Sub-Riemannian Problem on Lie Group of Motions of Pseudo Euclidean Plane

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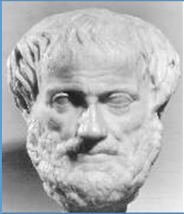


"Those who educate children well are more to be honored than they who produce them; for these only gave them life, those the art of living well."
— *Aristotle*

Introduction

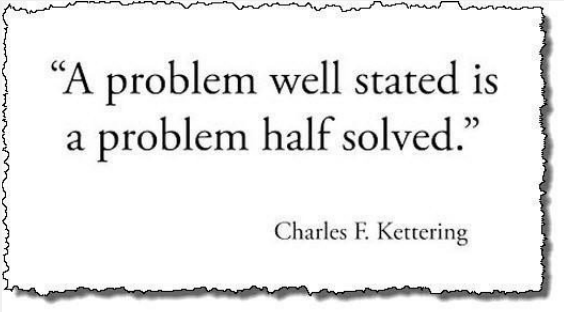
- Work in Mathematical control theory
- Precisely it belongs to Geometric Control Theory based on:
  - Calculus of Variations
  - Pontryagin's Maximum Principle
  - Hermann Nagano - Orbit Theorem
  - Rashevsky-Chow's Theorem
- Geometric control is a nonlinear optimal control technique that draws its strength from:
  - Hamiltonian Mechanics
  - Lie Groups Theory
  - Differential Geometry

Introduction Continued...

- Competing notion of nonlinear optimal control - HJB equations
  - Based on good heuristics and not a mathematical formulation
  - Leads to a set of nonlinear partial differential equations
- Applications are numerous
  - Robotics
  - Quantum Control
  - Image processing
  - Economics and finance
- Language and tools of research - Pure Mathematics
- Theoretical research with far reaching implications and applications

Outline

Problem Statement



"A problem well stated is a problem half solved."

Charles F. Kettering

Sub-Riemannian Manifold

Consider a driftless control system:

$$\dot{q} = \sum_{i=1}^{m} u_i(t) f_i(q), \quad q \in M,$$

- M is n-dimensional manifold
- $f_i(q)$ are the control vector fields (directions along which system can move)
- M is a sub-Riemannian manifold *iff*:
  - the distribution $\Delta = \mathrm{span}\{f_1(q), \ldots, f_m(q)\}$ has dimension $m < n$ and is completely non-integrable
  - it is endowed with smooth inner product g defined on $\triangle$ to measure distances on the manifold

Sub-Riemannian Manifold Example

Steering of a Car Problem

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = u_1 f_1(q) + u_2 f_2(q) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2,$$

$$\Delta = \mathrm{span}\{f_1(q), f_2(q)\}.$$

- Dimension of State Space = 3 and dimension of $\Delta = 2$

- Intuitively
  - $m < n$ implies that on state space of dimensions n, the system can move infinitesimally along only m directions
  - Motion along other $n - m$ directions is constrained
- **With motion constrained along n-m direction, can we still traverse the whole n dimensional manifold is the important question**

Sub-Riemannian Problem in General

- Recall we also need g to completely define a sub-Riemannian manifold:

$$g(u_1, \ldots, u_m) : \mathbb{R}^m \to \mathbb{R}.$$

- The Sub-Riemannian length is given as:

$$l = \int_0^{t_1} \sqrt{g(u_1, \ldots, u_m)} dt.$$

- Given a control system defined by $(M, \triangle, g)$ the sub-Riemannian problem is to find the control and corresponding trajectories that minimize the sub-Riemannian length functional l
- **Sub-Riemannian problem is the optimal control problem of nonholonomic systems with sub-Riemannian length as the cost to be optimized**

Pseudo Euclidean Plane

- Surfaces with constant negative curvature
- It looks like a saddle everywhere
- Polar coordinates of a point (x, y) on the pseudo Euclidean plane:

$$x = r \cosh \phi$$
$$y = r \sinh \phi$$

- Therefore, it is also called hyperbolic plane.

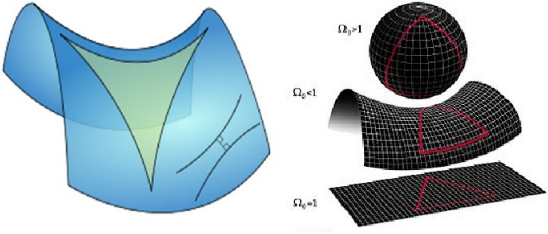


Figure : 1 - Types of Physical Surfaces

Lie Groups SH(2) of Motions of Hyperbolic Plane

- A smooth manifold that also has a group structure is called a Lie group.
- Lie groups are ubiquitous and serve as the state manifold of numerous practical systems
  - $\mathrm{SE}(3)$
    - The group of 6 DOF rotational and translational motions of a rigid body moving in space
    - Represented by 4x4 homogeneous matrices
  - $\mathrm{SE}(2)$
    - The group of 3 DOF rotational and translational motions of a rigid body moving on a plane
    - Represented by 3x3 homogeneous matrices

Lie Groups SH(2) and Lie Algebra sh(2)

- $\mathrm{SH}(2)$ – The group of 3 DOF rotational and translational motions of a rigid body moving on hyperbolic plane
- Matrix Representation of the Lie group $\mathrm{SH}(2)$

$$M = \mathrm{SH}(2) = \begin{pmatrix} \cosh\phi & \sinh\phi & a \\ \sinh\phi & \cosh\phi & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, \phi \in \mathbb{R}.$$

- Tangent space to Lie Group at the identity – Lie Algebra
- The basis A_i for Lie Algebra $\mathrm{sh}(2)$ are given as:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sub-Riemannian Problem on SH(2)

- Sub-Riemannian problem on the Lie group $\mathrm{SH}(2)$ is stated as:

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \quad q \in M = \mathrm{SH}(2), \quad (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min, \tag{3}$$

$$q = (x, y, z), \quad f_1(q) = qA_3, \qquad f_2(q) = qA_1. \tag{4}$$

- Dimension of $M = \mathrm{SH}(2) = 3$ and dimension of $\Delta = \mathrm{span}\{f_1, f_2\}$ and $g = \sqrt{u_1^2 + u_2^2}$
- Structure (M, Δ, g) is complete and defines sub-Riemannian problem on the Lie group $\mathrm{SH}(2)$
- Note that the vector fields $f_i(q)$ are matrices.

A Physical Model for SR Problem on SH(2)

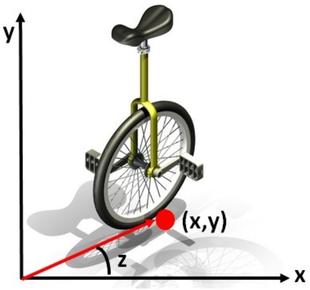
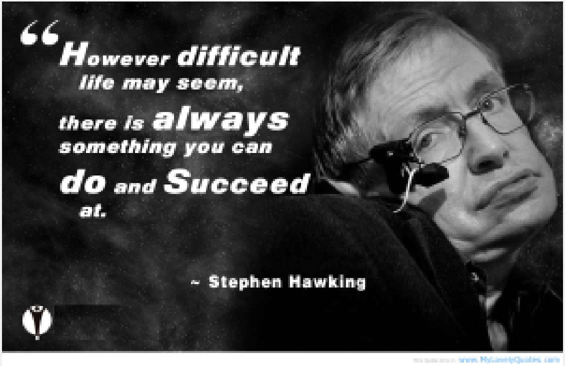


Figure : 2 - SR Problem Modeled as Unicycle Moving on Hyperbolic Plane

- Unicycle configuration - $q = (x, y, z)$
- (x, y) - position vector of point of contact with hyperbolic plane
- z - orientation of position vector
- u_1 - translational velocity, u_2 - rotational velocity

Research Objectives



Research Objectives

1. Prove controllability and integrability of the dynamical system
2. Obtain complete parametrization of extremal trajectories
   Second Order Optimality Analysis
3. Describe symmetries of the system and corresponding Maxwell sets
4. Computation of conjugate loci
5. Geometric view of extremal trajectories and Maxwell Strata through 3D plots sub-Riemannian spheres
6. Obtain complete characterization of the cut locus
7. Describe the global structure of the exponential mapping and the optimal synthesis
8. Geometric analysis of the cut locus and conjugate locus through 3D plot of sub-Riemannian caustic

Controllability and Integrability of the Control System

- Controllability
  - Easy for linear systems
  - Extremely difficult, often impossible to prove for nonlinear systems
- Integrability – Prove that nonlinear differential equations can be explicitly integrated in the form of elementary or special mathematical functions
  - Pontryagin's Maximum Principle – Necessary optimality conditions
  - The trajectories that satisfy PMP are candidate optimal only called extremal trajectories
  - Further analysis based on Maxwell strata and conjugate loci is necessary to establish optimality
  - Explicit integration of the nonlinear differential equations is necessary

Parametrization of Extremal Trajectories

- The process of integrating the nonlinear differential equations is called parametrization of extremal trajectories
- Essentially it is equivalent to finding the parametric equations of state variables of the system
- Consider a linear system:

$$\dot{x} = Ax + Bu.$$

- After integration, the solution to this ODE is:

$$x = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau.$$

- Note, all of the state variables are expressed in a common parameter
- A trivial problem for linear system but usually hopeless for nonlinear systems

Second Order Optimality Analysis

- After parametrization we have candidate optimal trajectories q_s
- The question whether q_s is indeed optimal is one of the most challenging questions in the sub-Riemannian problems
  - If $q(0)$ is very close to $q(t_1)$, q_s is the shortest curve connecting $q(0)$ to $q(t_1)$
- The set of points where q_s loses optimality is called cut locus
- q_s can lose optimality for two reasons
  - If there exist Maxwell Points along q_s
  - If there exist conjugate points along q_s

Maxwell Strata

- The locus of the intersection points of geodesics of equal lengths
- These are the set of points q_{t_1} connected by more than one extremal trajectory with the initial point q_0
- The Maxwell set is closely related to optimality of geodesics: a geodesic cannot be optimal after an intersection with another geodesic of the same length

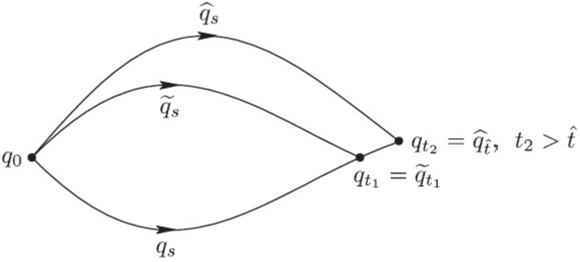


Figure : 3 - Concept of Maxwell Point

Conjugate Loci

- Conjugate locus is the set of points where extremal trajectories lose local optimality (i.e., optimality with respect to infinitesimally close extremal trajectories)
- At conjugate points the extremal trajectories lose uniqueness property and therefore are non-optimal
- Geometrically, at conjugate points extremal trajectories have an envelope

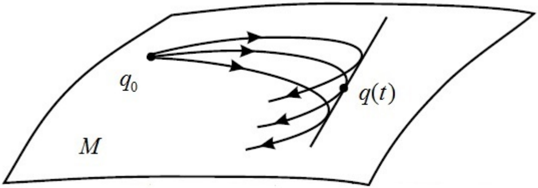


Figure : 4 - Concept of Conjugate Point

Motivation

"The majority see the obstacles; the few see the objectives; history records the successes of the latter, while oblivion is the reward of the former."

Alfred Armand Montapert

Why Research in Geometric Control Theory is Important?

- Theoretical results in mathematics have far reaching implications
- Geometric control theory brings together and builds on tools of various mathematical disciplines
- Geometric control is inherently an optimal control and path planning technique for nonlinear systems
- Optimal control for nonlinear systems is indispensable for all man made processes e.g.,
  - Network traffic routing, Economics, Management,
- Geometry and differential equations are intertwined and closely related
- Analysis of nonlinear systems relies on geometric techniques

Why Sub-Riemannian Problem on SH(2) is Important?

- Sub-Riemannian problems are the optimal control problems of nonholonomic systems e.g.,
  - Robotics, Quantum Mechanics, Economics
- Real world surfaces are non-flat
- Shape of universe, space-time and innumerable real world objects is hyperbolic
- SR problem on SH(2) represents a unicycle moving on hyperbolic plane
- Unicycle is used to model large class of nonholonomic systems in robotics e.g.,
  - Differential drive robots, parking of car, aircraft
  - Kinematic model of a helicopter lateral motion is mathematically a unicycle model

Literature Survey

Literature Survey - Theoretical Results

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Research Methodology and Results

Geometric Control Machinery

- Obtain the vectorial (Wei–Norman) representation of the control system
- Investigate the controllability of the system via Lie theory
- Define control dependent Hamiltonian
- Apply PMP and form the Hamiltonian system. This also gives the open loop optimal control input
- Prove integrability of the system
- Define a suitable transformation to parametrize the extremal trajectories
- Perform the qualitative analysis of the extremal trajectories
- Use the symmetries of the vertical subsystem to characterize Maxwell Strata
- Describe conjugate locus where the exponential mapping is degenerate

Wei–Norman Representation of the Dynamical System

Wei- Norman Representation of Control System

- Also known as representation in canonical coordinates of second kind
- Vector fields $f_i(q)$ are transformed from matrix to vector form
- According to Wei-Norman representation the solution of a driftless control system (1) may be expressed in the form:

$$q = e^{x_1 A_1} e^{x_2 A_2} \dots e^{x_n A_n}. \tag{5}$$

- The control system may be represented in new coordinates x_i in vector form as:

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = F(x_1, \dots, x_n) \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \tag{6}$$

where F is analytic in coordinates x_i

Wei- Norman Representation of Control System

- The process considers $\dot{q}$ and using Baker-Capmbell-Hausdorf formula we have

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = u_1 f_1 + u_2 f_2 = u_1 \begin{pmatrix} \cosh z \\ \sinh z \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- It is now easier to work out controllability and integrability of the system
- This forms part of our paper to be presented in ACC-2015 on July 3^{rd}, 2015

Controllability

- Lie Bracket

$$f_0 = [f_1, f_2] = \frac{\partial f_2}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_2 = -\begin{pmatrix} 0 & 0 & \sinh z \\ 0 & 0 & \cosh z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$f_0 = -\begin{pmatrix} \sinh z \\ \cosh z \\ 0 \end{pmatrix}.$$

- The Lie Algebra spanned by the distribution is:

$$\mathscr{L}_q \Delta = \mathrm{span}\{f_1, f_2, [f_1, f_2]\} = T_q M$$

- Rashevskii–Chow's Theorem - For a connected manifold and corresponding bracket generating control distribution, the system is completely controllable.

PMP Statement

Let $\tilde{u}(t)$ be optimal control and $\tilde{q}(t)$ be optimal trajectory for $t \in [0, t_1]$ and $h_u^\nu(\lambda)$ be the Hamiltonian function. Then, there exists a nontrivial pair:

$$(\nu, \lambda_t) \neq 0, \qquad \nu \in \mathbb{R}, \quad \lambda_t \in T^*_{\tilde{q}(t)} M, \quad \pi(\lambda_t) = \tilde{q}(t).$$

where λ_t is a Lipschitzian curve and a costate variable and $\nu \in \{-1, 0\}$ is a number such that following conditions hold for almost all time $t \in [0, t_1]$:

$$\dot{\lambda}_t = \overrightarrow{h}^{\nu}_{\tilde{u}(t)}(\lambda_t), \tag{7}$$

$$h^\nu_{\tilde{u}(t)}(\lambda_t) = \max_{u \in \mathbb{R}^2} h^\nu_{u(t)}(\lambda_t), \tag{8}$$

$$(\nu, \lambda_t) \neq 0. \tag{9}$$

where $\overrightarrow{h}^{\nu}_{\tilde{u}(t)}(\lambda_t)$ is the Hamiltonian vector field corresponding to the maximized Hamiltonian function $h^\nu_{\tilde{u}(t)}$.
Equations (7)–(9) are called costate or adjoint equation, maximization condition and non-triviality condition respectively.

Control Dependent Hamiltonian - $\nu = -1$

- For $\nu = -1$ the trajectories given by PMP are called normal extremal trajectories.
- The Hamiltonian in this case is given as:

$$H = h_u^{-1}(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \frac{1}{2}\left(u_1^2 + u_2^2\right), \qquad u \in \mathbb{R}^2.$$

- Applying the first order optimality conditions w.r.t the controls:

$$\frac{\partial H}{\partial u} = \begin{pmatrix} h_1 - u_1 \\ h_2 - u_2 \end{pmatrix} = 0,$$

$$\implies u_1 = h_1, \quad u_2 = h_2.$$

- Remember $h_i = \langle \lambda, f_i(q) \rangle$

Hamiltonian System

- Using Poisson bracket:

$$\begin{aligned}
\dot{h}_1 &= \{H, h_1\} = \left\{\frac{1}{2}\left(h_1^2 + h_2^2\right), h_1\right\} = h_2\{h_2, h_1\} = h_2 h_0,\\
\dot{h}_2 &= \{H, h_2\} = \left\{\frac{1}{2}\left(h_1^2 + h_2^2\right), h_2\right\} = h_1\{h_1, h_2\} = -h_1 h_0,\\
\dot{h}_0 &= \{H, h_0\} = \left\{\frac{1}{2}\left(h_1^2 + h_2^2\right), h_0\right\} = h_1\{h_1, h_0\} + h_2\{h_2, h_0\} = h_1 h_2.
\end{aligned}$$

- Hence, complete Hamiltonian system in normal case is given as:

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_0 \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} h_2 h_0 \\ -h_1 h_0 \\ h_1 h_2 \\ h_1 \cosh z \\ h_1 \sinh z \\ h_2 \end{pmatrix}. \tag{10}$$

Liouville Integrability of the Hamiltonian System

- Prove that Hamiltonian system can be integrated and has a closed form solution
- Proof is based on the existence of constants or invariants of motion i.e., quantities that remain conserved along the system trajectories

Liouville's Theorem

If a 2n-dimensional Hamiltonian system with n degrees of freedom has n integrals of motion $f_1;;f_n$ in involution, $\{f_i;f_j\}=0$ and functionally independent on the (intersection of) level sets of the n functions, $f_i=F_i$, then the solutions of the corresponding Hamiltonian system can be found by quadratures.

Liouville Integrability of the Hamiltonian System

Theorem

The normal Hamiltonian system (10) in sub-Riemannian problem on Lie group SH(2) is integrable by quadratures.

- We compute the right invariant Hamiltonians
- Prove that they Poisson commute and are functionally independent
- Yasir Awais Butt, Aamer Iqbal Bhatti, Yuri L. Sachkov, "Integrability by Quadratures in Optimal Control of a unicycle on a hyperbolic plane", Accepted for presentation in American Control Conference, Chicago, Illinois, 1–3 Jul 2015

Parametrization of Extremal Trajectories

Vertical Subsystem

Theorem

Vertical subsystem of the Hamiltonian system (10) in normal case is a mathematical pendulum.

- Introduce following coordinates transformation:

$$h_1 = \cos\alpha, \quad h_2 = \sin\alpha.$$

- Another change of coordinates:

$$\gamma = 2\alpha \in 2S^1 = \mathbb{R}/4\pi\mathbb{Z}, \quad c = -2h_0 \in \mathbb{R},$$

$$\begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c \\ -\sin\gamma \end{pmatrix}.$$

- Hence $h_1 = \cos\gamma/2,\ h_2 = \sin\gamma/2$

Hamiltonian System

- Hence, complete Hamiltonian system in normal case is given as:

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_0 \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} h_2 h_0 \\ -h_1 h_0 \\ h_1 h_2 \\ \cos\frac{\gamma}{2}\cosh z \\ \cos\frac{\gamma}{2}\sinh z \\ \sin\frac{\gamma}{2} \end{pmatrix}. \tag{11}$$

Decomposition of Phase Cylinder of the Vertical Subsystem

- Energy of the pendulum is given as:

$$E = \frac{c^2}{2} - \cos\gamma.$$

- Phase cylinder of the pendulum may be decomposed according to the energy of the system as:

$$\begin{aligned}
C &= \bigcup_{i=1}^{5} C_i, \\
C_1 &= \{\lambda \in C | E \in (-1, 1)\}, \\
C_2 &= \{\lambda \in C | E \in (1, \infty)\}, \\
C_3 &= \{\lambda \in C | E = 1, c \neq 0\}, \\
C_4 &= \{\lambda \in C | E = -1\} = \{(\gamma, c) \in C | \gamma = 2\pi n, c = 0\}\}, \quad n \in \mathbb{N}, \\
C_5 &= \{\lambda \in C | E = 1\} = \{(\gamma, c) \in C | \gamma = 2\pi n + \pi, c = 0\}, \quad n \in \mathbb{N}.
\end{aligned}$$

Decomposition of Phase Cylinder of the Vertical Subsystem

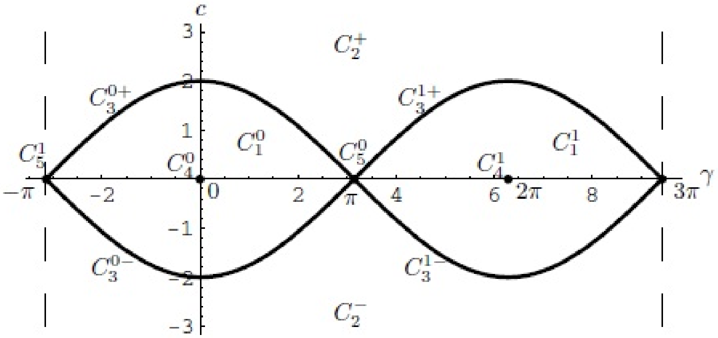


Figure : 5 - Decomposition of the Phase Cylinder and the Connected Subsets

Elliptic Coordinates on The Phase Cylinder of Vertical Subsystem

- $\lambda = (\varphi, k) \in C_1$

$$
\begin{aligned}
k &= \sqrt{\frac{E+1}{2}} = \sqrt{\sin^2\frac{\gamma}{2} + \frac{c^2}{4}} \in (0,1), \\
\sin\frac{\gamma}{2} &= s_1 k \operatorname{sn}(\varphi, k), \quad s_1 = sgn\left(\cos\frac{\gamma}{2}\right), \\
\cos\frac{\gamma}{2} &= s_1 \operatorname{dn}(\varphi, k), \\
\frac{c}{2} &= k \operatorname{cn}(\varphi, k), \quad \varphi \in [0, 4K(k)].
\end{aligned}
$$

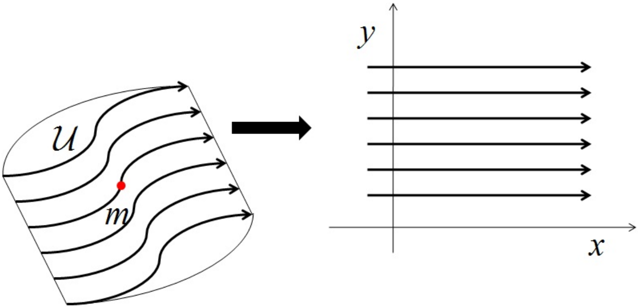
where k is the re-parametrized energy and φ is the re-parametrized time of motion of pendulum

Rectification of Flow in Elliptic Coordinates

Theorem

The elliptic coordinates cause the flow of vertical subsystem to be rectified i.e., the flow lines become parallel to each other.

$$\dot{k} = 0.$$
$$\dot{\varphi} = 1.$$



Theorem

In case 1 extremal trajectories are parametrized as follows:

$$\begin{pmatrix} x_t \\ \\ y_t \\ \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left(w+\frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t)-\mathrm{E}(\varphi)]+ \\ \frac{s_1}{2}\left(\frac{k}{w(1-k^2)}-kw\right)[\operatorname{sn}\varphi_t-\operatorname{sn}\varphi] \\ \frac{1}{2}\left(w-\frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t)-\mathrm{E}(\varphi)]- \\ \frac{1}{2}\left(\frac{k}{w(1-k^2)}+kw\right)[\operatorname{sn}\varphi_t-\operatorname{sn}\varphi] \\ s_1 \ln[(\operatorname{dn}\varphi_t-k\operatorname{cn}\varphi_t).w] \end{pmatrix}$$

where $w=\frac{1}{\operatorname{dn}\varphi-k\operatorname{cn}\varphi}$.

Parametrization of Extremal Trajectories

Theorem

In case 2 extremal trajectories are parametrized as follows:

$$\begin{pmatrix} x_t \\ \\ y_t \\ \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{1}{w(1-k^2)} - w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ \frac{1}{2}\left(kw + \frac{k}{w(1-k^2)}\right)\left[\operatorname{sn}\psi_t - \operatorname{sn}\psi\right], \\ -\frac{s_2}{2}\left(\frac{1}{w(1-k^2)} + w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ \frac{s_2}{2}\left(kw - \frac{k}{w(1-k^2)}\right)\left[\operatorname{sn}\psi_t - \operatorname{sn}\psi\right], \\ s_2 \ln[(\operatorname{dn}\psi_t - k\operatorname{cn}\psi_t).w], \end{pmatrix}$$

where $w = \frac{1}{\operatorname{dn}\psi - k\operatorname{cn}\psi}$.

Parametrization of Extremal Trajectories

Theorem

In case 3 extremal trajectories are parametrized as follows:

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left[\frac{1}{w}(\varphi_t - \varphi) + w(\tanh\varphi_t - \tanh\varphi)\right] \\ \frac{s_2}{2}\left[\frac{1}{w}(\varphi - \varphi_0) - w(\tanh\varphi_t - \tanh\varphi)\right] \\ -s_1 s_2 \ln[w \operatorname{sech}\varphi_t] \end{pmatrix}$$

where $w = \cosh\varphi$.

Parametrization of Extremal Trajectories

Theorem

In case 4, 5 extremal trajectories are parametrized as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}\left(\cos\frac{\gamma}{2}\right) t \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{sgn}\left(\sin\frac{\gamma}{2}\right) t \end{pmatrix}.$$

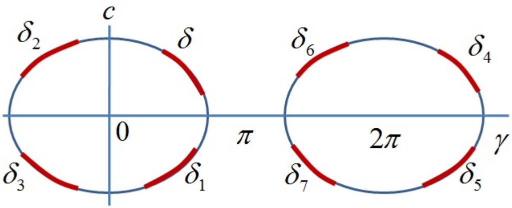
- Y. A. Butt, Yuri L. Sachkov, A. I. Bhatti, "Extremal Trajectories and Maxwell Strata in SR Problem on Group of Motions of Pseudo-Euclidean Plane", JDCS, (2014)
- Y. A. Butt, Yuri L. Sachkov, A. I. Bhatti, "Extremal Trajectories and Maxwell Points in SR Problem on the Group SH(2)", International Conference on Mathematical Control Theory and Mechanics, Suzdal,

Complete Description of Maxwell Strata

Description of Maxwell Strata

- The vertical subsystem of the Hamiltonian system is a mathematical pendulum which has reflection symmetries
- We use the reflection symmetries to find the fixed points in the image and in the preimage of the exponential mapping
- These points form the Maxwell set corresponding to the reflection symmetries

$$MAX^i = \{\nu = (\lambda, t) \in N = C \times \mathbb{R}^+ \quad | \quad \lambda \neq \lambda^i, \quad Exp(\lambda, t) = Exp(\lambda^i, t)\},$$
$$Max^i = Exp(MAX^i) \subset M.$$



Description of Maxwell Strata

- We computed the manifold in which the Maxwell points are located

$$R_1 = y\cosh\frac{z}{2} - x\sinh\frac{z}{2} = 0, \qquad R_2 = x\cosh\frac{z}{2} - y\sinh\frac{z}{2} = 0, \qquad z = 0.$$

Theorem

First Maxwell time is bounded as:

$$\begin{aligned} \lambda \in C_1 \quad &\Longrightarrow \quad t_1^{MAX}(\lambda) = 4K(k), \\ \lambda \in C_2 \quad &\Longrightarrow \quad t_1^{MAX}(\lambda) = 4kK(k), \\ \lambda \in C_3 \cup C_4 \cup C_5 \quad &\Longrightarrow \quad t_1^{MAX}(\lambda) = +\infty. \end{aligned}$$

- In our later work we have also proved that the cut time is equal to the first Maxwell time
- This involves the proof of the conjecture that the exponential mapping is a diffeomorphism
- Results have important implications in optimal control of unicycle like systems on hyperbolic plane

Conjugate Locus in SR Problem on SH(2)

Conjugate Locus

- A point $q_t = \mathrm{Exp}(\lambda, t)$ is called a conjugate point for q_0 if $v = (\lambda, t) = (\gamma, c, t)$ is a critical point of the exponential mapping, q_t being its critical value i.e.,

$$d_v \mathrm{Exp} : \; T_v N \to T_{q_t} M \text{ is degenerate,}$$

where $d_v\mathrm{Exp}$ amounts to the Jacobian J of the exponential mapping i.e.,

$$J = \frac{\partial(x_t, y_t, z_t)}{\partial(\gamma, c, t)} = \begin{vmatrix} \frac{\partial x_t}{\partial \gamma} & \frac{\partial x_t}{\partial c} & \frac{\partial x_t}{\partial t} \\ \frac{\partial y_t}{\partial \gamma} & \frac{\partial y_t}{\partial c} & \frac{\partial y_t}{\partial t} \\ \frac{\partial z_t}{\partial \gamma} & \frac{\partial z_t}{\partial c} & \frac{\partial z_t}{\partial t} \end{vmatrix} = 0.$$

Conjugate Locus

- The proof is based on several concepts such as homotopy
- An important fact is that first conjugate point is upper bounded by the same functions that correspond to the second Maxwell times

Theorem

The n^{th} conjugate times are bounded as:

$$
\begin{aligned}
\lambda \in C_1 \quad &\Longrightarrow \quad 4nK(k) \leq t_{2n-1}^{\text{conj}} \leq 2p_1^n(k), \quad 2p_1^n(k) \leq t_{2n}^{\text{conj}} \leq 4(n+1)K(k), \\
\lambda \in C_2 \quad &\Longrightarrow \quad 4nkK(k) \leq t_{2n-1}^{\text{conj}} \leq 2kp_1^n(k), \quad 2kp_1^n(k) \leq t_{2n}^{\text{conj}} \leq 4(n+1)kK(\\
\lambda \in C_4 \quad &\Longrightarrow \quad t_1^{\text{conj}}(\lambda) = 2\pi.
\end{aligned}
$$

- Yasir Awais Butt, Yuri L. Sachkov, Aamer Iqbal Bhatti, "Maxwell Strata and Conjugate Points in SR Problem on Group SH(2)", International Youth Conference Geometry and Control, Moscow, 2014
- Yasir Awais Butt, Yuri L. Sachkov, Aamer Iqbal Bhatti, "Maxwell Strata and Conjugate Points in the SR Problem on the Lie Group

Generalized Rolle's Theorem

Between any two Maxwell points there is one conjugate point, along any geodesic.

- We proved in our final work that this conjecture holds for SR problem on SH(2).

Plot of Sub-Riemannian Sphere

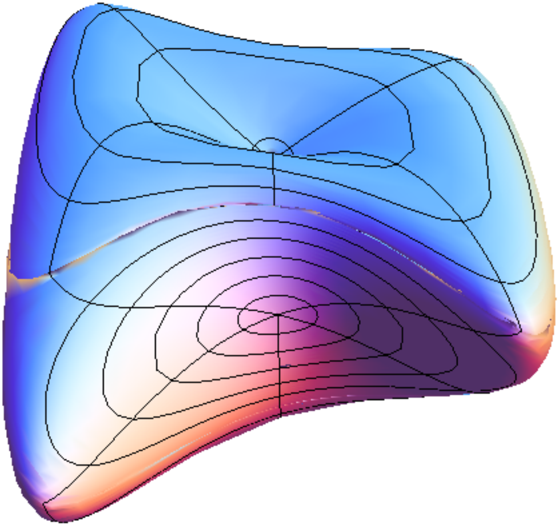


Figure : 8 - Sub-Riemannian sphere of radius R=2

Plot of Sub-Riemannian Wavefront

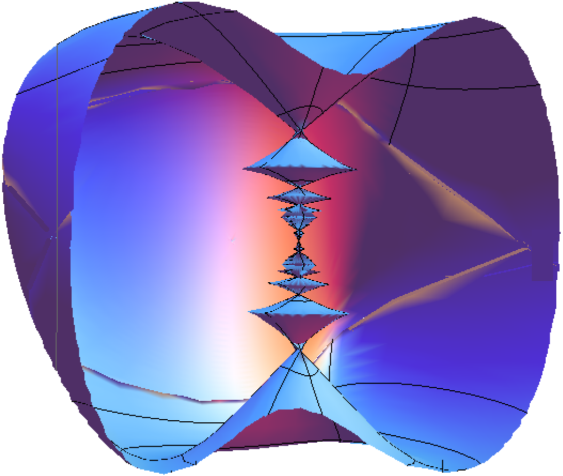
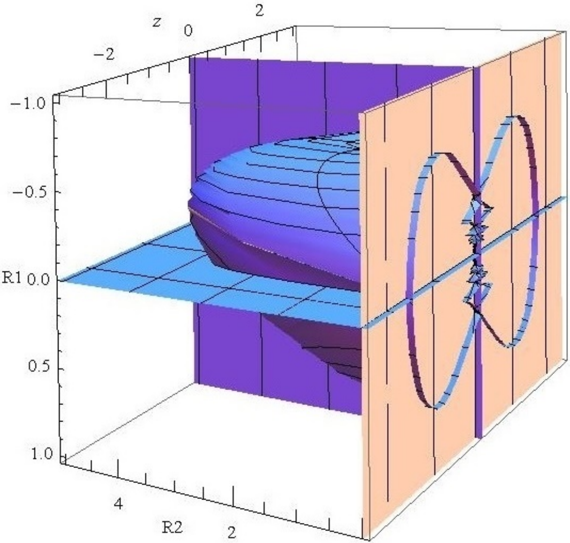
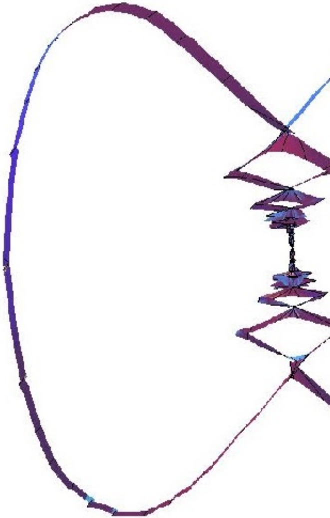


Figure : 9 - Cutout of the sub-Riemannian wavefront for $R = 2$

Plot of Sub-Riemannian Wavefront





Plot of Matryoshka of Sub-Riemannian Wavefront

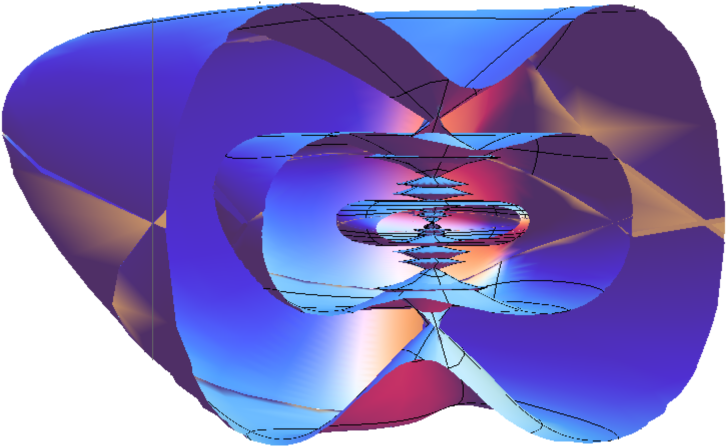
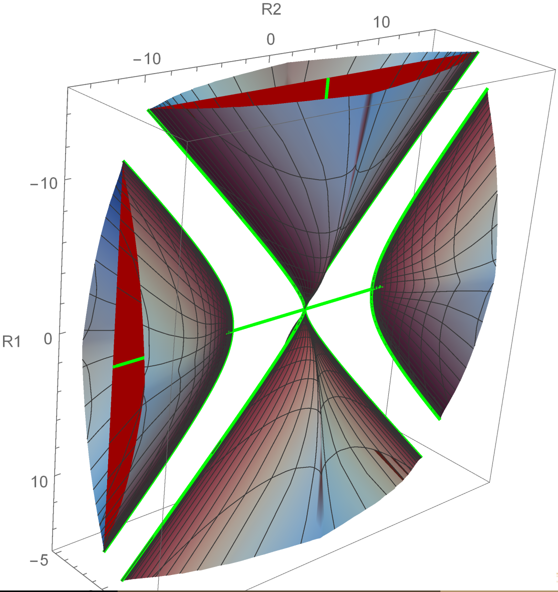


Figure : 13 - Matryoshka of sub-Riemannian wavefronts W_R for $R = 1, 2, 3$

Plot of Sub-Riemannian Caustic



Plot of Sub-Riemannian Caustic - Local Component

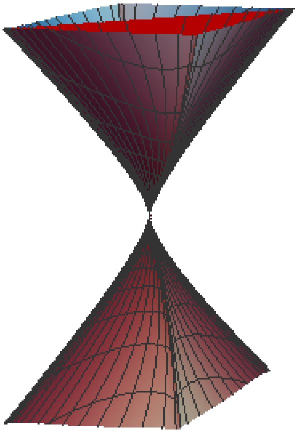
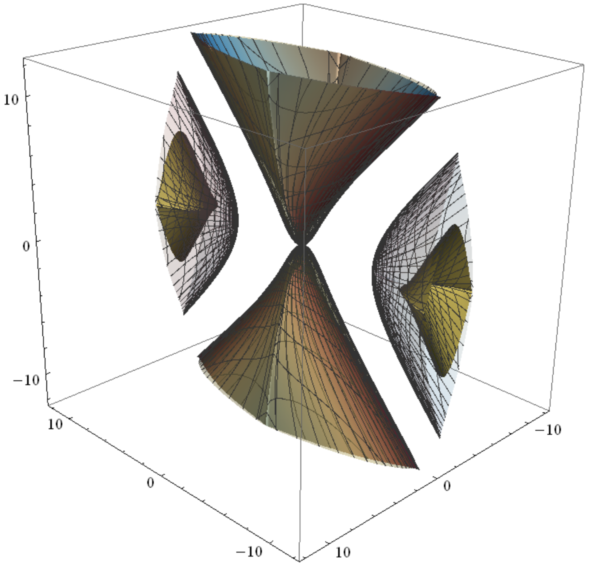


Figure : 13 - Local Component of the sub-Riemannian caustic

Plot of Sub-Riemannian First and Second Caustic



Novelty of Research Problem and Results

Two aspects of novelty:

- Research discipline - Geometric control is a relatively new field with only handful of people involved in this research
- SR problem has not been considered on Lie group SH(2) prior to this work
  - In depth analysis of extremal trajectories on SH(2)
  - Integrability of of the Hamiltonian system on Lie group SH(2)
  - Description of the Maxwell points and calculation of the upper bound on cut time
  - Characterization of conjugate loci
  - Global explicit description of cut time and cut locus and optimal synthesis
  - Transformation between SH(2) and SOLV^- has been obtained
  - Geometric description of global component of the sub-Riemannian caustic
  - Proof of Generalized Rolle's theorem

Main Contributions of Research Work

1. Proving the controllability and the integrability of the dynamical system
2. Obtaining parametrization of extremal trajectories in elliptic coordinates and Jacobi elliptic functions.
3. Complete description of Maxwell strata
4. Characterization of conjugate loci
5. Geometrical view of sub-Riemannian wavefront and sphere
6. Complete description of cut loci
7. Transformation between SH(2) and SOLV^- has not been obtained
8. Geometric description of global component of the sub-Riemannian caustic
9. Proof of Generalized Rolle's theorem

Research Publications

6. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Cut Locus and Optimal Synthesis in SR Problem on the Lie Group SH(2)", To be Submitted
5. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Maxwell Strata and Conjugate Points in the SR Problem on the Lie Group SH(2)", Submitted to SICON, 2014
4. Y. A. Butt, A. I. Bhatti, Y. L. Sachkov, "Integrability by Quadratures in Optimal Control of a unicycle on a hyperbolic plane", Accepted for presentation in ACC, 1–3 Jul 2015, Chicago, Illinois
3. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Extremal Trajectories and Maxwell Strata in SR Problem on Group of Motions of Pseudo-Euclidean Plane", JDCS, 2014
2. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Maxwell Strata and Conjugate Points in SR Problem on Group SH(2)", International Youth Conference Geometry and Control, Moscow, 2014
1. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Extremal Trajectories and Maxwell Points in SR Problem on the Group SH(2)", International Conference on Mathematical Control Theory and Mechanics, Suzdal, Russia, 2013

Future Work

- Apply methods to a physical system such as differential drive robot or a car with trailer system
- Develop software for computation of optimal trajectories
- Work on another exciting research problem i.e.,

Generalized Dido Problem

Questions

Conclusion

- Sub-Riemannian problem on Lie Group $\mathrm{SH}(2)$ was considered
- All of the proposed research objectives have been achieved
- Analysis techniques developed in previous similar works have been extended
- Three conference papers and three journal papers have been produced
- Some advanced results have been achieved, i.e.,
  1. Orthogonal transformation between Lie groups $\mathrm{SH}(2)$ and SOLV^-
  2. Generalized Rolle's theorem
  3. Description of the global component of the sub-Riemannian caustic
  4. Global explicit description of cut time and cut locus and optimal synthesis
- Overall learning experience has been overwhelming

"The only true wisdom is in knowing that you know nothing".

Socrates