Low-Dimensional Control of the 2D Navier–Stokes and Euler Equations

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We consider the Navier–Stokes and Euler equations on the 2-dimensional Riemannian surface M homeomorphic to the sphere, torus or disc. In the last case we assume that ∂M is a piecewise smooth curve and impose Lions boundary condition. The equations written in terms of the vorticity w and the stream functions ψ read:

$$\frac{\partial w}{\partial t} + \{\psi, w\} - \nu \Delta w = f(t,x), \quad \Delta \psi = w, \quad (1)$$

$$0 \le t \le T,\ x \in M, \quad \psi\Big|_{\partial M} = w\Big|_{\partial M} = 0,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket, Δ the Laplace–Beltrami operator, ν a nonnegative real number, and the right-hand side f is the vorticity of the external force. We assume that the right-hand side has the form:

$$f(t,x) = f_0(x) + \sum_{i=1}^{k} v_i(t) f_i(x),$$

where $f_0, f_1, \dots, f_k$ are fixed smooth functions and $v_1(\cdot), \dots, v_k(\cdot)$ are control functions at our disposal. We assume that $v_i(\cdot)$ belong to the space of admissible controls $V \subset L_\infty[0,T]$ and that V is an everywhere dense vector subspace of $L_1[0,T]$.

Given $\varphi_0 \in H_2(M)$, we say that $\varphi_T \in H_2(M)$ is reachable from φ_0 if there exist admissible control functions $v_1(\cdot), \ldots, v_k(\cdot)$ such that the solution of system (1) with the initial condition $w(0,\cdot) = \varphi_0$ satisfies the equation $w(T,\cdot) = \varphi_T$. Let $\mathcal{R}(\varphi_0) \subset H_2(M)$ be the set of all reachable functions. We say that the system is L_2-approximately controllable if $\mathcal{R}_T(\varphi_0)$ is everywhere dense in $L_2(M)$ for any $\varphi_0 \in H_2(M)$. The system is controllable in finite dimensional projections if the L_2-orthogonal projection of $\mathcal{R}_T(\varphi_0)$ on any finite dimensional subspace of $H_2(M)$ is surjective.

The input–state map $S_{\varphi_0} : V^k \to H_2(M)$ sends a control vector-function $(v_1, \ldots, v_k)$ to $w(T,\cdot)$.

In particular, $\mathcal{R}(\varphi) = S_{\varphi_0}(V^k)$. Given a finite dimensional subspace E of $H_2(M)$ we denote by $P_E : L_2(M) \to E$ the orthogonal projector. The system is controllable in finite dimensional projections iff the mapping $P_E \circ S_{\varphi_0}$ is surjective for any E and φ_0.

Solid controllability in finite dimensional projections is a robust version of the usual one. We say that the mapping $P_E \circ S_{\varphi_0}$ is robustly surjective if for any ball B in E there exists a finite dimensional ball $\mathcal{B}$ in V^k such that $\Phi(\mathcal{B}) \supset B$ for any sufficiently close to $P_E \circ S_{\varphi_0}|_{\mathcal{B}}$ in C^0-topology continuous mapping $\Phi : \mathcal{B} \to E$. The system is solidly controllable in finite dimensional projections if $P_E \circ S_{\varphi_0}$ is robustly surjective for any E and φ_0.

Assume that $f_1, \ldots, f_l$ are steady states of the Euler equation:

$$\{\Delta^{-1} f_i, f_i\} = 0, \quad i = 1, \ldots, l, \ l \leq k.$$

We denote $D_{f_i} = \{\Delta^{-1}\cdot, f_i\} + \{\Delta^{-1} f_i, \cdot\}$, the operator obtained by the linearization of the Euler equation at the steady state f_i.

Theorem 1. Let $\mathcal{F}$ be the minimal common invariant subspace of the operators $D_{f_1}, \ldots, D_{f_l}$ which contains $f_1, \ldots, f_k$. If $\mathcal{F}$ is everywhere dense in $L_2(M)$, then the system is L_2-approximately controllable and solidly controllable in finite dimensional projections.

In all applications below $f_1, \ldots, f_k$ are eigenfunctions of Δ and $l = k$.

Examples.

1. Torus $S^1 \times S^1$. Eigenfunctions of Δ:

$$\sin(n_1 x_1 + n_2 x_2),\ \cos(n_1 x_1 + n_2 x_2),$$

$n_1, n_2 \in \mathbb{Z}_+$. Take $k = 4$, $\quad \{f_1, \ldots, f_4\} =$

$$\{\sin x_1, \cos x_1, \sin(x_1 + x_2), \cos(x_1 + x_2)\}.$$

2. Square $[0, \pi] \times [0, \pi]$. Eigenfunctions of Δ:

$$\sin(n_1 x_1) \sin(n_2 x_2), \ n_1, n_2 \in \mathbb{Z}_+.$$

Take $k = 8$,

$$\{f_1, \dots, f_8\} = \{\sin(n_1 x_1) \sin(n_2 x_2) :$$

$$n_1, n_2 \leq 3, \ (n_1, n_2) \neq (3, 3)\}.$$

3. Sphere S^2. Eigenfunctions of Δ are homogeneous harmonic polynomials of 3 variables. Take $k = 5$ and the set $\{f_1, \dots, f_5\}$ containing three linear, one quadratic and one cubic polynomials.

Proposition. Given $k > 0$ assume that for some Riemannian structure on M $\exists$ eigenfunctions $f_1, \dots, f_k$ of Δ which satisfy conditions of Theorem 1. Then the eigenfunctions of Δ with such a property do exist for generic Riemannian structure on M.

Sketch of the proof:

The set of appropriate Riemannian structures is the intersection of a countable number of open subsets in the space of all Riemannian structures. It remains to prove that this is a everywhere dense subset.

Given Riemannian structures μ_0, μ_1, connect them by a continuous family μ_t, $0 \le t \le 1$ that is analytic w. r. t. t on the interval $(0,1)$. Then any eigenfunction f^0 of the Laplace–Beltrami operator Δ_{μ_0} is included in the continuous family f^t of the eigenfunctions of Δ_{μ_t}, $0 \le t \le 1$, and the family f^t is analytic on the interval $(0,1)$. Let $f_1^0, \ldots, f_k^0$ be eigenfunctions of Δ_{μ_0}; it is not hard to show that the set

$$\{t \in [0,1] : (f_1^t, \ldots, f_k^t) \text{ satisfies Th. 1}\}$$

is either empty or the complement of a countable subset of $[0,1]$.

Any homeomorphic to the disc Riemannian surface is isometric to the disc endowed with a Riemannian structure of the form

$$e^{a(x_1,x_2)}(dx_1^2 + dx_2^2).$$

This Riemannian disc is isometric to a simply connected domain in $\mathbb{R}^2$ iff $\Delta a = 0$.

The specification of the above proof: take $\mu_t = e^{a_t}(dx_1^2 + dx_2^2)$, $\Delta a_t = 0$. We obtain:

Proposition. Given $k \geq 0$, assume that for some bounded simply connected domain $M \subset \mathbb{R}^2$ there exist eigenfunctions $f_1, \ldots, f_k$ of Δ which satisfy conditions of Theorem 1. Then the eigenfunctions of Δ with such a property do exist for generic domain.

Outline of the proof of Th. 1.

The control system: $\frac{\partial w}{\partial t}+\{\Delta^{-1}w,w\}-\nu\Delta w=$

$$=f_0+\sum_{i=1}^{k}v_i(t)f_i,\quad w(0,\cdot)=\varphi_0.$$

We use fast oscillating control functions $v_i(t)$. Our method is based on the continuity of the input–state map $S_{\varphi_0}:V^k\to H_2(M)$ w. r. t. controls endowed with the 'relaxation norm'

$$\|v(\cdot)\|_{\mathrm{rx}}\stackrel{\mathrm{def}}{=}\max_{t\in[0,T]}\left|\int_0^t v(\tau)\,d\tau\right|.$$

We show that controllability of the extended system $\frac{\partial w}{\partial t}+\{\Delta^{-1}w,w\}-\nu\Delta w=$

$$=f_0+\sum_{i=1}^{k}\left(v_i(t)f_i+\sum_{j=1}^{l}v_{ij}(t)D_{f_j}f_i\right)$$

implies controllability of the original system and then iterate the procedure: substitute $\{f_i:1\leq i\leq k\}$ by $\{f_i,D_{f_j}f_i:1\leq i\leq k,\ 1\leq j\leq l\}$ e. t. c.

Induction step.

To simplify notations, we make calculations for the case $l = 1,\ k = 2$.

1. Take Lipschitzian functions $\widehat{v}_1(t), \widehat{v}_2(t)$ and substitute v_1, v_2 by $\frac{d\widehat{v}_1}{dt} + v_1$ and $\frac{d\widehat{v}_2}{dt} + v_2$. Let $q = w - \widehat{v}_1 f_1 - \widehat{v}_2 f_2$; then:

$$\frac{\partial q}{\partial t} + \{\Delta^{-1}(q + \widehat{v}_1 f_1 + \widehat{v}_2 f_2), q + \widehat{v}_1 f_1 + \widehat{v}_2 f_2\} -$$

$$\nu\Delta(q + \widehat{v}_1 f_1 + \widehat{v}_2 f_2) = f_0 + v_1 f_1 + v_2 f_2.$$

Write it slightly differently: $\frac{\partial q}{\partial t} + \{\Delta^{-1} q, q\} - \nu\Delta q$

$$+\widehat{v}_1(D_{f_1} q - \nu\Delta f_1) + \widehat{v}_2(D_{f_2} q - \nu\Delta f_2)$$

$$= f_0 + v_1 f_1 + v_2 f_2 - \widehat{v}_1 \widehat{v}_2 D_{f_1} f_2 - \frac{\widehat{v}_2^2}{2} D_{f_2} f_2.$$

If $\widehat{v}_1(T) = \widehat{v}_2(T) = 0$, then

$$q_T = S_{\varphi_0}\left(\frac{d\widehat{v}_1}{dt} + v_1, \frac{d\widehat{v}_2}{dt} + v_2\right).$$

2. Substitute $\widehat{v}_i(t)$ by sgn(sin$(t/\varepsilon))\widehat{v}_i(t)$, $\varepsilon \to 0$; this kills linear terms $\widehat{v}(2D_{f_i}q - \nu\Delta f_i)$ without affecting quadratic terms. We arrive to the system:

$$\frac{\partial q}{\partial t} + \{\Delta^{-1}q, q\} - \nu\Delta q =$$

$$f_0 + v_1 f_1 + v_2 f_2 - \widehat{v}_1\widehat{v}_2 D_{f_1} f_2 - \frac{\widehat{v}_2^2}{2} D_{f_2} f_2.$$

Solid controllability of this system implies solid controllability of the original one.

3. Substitute $\widehat{v}_1$ and $\widehat{v}_2$ by $\frac{\widehat{v}_1}{\varepsilon}$ and $\varepsilon\widehat{v}_2$, and set $v_{12} = -\widehat{v}_1\widehat{v}_2$. We obtain:

$$\frac{\partial q}{\partial t} + \{\Delta^{-1}q, q\} - \nu\Delta q =$$

$$f_0 + v_1 f_1 + v_2 f_2 + v_{12} D_{f_1} f_2 + O(\varepsilon^2).$$

Go to the limit as $\varepsilon \to 0$. Solid controllability of the limit system implies solid controllability of the original one.