, \quad \beta \in \mathbb{R} \setminus \{0\}, \tag{86}$$

where $\beta \neq 0$, since $g \in L_c^{(1)}$.

Now we choose the new base vectors

$$e_1' = \beta e_1, \quad f_1' = \beta f_1, \quad \text{and} \quad g' = \beta g,$$

denote them as before by e_1, f_1, and g, respectively, and obtain

$$[e_2, f_2] = ig$$

instead of (86).

Finally, we pass to the real basis (79)–(81) in L and obtain $L = L_{6,III}(i)$; see Construction 6.3.

Therefore, all possible cases of disposition of the spectrum $\mathrm{Sp}^{(1)}$ in the complex plane are considered, and in all these cases, the Lie algebra L has one of the types $L_{6,I}$–$L_{6,VIII}$. The necessity follows.

Sufficiency. All Lie algebras listed in (1)–(8) of Theorem 6.7 are controllable by Theorems 6.1–6.6, (*c*).

Theorem 6.7 is completely proved. □

6.4. Isomorphisms of controllable Lie algebras.

Theorem 6.8. *Any two six-dimensional controllable solvable Lie algebras that belong to distinct classes (1)–(8) given in Theorem 6.7 are not isomorphic one to another. All isomorphisms inside these classes are as follows.*

(1) $L_{6,I}(\lambda_1, \mu_1) \simeq L_{6,I}(\lambda_2, \mu_2)$, $\lambda_j, \mu_j \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_j \neq \mu_j, \bar{\mu}_j$, $j = 1, 2$, *if and only if* $\{\lambda_2, \bar{\lambda}_2\} = k\{\lambda_1, \bar{\lambda}_1\}$ *and* $\{\mu_2, \bar{\mu}_2\} = k\{\mu_1, \bar{\mu}_1\}$ *for some* $k \in \mathbb{R} \setminus \{0\}$.

(2) $L_{6,II}(\lambda_1,\mu_1) \simeq L_{6,II}(\lambda_2,\mu_2)$, $\lambda_j,\mu_j \in \mathbb{C}\setminus\mathbb{R}$, $\operatorname{Re}\lambda_j = \operatorname{Re}\mu_j$, $\lambda_j \neq \mu_j, \bar{\mu}_j$, $j = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1, \mu_1, \bar{\mu}_1\} \sim \{\lambda_2, \bar{\lambda}_2, \mu_2, \bar{\mu}_2\}$.
(3) $L_{6,III}(\lambda_1) \simeq L_{6,III}(\lambda_2)$, $\lambda_j \in \mathbb{C}\setminus\mathbb{R}$, $j = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.
(4) $L_{6,IV}(\lambda_1) \simeq L_{6,IV}(\lambda_2)$, $\lambda_j \in \mathbb{C}\setminus(\mathbb{R}\cup i\,\mathbb{R})$, $j = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.
(5) $L_{6,V}(\lambda_1) \simeq L_{6,V}(\lambda_2)$, $\lambda_j \in \mathbb{C}\setminus\mathbb{R}$, $j = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.
(6) $L_{6,VI}(\lambda_1) \simeq L_{6,VI}(\lambda_2)$, $\lambda_j \in \mathbb{C}\setminus\mathbb{R}$, $j = 1, 2$, *if and only if* $\{\lambda_1, \bar{\lambda}_1\} \sim \{\lambda_2, \bar{\lambda}_2\}$.

Proof. In this proof, we denote the Lie algebra $L_{6,III}(bi) \simeq L_{6,III}(i)$, $b \in \mathbb{R}\setminus\{0\}$, by $L_{6,IX}$.

(1) Lie algebras $L_{6,III}(\lambda)$, $\lambda \in \mathbb{C}\setminus(\mathbb{R}\cup i\,\mathbb{R})$, are not isomorphic to Lie algebras of all other classes $L_{6,I}$, $L_{6,II}$, $L_{6,IV}$–$L_{6,IX}$, since $\dim L^{(2)} = 3$ and the spectrum $\operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = \{\lambda, \bar{\lambda}, \lambda + 2a, \bar{\lambda} + 2a, 2a\}$ is algebraically simple for $L = L_{6,III}(\lambda)$, $\lambda \in \mathbb{C}\setminus(\mathbb{R}\cup i\,\mathbb{R})$, which is not the case for Lie algebras of all other classes.

(2) We show that $L_{6,I}(\lambda_1,\mu_1) \not\simeq L_{6,II}(\lambda_2,\mu_2)$. On the contrary, we suppose that $L_{6,I}(\lambda_1,\mu_1) \simeq L_{6,II}(\lambda_2,\mu_2)$. We identify both these Lie algebras with $L = L_{6,I}(\lambda,\mu)$, and fix a basis $\{x, y, z, u, v, w\}$ in L as in Construction 6.1. There exists another basis $\{x', y', z', u', v', w'\}$ in L with multiplication rules as in Construction 6.2. By Lemma 10.2, $\operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) \sim \operatorname{Sp}(\operatorname{ad} x'|_{L^{(1)}})$. Rescaling and renumbering the base vectors, if necessary, we can obtain $\lambda = \lambda_1 = \lambda_2$, and $\mu = \mu_1 = \mu_2$.

We pass to the corresponding complex bases in L_c:

$$x,\ e_1 = y + iz,\ f_1 = y - iz,\ e_2 = u + iv,\ f_2 = u - iv,\ g = w,$$
$$x',\ e_1' = y' + iz',\ f_1' = y' - iz',\ e_2' = u' + iv', f_2' = u' - iv',\ g' = w'.$$

Then

$$\operatorname{ad} x|_{\operatorname{span}(e_1,f_1,e_2,f_2,g)} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \bar{\mu} & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix},$$
$$[e_1, f_1] = -2ig,$$
$$[e_2', f_2'] = -2ig'. \tag{87}$$

Further, we have the decomposition

$$x' = x + \alpha e_1 + \overline{\alpha} f_1 + \beta f_1 + \overline{\beta} f_2 + \gamma g, \quad \alpha, \beta \in \mathbb{C}, \quad \gamma \in \mathbb{R};$$

hence

$$\operatorname{ad} x'|_{\operatorname{span}(e_1,f_1,e_2,f_2,g)} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \bar{\mu} & 0 \\ -2i\alpha & 2i\overline{\alpha} & 0 & 0 & 2a \end{pmatrix}.$$

Thus,

$$e_1' = p(e_1 - 2i(\alpha/\lambda)g), \quad e_2' = qe_2, \quad p,\ q \in \mathbb{C} \setminus \{0\}; \tag{88}$$

$$f_1' = \overline{p}(f_1 + 2i(\overline{\alpha}/\overline{\lambda})g), \quad f_2' = \overline{q}e_2, \tag{89}$$

$$g' = rg, \quad r \in \mathbb{R} \setminus \{0\}. \tag{90}$$

Now

$$[e_2', f_2'] = 0,$$

which contradicts (87). Thus $L_{6,I}(\lambda_1,\mu_1) \not\simeq L_{6,II}(\lambda_2,\mu_2)$.

(3) We show that $L_{6,I}(\lambda_1,\mu_1) \not\simeq L_{6,IV}(\lambda_2)$. As in **(2)**, we have

$$[e_1', f_2'] = -2ig'$$

from Construction 6.4 and

$$[e_1', f_2'] = 0$$

from (88) and (89), a contradiction.

(4) The Lie algebras $L_{6,I}(\lambda,\mu)$, $\lambda,\mu \in \mathbb{C}\backslash\mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, are not isomorphic to any of the Lie algebras $L_{6,V}$–$L_{6,IX}$, since the spectrum $\{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}$ of the operator $\operatorname{ad} x|_{L^{(1)}}$ is algebraically simple for $L = L_{6,I}(\lambda,\mu)$, $\lambda,\mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, which is not the case for Lie algebras of the classes $L_{6,V}$–$L_{6,IX}$.

(5) $L_{6,II}(\lambda_1,\mu_1) \not\simeq L_{6,IV}(\lambda_2)$, $\lambda_1,\lambda_2,\mu_1 \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda_1 = \operatorname{Re}\mu_1$, $\lambda_1 \neq \mu_1, \bar{\mu}_1$, since $\{\lambda_1, \bar{\lambda}_1, \mu_1, \bar{\mu}_1, 2a_1\} \not\simeq \{\lambda_2, \bar{\lambda}_2, -\lambda_2, -\bar{\lambda}_2, 0\}$.

(6) The Lie algebras $L_{6,II}(\lambda,\mu)$, $\lambda,\mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda = \operatorname{Re}\mu$, $\lambda \neq \mu, \bar{\mu}$, are not isomorphic to any of the Lie algebras $L_{6,V}$–$L_{6,IX}$, since the spectrum $\{\lambda, \bar{\lambda}, \mu, \bar{\mu}\}$ is algebraically simple for $L = L_{6,II}(\lambda,\mu)$, which is not the case for Lie algebras of the classes $L_{6,V}$–$L_{6,IX}$.

(7) The Lie algebras $L_{6,IV}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, are not isomorphic to any of the Lie algebras $L_{6,V}$–$L_{6,IX}$ by the same argument as in **(6)**.

(8) We show that $L_{6,V}(\lambda_1) \not\simeq L_{6,VI}(\lambda_2)$, $\lambda_1,\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Assume that $L_{6,V}(\lambda_1) \simeq L_{6,VI}(\lambda_2)$. Choose the canonical bases as in Constructions 6.5, 6.6:

$$L_{6,V}(\lambda_1) = \operatorname{span}(x_1, y_1, z_1, u_1, v_1, w_1);$$

$$L_{6,VI}(\lambda_2) = \operatorname{span}(x_2, y_2, z_2, u_2, v_2, w_2).$$

The derived subalgebra $L_{6,V}^{(1)}(\lambda_1)$ contains the 3-dimensional subspace

$$I_1 = \operatorname{span}(u_1, v_1, w_1)$$

in its center. Thus, there is a 3-dimensional subspace I_2 in the center of $L^{(1)}_{6,VI}(\lambda_2)$. We have

$$\dim(I_2 \cap \operatorname{span}(y_2, z_2, u_2, v_2)) \geq 1.$$

Take any vector

$$0 \neq f = ay_2 + bz_2 + cu_2 + dv_2 \in I_2.$$

We multiply this decomposition by vectors y_2, z_2, u_2, and v_2, and obtain

$$a = b = c = d = 0,$$

a contradiction.

(9) The Lie algebras $L_{6,V}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are not isomorphic to any one of the Lie algebras $L_{6,VII}$, $L_{6,VIII}$, and $L_{6,IX}$, since $\dim L^{(2)}_{6,V}(\lambda) = 1$ but $\dim L^{(2)}_{6,VII} = \dim L^{(2)}_{6,VIII} = \dim L^{(2)}_{6,IX} = 3$.

(10) We show that $L_{6,VI}(\lambda) \not\simeq L_{6,VII}$, $L_{6,VIII}$, $L_{6,IX}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, as in items **(8)** and **(9)** above.

(11) We show that $L_{6,VII} \not\simeq L_{6,VIII}$. We suppose the contrary, identify $L = L_{6,VII} = L_{6,VIII}$, and choose the canonical bases as in Constructions 6.7 and 6.8:

$$\begin{aligned}
&L_{6,VII} = \operatorname{span}(x, y, z, u, v, w);\\
&L_{6,VIII} = \operatorname{span}(x', y', z', u', v', w');\\
&e_1 = y + iz,\ f_1 = y - iz,\ e_2 = u + iv,\ f_2 = u - iv,\ g = w;\\
&e_1' = y' + iz',\ f_1' = y' - iz',\ e_2' = u' + iv',\ f_2' = u' - iv',\ g' = w'.
\end{aligned}$$

Then

$$\begin{aligned}
&\operatorname{ad} x|_{\operatorname{span}(e_1, f_1, e_2, f_2, g)} = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\\
&[g, e_1] = ie_2, \quad [g, f_1] = -if_2,\\
&[g', e_1'] = -ie_2', \quad [g', f_1'] = if_2'.
\end{aligned} \tag{91}$$

Further,

$$x' = x + \alpha e_1 + \beta e_2 + \overline{\alpha} f_1 + \overline{\beta} f_2 + \gamma g, \quad \alpha,\ \beta \in \mathbb{C}, \quad \gamma \in \mathbb{R},$$

consequently,

$$\operatorname{ad} x'|_{\operatorname{span}(e_1,f_1,e_2,f_2,g)} = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 1+i\gamma & i & 0 & 0 & i\alpha \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 1-i\gamma & -i & -i\overline{\alpha} \\ 2i\overline{\alpha} & 0 & -2i\alpha & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} &e_1' = k(e_1 + 2\overline{\alpha}g), \quad e_2' = k(1+i\gamma)e_2, \quad k \in \mathbb{C}\setminus\{0\}, \\ &f_1' = \overline{k}(f_1 + 2\alpha g), \quad f_2' = \overline{k}(1-i\gamma)f_2, \\ &g' = -1/(2i)[e_1', f_1'] = k\overline{k}(g + \alpha e_2 + \overline{\alpha} f_2), \\ &[g', e_1'] = \frac{ik\overline{k}}{1+i\gamma}e_2'. \end{aligned}$$

We compare the last relation with (91) and obtain

$$\frac{k\overline{k}}{1+i\gamma} = -1, \tag{92}$$

which is impossible. The contradiction shows that $L_{6,VII} \not\simeq L_{6,VIII}$.

(12) We show that $L_{6,VII} \not\simeq L_{6,IX}$. By an argument similar to that of item **(11)**, we obtain from multiplication rules in $L_{6,IX}$ that

$$[g', e_1'] = e_2'$$

instead of (91). Now

$$\frac{ik\overline{k}}{1+i\gamma} = 1$$

instead of (92), a contradiction.

(13) By a similar argument, we show that $L_{6,VIII} \not\simeq L_{6,IX}$.

Therefore, we proved that all classes of Lie algebras $L_{6,I}$–$L_{6,IX}$ are pairwise nonisomorphic. Now we pass to the study of isomorphisms inside these classes.

(14) We prove statement (1) of Theorem 6.8.

Necessity. By Lemma 10.2, there exists a number $k \in \mathbb{R}\setminus\{0\}$ such that

$$\{\lambda_2, \bar{\lambda}_2\,\mu_2, \bar{\mu}_2\} = k\{\lambda_1, \bar{\lambda}_1\,\mu_1, \bar{\mu}_1\}.$$

It follows from the multiplication rules in Construction 6.1 that in any Lie algebra $L_{6,I}(\lambda,\mu) = \operatorname{span}(x,y,z,u,v,w)$, the ideal $I = \operatorname{span}(u,v) = L^{(1)}(\mu)$ is invariantly defined (see expressions for e_2', and f_2' in (88), and (89)). Let $I_j \subset L_{6,I}(\lambda_j,\mu_j)$, $j = 1,2$, be such ideals in the isomorphic Lie algebras. Then

$$L_4(\lambda_1) = L_{6,I}(\lambda_1,\mu_1)/I_1 \simeq L_{6,I}(\lambda_2,\mu_2)/I_2 = L_4(\lambda_2).$$

By Theorem 4.3,

$$\{\lambda_2, \bar{\lambda}_2\} = k\{\lambda_1, \bar{\lambda}_1\}.$$

Necessity follows.

Sufficiency. Let

$$\{\lambda_2, \bar{\lambda}_2\} = k\{\lambda_1, \bar{\lambda}_1\}, \quad \{\mu_2, \bar{\mu}_2\} = k\{\mu_1, \bar{\mu}_1\}, \quad k \in \mathbb{R} \setminus \{0\}.$$

There can be the following four cases:

(1) $\lambda_2 = k\lambda_1$, $\mu_2 = k\mu_1$;
(2) $\lambda_2 = k\lambda_1$, $\mu_2 = k\bar{\mu}_1$;
(3) $\lambda_2 = k\bar{\lambda}_1$, $\mu_2 = k\mu_1$;
(4) $\lambda_2 = k\bar{\lambda}_1$, $\mu_2 = k\bar{\mu}_1$.

In each of these cases, the required correspondence between the canonical base vectors in $L_{6,I}(\lambda_1, \mu_1) = \operatorname{span}(x_1, y_1, z_1, u_1, v_1, w_1)$ and $L_{6,I}(\lambda_2, \mu_2) = \operatorname{span}(x_2, y_2, z_2, u_2, v_2, w_2)$ is as follows:

(1) $x_2 \mapsto kx_1$, $y_2 \mapsto y_1$, $z_2 \mapsto z_1$, $u_2 \mapsto u_1$, $v_2 \mapsto v_1$, $w_2 \mapsto w_1$;
(2) $x_2 \mapsto kx_1$, $y_2 \mapsto y_1$, $z_2 \mapsto z_1$, $u_2 \mapsto v_1$, $v_2 \mapsto u_1$, $w_2 \mapsto w_1$;
(3) $x_2 \mapsto kx_1$, $y_2 \mapsto z_1$, $z_2 \mapsto y_1$, $u_2 \mapsto u_1$, $v_2 \mapsto v_1$, $w_2 \mapsto w_1$;
(4) $x_2 \mapsto kx_1$, $y_2 \mapsto z_1$, $z_2 \mapsto y_1$, $u_2 \mapsto v_1$, $v_2 \mapsto u_1$, $w_2 \mapsto w_1$.

(15) We prove statement (2) of Theorem 6.8.

Necessity follows from Lemma 10.2.

Sufficiency is proved by constructing a correspondence between canonical bases as in item **(14)**.

(16) Statements (3)–(6) are proved by the argument used in the previous item. □

7. Low-dimensional solvable Lie algebras

In this section, we collect several propositions that are valid for all controllable low-dimensional solvable Lie algebras.

Theorem 7.1. *Let L be a controllable solvable Lie algebra,* $\dim L \leq 6$. *Then the following assertions hold.*

(a) *The only codimension one subalgebra in L is its derived subalgebra $L^{(1)}$.*
(b) *Let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold:*
   (1) $B \notin L^{(1)}$;
   (2) $\operatorname{Lie}(A, B) = L$.
(c) *Let $B \in L \setminus L^{(1)}$. Then the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable for almost all $A \in L$.*

Proof. This proposition follows directly from results of Secs. 1–6. □

The simple description of all codimension one subalgebras provided by the previous theorem leads to the following controllability test for *arbitrary* (not just control-affine and single-input) right-invariant systems.

Theorem 7.2. *Let L be controllable solvable Lie algebra with* $\dim L \leq 6$. *Then an arbitrary right-invariant system $\Sigma \subset L$ is controllable if and only if the following conditions hold*:

(1) $\mathrm{Lie}(\Sigma) = L$;
(2) *the system Σ is not contained in any one of the two half-spaces in L bounded by the hyperplane $L^{(1)}$.*

Proof. Apply Proposition 1 and Theorem 7.1, (a). □

8. Controllability of segments

The preceding results of this paper were related to right-invariant systems of the form

$$\Gamma = A + \mathbb{R}B = \{A + uB \mid u \in \mathbb{R}\} \subset L, \tag{93}$$

i.e., affine lines in a Lie algebra L. Moreover, we define a controllable Lie algebra as a Lie algebra that contains at least one controllable line Γ. Now we pass to right-invariant systems of the form

$$S = \{(1-u)A + uB \mid u \in [0, 1]\} \subset L, \tag{94}$$

i.e., *segments* in L. We obtain complete controllability conditions for segments and show that the definition of a controllable Lie algebra L can equivalently be given in terms of controllable segments in L.

Remark. In the classical notation, segment (94) is written as the control system

$$\dot{X} = (1-u)A(X) + uB(X), \quad u \in [0, 1], \quad X \in G,$$

where the state space G is the connected simply connected Lie group corresponding to the Lie algebra L.

For a subset Σ of a vector space L, denote by $\mathrm{cone}(\Sigma)$ the closed convex positive cone generated by the set Σ. Recall that a right-invariant system Σ in a Lie algebra L is controllable iff the system $\mathrm{cone}(\Sigma)$ is controllable.

For an arbitrary Lie algebra L, controllability of a segment $S \subset L$ obviously implies controllability of any line $\Gamma \subset L$ such that $\mathrm{cone}(S) \subset \mathrm{cone}(\Gamma)$. The inverse implication holds in solvable Lie algebras.

Theorem 8.1. *Let L be a real solvable Lie algebra. A segment $S \subset L$ is controllable iff any line $\Gamma \subset L$ with* $\mathrm{cone}(S) \subset \mathrm{cone}(\Gamma)$ *is controllable.*

Proof. Necessity is already known, and we pass to sufficiency. Let $S \subset L$ be a noncontrollable segment. To prove this theorem, we construct a noncontrollable line $\Gamma \subset L$ with $\operatorname{cone}(S) \subset \operatorname{cone}(\Gamma)$. By Proposition 1, we have

(1) $\operatorname{Lie}(S) \neq L$ or
(2) there exists a codimension one subalgebra $l \subset L$ such that S is contained in a half-space $\Pi \subset L$ bounded by the hyperplane l.

In case (1), the line $\Gamma \supset S$ is the required one:

$$\operatorname{Lie}(\Gamma) = \operatorname{Lie}(A, B) = \operatorname{Lie}(S) \neq L,$$
$$\operatorname{cone}(S) \subset \operatorname{cone}(\Gamma).$$

Consider case (2). If the subalgebra l contains the space $\operatorname{span}(S)$, then

$$\operatorname{Lie}(S) \subset l \neq L,$$

and we proceed as in case (1). Let $\operatorname{span}(S) \not\subset l$. If $\dim \operatorname{span}(S) = 1$, then $\operatorname{Lie}(S) = \operatorname{span}(S) \neq L$. Therefore, $\dim \operatorname{span}(S) = 2$, i.e., the hyperplane l and the plane $\operatorname{span}(S)$ intersect transversally. The intersection $\Pi_1 = \operatorname{span}(S) \cap \Pi$ is a half-plane and obviously $S \subset \Pi_1$. Thus, $\operatorname{cone}(S) \subset \operatorname{cone}(\Pi_1) = \Pi_1$. Take any line Γ in the half-plane Π_1 such that $\operatorname{cone}(\Gamma) = \Pi_1$. The line Γ is the required one, since

$$\operatorname{cone}(S) \subset \Pi_1 = \operatorname{cone}(\Gamma)$$

and

$$\Gamma \subset \Pi_1 \subset \Pi \ \Rightarrow\ \Gamma \text{ is noncontrollable.} \qquad \square$$

Theorem 8.2. *A Lie algebra L is controllable* (*i.e, L contains a controllable line* (93)) *iff L contains a controllable segment* (94).

Proof. Sufficiency. If a segment $S \subset L$ is controllable, then the line $\Gamma \supset S$ is also controllable.

Necessity. If a line $\Gamma \subset L$ is controllable, then a sufficiently long segment $S \subset \Gamma$ is also controllable. Indeed, let $O \subset \operatorname{span}(\Gamma)$ be a circle centered at the origin. Since the line Γ is controllable, the arc

$$A = O \cap \operatorname{cone}(\Gamma)$$

is controllable because $\operatorname{cone}(A) = \operatorname{cone}(\Gamma)$. Further, controllability of right-invariant systems is preserved under small perturbations (see, e.g., Theorem 2.10 in 8); therefore, any arc A_1 contained in interior of A and sufficiently close to A is controllable. Then the segment

$$S = \Gamma \cap \operatorname{cone}(A_1) \subset \Gamma$$

is controllable since $\operatorname{cone}(S) = \operatorname{cone}(A_1)$. $\square$

Therefore, if a solvable Lie algebra L is noncontrollable, then any segment $S \subset L$ is noncontrollable. If L is controllable, then it contains a controllable segment S. A controllability test for segments (as well as for arbitrary right-invariant systems) in controllable Lie algebras is provided by Theorem 7.2.

9. Concluding remarks

The complete description of controllable solvable Lie algebras up to dimension 6 obtained in this paper is possible mainly due to the necessary and sufficient controllability conditions on solvable Lie groups of Theorems 1 and 2 in 7. The most essential gap between these conditions, the absence of N-pairs of eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$, almost vanishes in dimensions 1–6. However, in dimension 7, there appear families of Lie algebras L and vectors $A, B \in L$ for which necessary controllability conditions hold, but sufficient controllability conditions are violated (see Construction 9.1 below). Therefore, the approach of this paper fails starting from dimension 7. Although, this bound seems to be technical only: the results of Sec. 7 common for all dimensions 1–6 may well be extended to higher dimensions. Intrinsically, the result of Jimmie Lawson (Proposition 1) states that in solvable Lie algebras, codimension one subalgebras (together with the rank condition) are responsible for controllability. It seems that in solvable Lie algebras L, the following alternative holds:

(1) either the derived subalgebra $L^{(1)}$ is the only codimension one subalgebra;
(2) or there is an infinite number of codimension one subalgebras.

If this alternative is true, then controllable solvable Lie algebras are exactly Lie algebras with one subalgebra of codimension one, the derived subalgebra. The theory of K.H. Hoffmann of codimension one subalgebras 2 can be helpful in this direction.

We conclude by the seven-dimensional example, a gap between necessary and sufficient controllability conditions.

Construction 9.1. The Lie algebra $L_7(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$; see Fig. 11.

$$L_7(\lambda, \mu) = \operatorname{span}(x, y, z, u, v, s, t);$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,s,t)} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & -d & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c \end{pmatrix};$$

$$\lambda = a + bi, \ \mu = c + di;$$

$$[y, z] = s, \quad [u, v] = t.$$

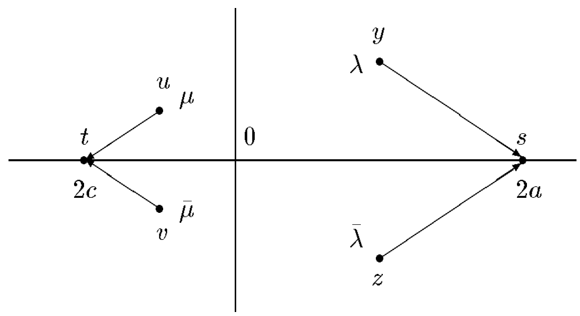

Fig. 11. $L_7(\lambda, \mu)$.

Let $L = L_7(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ with $c = \operatorname{Re} \mu < 0 < \operatorname{Re} \lambda = a$, and let $A, B \in L$ be any elements such that $B \notin L^{(1)}$ and $A(B_x \lambda) \neq 0$, $A(B_x \mu) \neq 0$. Then all conditions of Theorem 1 in 7 are satisfied. On the other hand, the pair $(B_x \cdot 2c, B_x \cdot 2a)$ is an N-pair of eigenvalues; thus, condition (6) of Theorem 2 in 7 is violated.

Remark. After this work was completed, the author found out that Dirk Mittenhuber 10 obtained a purely algebraic description of controllable solvable Lie algebras in arbitrary dimensions. It follows from this description that the Lie algebra $L_7(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, is controllable.

10. Appendix: auxiliary propositions

Lemma 10.1. *Let L be a real Lie algebra and $B \in L \setminus L^{(1)}$. Then*

(1) $[L_c^{(1)}(a), L_c^{(1)}(b)] \subset L_c^{(1)}(a+b)$ *if* $a, b \in \operatorname{Sp}^{(1)}$;

(2) $$[L^{(1)}(a), L^{(1)}(b)] \subset \begin{cases} L^{(1)}(a+b) & \text{if } a, b \in \operatorname{Sp}_r^{(1)}, \\ L^{(1)}(a+b) & \text{if } a \in \operatorname{Sp}_c^{(1)}, \\ & \quad b \in \operatorname{Sp}_r^{(1)}, \\ L^{(1)}(a+b) + L^{(1)}(a+\bar{b}) & \text{if } a, b \in \operatorname{Sp}_c^{(1)}; \end{cases}$$

(3) $[L^{(1)}(a), L^{(1)}(a)] \subset L^{(1)}(2 \operatorname{Re} a)$ *if* $a \in \operatorname{Sp}_c^{(1)}$.

Proof. (1) follows from the Jacobi identity.

(2) and (3) follow from realification of (1). $\square$

Lemma 10.2. *Let L be a real solvable Lie algebra with* $\dim L^{(1)} = \dim L - 1$. *For any elements $x, B \in L \setminus L^{(1)}$, consider the decomposition*

$$B = B_x x + B^1, \quad B^1 \in L^{(1)}.$$

Then

$$\mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}}) = B_x \cdot \mathrm{Sp}(\mathrm{ad}\, x|_{L^{(1)}});$$
$$\mathrm{Sp}(\mathrm{ad}\, B|_{L^{(2)}}) = B_x \cdot \mathrm{Sp}(\mathrm{ad}\, x|_{L^{(2)}}).$$

Proof. It is well known that in any complex solvable Lie algebra there exists a basis in which all inner derivation operators are triangular (see e. g. Theorem 3.7.3 in 9). We choose such a basis in the complexification L_c:

$$\mathrm{ad}_c\, z|_{L_c^{(1)}} = \begin{pmatrix} \lambda_1(z) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n(z) \end{pmatrix}, \quad z \in L_c;$$

then

$$\mathrm{Sp}(\mathrm{ad}_c\, z|_{L_c^{(1)}}) = \mathrm{Sp}(\mathrm{ad}\, z|_{L^{(1)}}) = \{\lambda_1(z), \dots, \lambda_n(z)\}, \quad z \in L.$$

But the Lie algebra $L_c^{(1)}$ is nilpotent as the derived subalgebra of a solvable Lie algebra; consequently, the operator $\mathrm{ad}_c\, B^1|_{L_c^{(1)}}$ is nilpotent, i.e.,

$$\lambda_1(B^1) = \dots = \lambda_n(B^1) = 0.$$

Therefore,

$$\mathrm{ad}_c\, B|_{L_c^{(1)}} = B_x\, \mathrm{ad}_c\, x|_{L_c^{(1)}} + \mathrm{ad}_c\, B^1|_{L_c^{(1)}} = B_x \begin{pmatrix} \lambda_1(z) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n(z) \end{pmatrix};$$

thus,

$$\mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}}) = B_x \cdot \{\lambda_1(z), \dots, \lambda_n(z)\} = B_x \cdot \mathrm{Sp}(\mathrm{ad}\, x|_{L^{(1)}}).$$

The relation for the spectra in $L^{(2)}$ is proved similarly. □

Lemma 10.3. *Let L be a real Lie algebra such that $L \neq L^{(1)} \neq L^{(2)}$. Let $B \in L \setminus L^{(1)}$ and $\mathrm{Sp}^{(1)} \subset \mathbb{R}$. Then $L_r^{(1)} \neq L_r^{(2)}$.*

Proof. $L_r^{(1)} = L^{(1)} \neq L^{(2)} = L_r^{(2)}$. □

Lemma 10.4. *Let L be a real Lie algebra,* $\dim L^{(1)} = \dim L - 1$, $A, B \in L$, $B \notin L^{(1)}$, *and let the spectrum* $\mathrm{Sp}^{(1)} = \mathrm{Sp}(\operatorname{ad} B|_{L^{(1)}})$ *be geometrically simple. Consider the decomposition*

$$A = A_B B + A^1, \quad A_B \in \mathbb{R},\ A^1 \in L^{(1)}. \tag{95}$$

Then the following conditions are equivalent:

(1) $\operatorname{top}(A, \lambda) \neq 0$ *for all* $\lambda \in \mathrm{Sp}^{(1)}$;
(2) *the vector* A^1 *does not belong to any proper invariant subspace of the operator* $\operatorname{ad} B|_{L^{(1)}}$;
(3) $\operatorname{span}(B, A, (\operatorname{ad} B)A, \dots, (\operatorname{ad} B)^{n-2}A) = L$, $n = \dim L$.

Proof. By Lemma 5.1 in 7, condition (1) is equivalent to the following one:

$$\operatorname{rank}(A^1, (\operatorname{ad} B)A^1, \dots, (\operatorname{ad} B)^{n-2}A^1) = n - 1 \tag{96}$$

(one should just replace in that lemma $\mathbb{R}^n$, b, and A respectively by $L^{(1)}$, A^1, and $\operatorname{ad} B$). Further, (96) is equivalent to

$$\operatorname{span}(A^1, (\operatorname{ad} B)A^1, \dots, (\operatorname{ad} B)^{n-2}A^1) = L^{(1)}, \tag{97}$$

which holds iff

$$\operatorname{span}(B, A^1, (\operatorname{ad} B)A^1, \dots, (\operatorname{ad} B)^{n-2}A^1) = L.$$

In view of (95) and since $(\operatorname{ad} B)A^1 = (\operatorname{ad} B)A$, the above relation is equivalent to condition (3) of this lemma. The equivalence (1) $\Leftrightarrow$ (3) is proved.

The proposition (2) $\Leftrightarrow$ (3) follows since (97) is equivalent to (2). $\square$

Lemma 10.5. *Let L be a real solvable Lie algebra with a codimension one derived subalgebra $L^{(1)}$. Assume that for any element $B \in L \setminus L^{(1)}$, there exists an element $A \in L$ such that the system $\Gamma = A + \mathbb{R}B \subset L$ is controllable. Then the derived subalgebra $L^{(1)}$ is the only codimension one subalgebra of L.*

Proof. Suppose that $l \neq L^{(1)}$ is another codimension one subalgebra in L. Take any element $B \in l \setminus L^{(1)}$. By Proposition 1, for any $A \in L$, the system $\Gamma = A + \mathbb{R}B$ is not controllable, since Γ is contained in a half-space bounded by the subalgebra l. The contradiction with the hypothesis of this lemma completes the proof. $\square$

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