Sub-Riemannian Problem on Lie Group SH(2)

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Lie Groups SH(2) and Lie Algebra sh(2)

- $\mathrm{SH}(2)$ – The group of 3 DOF rotational and translational motions of a rigid body moving on hyperbolic plane
- Matrix Representation of the Lie group $\mathrm{SH}(2)$

$$M = \mathrm{SH}(2) = \left\{ \begin{pmatrix} \cosh\phi & \sinh\phi & a \\ \sinh\phi & \cosh\phi & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, \phi \in \mathbb{R} \right\}.$$

- Lie Algebra basis A_i for Lie Algebra $\mathrm{sh}(2)$ are given as:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sub-Riemannian Problem on SH(2)

- Sub-Riemannian problem on the Lie group $\mathrm{SH}(2)$ is stated as:

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \quad q \in M = \mathrm{SH}(2), \quad (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \to \min, \tag{3}$$

$$q = (x, y, z), \quad f_1(q) = qA_3, \qquad f_2(q) = qA_1. \tag{4}$$

- $\Delta = \mathrm{span}\{f_1, f_2\}$ and $g(f_i, f_j) = \delta_{ij}$
- Structure (M, Δ, g) defines sub-Riemannian problem on the Lie group $\mathrm{SH}(2)$

A Physical Model for SR Problem on SH(2)

- In canonical coordinates of the second kind (Wei-Norman transformation):

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = u_1 f_1 + u_2 f_2 = u_1 \begin{pmatrix} \cosh z \\ \sinh z \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- From first principles, the model of a unicycle moving on hyperbolic plane has same representation with:
  - (x, y) - position vector of point of contact with hyperbolic plane
  - z - orientation of position vector
  - u_1 - translational velocity
  - u_2 - rotational velocity

A Physical Model for SR Problem on SH(2)

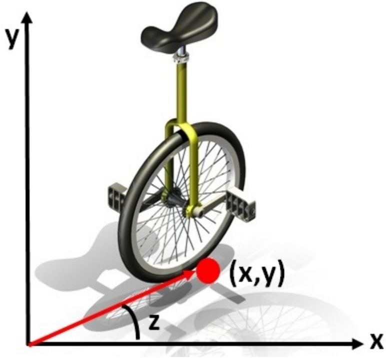


Figure: SR Problem Modeled as Unicycle Moving on Hyperbolic Plane

Objectives

1. Obtain complete parametrization of extremal trajectories
2. Describe symmetries of the exponential mapping and corresponding Maxwell sets
3. Computation of conjugate loci
4. Obtain complete characterization of the cut locus
5. Describe the global structure of the exponential mapping and the optimal synthesis
6. Geometric analysis of the cut locus and conjugate locus through 3D plot of sub-Riemannian caustic, spheres and wavefronts

Known Results for Sub-Riemannian Problems on Lie Groups

- 3-Dimensional Lie Groups
  - Heisenberg group – A.Vershik, V.Gershkovich, 1986
  - S^3, SO(3), SL(2) and Lens spaces – Ugo Boscain and F. Rossi, 2008
  - SE(2) – Yuri L. Sachkov, 2008
- Contact Problems in $\mathbb{R}^3$ – A. Agrachev, 1996, J. P. Gauthier, 1996
- 5-Dimensional nilpotent Lie group with growth vector (2,3,5) – Yuri L. Sachkov, 2006
- Euler Elasticae Problem – Yuri L. Sachkov, 2008
- 4-Dimensional nilpotent Lie group with growth vector (2,3,4) i.e., Engel group – A. A. Ardentov and Yu. L. Sachkov, 2011–2015

Controllability and Existence of Minimizers

- The Lie Algebra spanned by the distribution is:

$$\mathscr{L}_q\Delta = \text{span}\{f_1, f_2, [f_1, f_2]\} = T_qM \quad \forall q \in M$$

  - Rashevskii–Chow's Theorem - The system is completely controllable.

- Fillipov's theorem
  - Existence of optimal trajectories $q(t)$

Pontryagin Maximum Principle

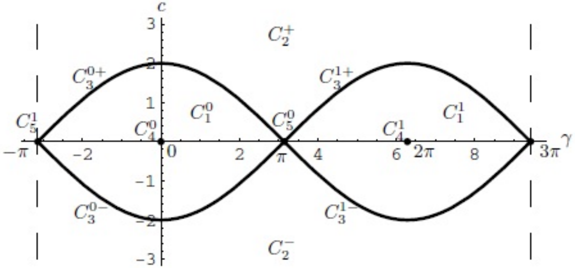
- Abnormal extremal trajectories are constant
- Normal extremals:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \quad (\gamma, c) \in C \cong \left(2S^1_\gamma\right) \times \mathbb{R}_c,$$
$$\dot{x} = \cos\frac{\gamma}{2}\cosh z, \quad \dot{y} = \cos\frac{\gamma}{2}\sinh z, \quad \dot{z} = \sin\frac{\gamma}{2}.$$

Decomposition of Phase Cylinder of the Pendulum $C = \bigcup_{i=1}^{5} C_i$,

- Energy of the pendulum is given as $E = \frac{c^2}{2} - \cos\gamma \in [-1, +\infty)$.

$$
\begin{aligned}
C_1 &= \{\lambda \in C | E \in (-1, 1)\},\\
C_2 &= \{\lambda \in C | E \in (1, \infty)\},\\
C_3 &= \{\lambda \in C | E = 1, c \neq 0\},\\
C_4 &= \{\lambda \in C | E = -1\} = \{(\gamma, c) \in C | \gamma = 2\pi n, c = 0\}\}, \quad n \in \mathbb{N},\\
C_5 &= \{\lambda \in C | E = 1\} = \{(\gamma, c) \in C | \gamma = 2\pi n + \pi, c = 0\}, \quad n \in \mathbb{N}.
\end{aligned}
$$



Parametrization of Extremal Trajectories

- $\lambda = (\gamma, c) \in C_1 \quad \Longrightarrow$

$$\begin{pmatrix} x_t \\ \\ y_t \\ \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left(w + \frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)] + \\ \frac{s_1}{2}\left(\frac{k}{w(1-k^2)} - kw\right)[\mathrm{sn}\,\varphi_t - \mathrm{sn}\,\varphi] \\ \frac{1}{2}\left(w - \frac{1}{w(1-k^2)}\right)[\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)] - \\ \frac{1}{2}\left(\frac{k}{w(1-k^2)} + kw\right)[\mathrm{sn}\,\varphi_t - \mathrm{sn}\,\varphi] \\ s_1 \ln[(\mathrm{dn}\,\varphi_t - k\,\mathrm{cn}\,\varphi_t).w] \end{pmatrix}$$

where $w = \frac{1}{\mathrm{dn}\varphi - k\mathrm{cn}\varphi}$.

Parametrization of Extremal Trajectories

- $\lambda = (\gamma, c) \in C_2 \quad \Longrightarrow$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{1}{w(1-k^2)} - w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ \frac{1}{2}\left(kw + \frac{k}{w(1-k^2)}\right)\left[\operatorname{sn}\psi_t - \operatorname{sn}\psi\right], \\ -\frac{s_2}{2}\left(\frac{1}{w(1-k^2)} + w\right)\left[\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - k'^2(\psi_t - \psi)\right] \\ \frac{s_2}{2}\left(kw - \frac{k}{w(1-k^2)}\right)\left[\operatorname{sn}\psi_t - \operatorname{sn}\psi\right], \\ s_2 \ln[(\operatorname{dn}\psi_t - k\operatorname{cn}\psi_t).w], \end{pmatrix}$$

where $w = \frac{1}{\operatorname{dn}\psi - k\operatorname{cn}\psi}$.

Parametrization of Extremal Trajectories

- $\lambda = (\gamma, c) \in C_3 \quad \Longrightarrow$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left[\frac{1}{w}(\varphi_t - \varphi) + w(\tanh\varphi_t - \tanh\varphi)\right] \\ \frac{s_2}{2}\left[\frac{1}{w}(\varphi - \varphi_0) - w(\tanh\varphi_t - \tanh\varphi)\right] \\ -s_1 s_2 \ln[w \operatorname{sech}\varphi_t] \end{pmatrix}$$

where $w = \cosh\varphi$.

- $\lambda = (\gamma, c) \in C_4 \quad \Longrightarrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}\left(\cos\frac{\gamma}{2}\right) t \\ 0 \\ 0 \end{pmatrix}.$$

- $\lambda = (\gamma, c) \in C_5 \quad \Longrightarrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{sgn}\left(\sin\frac{\gamma}{2}\right) t \end{pmatrix}.$$

Optimality Analysis of Extremal Trajectories

- The question whether $q(t)$ is indeed optimal is one of the most challenging questions in the sub-Riemannian problems
- Every $q(t)$ is locally optimal
- $q(t)$ can lose optimality due to existence of
  - Maxwell Points
  - Conjugate points
- The set of points where $q(t)$ loses optimality is called cut locus
- Cut time

$$t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s) \text{ is optimal for } s \in [0, t]\}.$$

Maxwell Strata

- The locus of the intersection points of geodesics of equal lengths

$$\exists \text{ extremal trajectory } \tilde{q}_s \not\equiv q_s : \quad q_0 = \tilde{q}_0, \quad q_t = \tilde{q}_t.$$

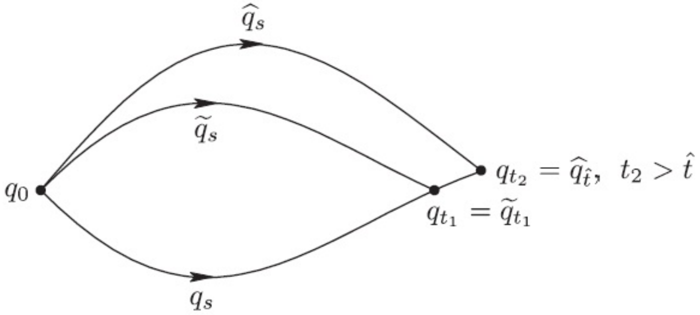


Figure: Maxwell Point $t_2 < t_1$

Conjugate Loci

- Extremal trajectories $q(t)$ lose local optimality (i.e., optimality with respect to infinitesimally close extremal trajectories)
- $q(t) \in$ envelope of family of extremal trajectories

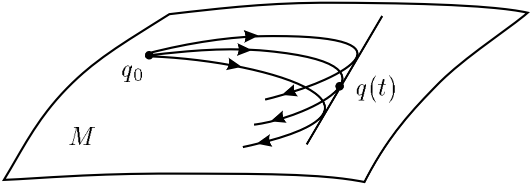


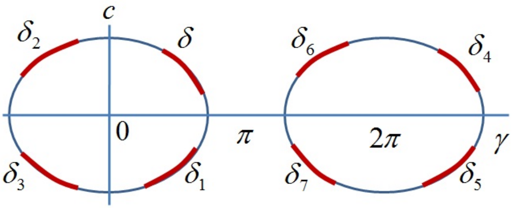
Figure: At conjugate point extremal trajectories have envelope

- Note that,

$$t_{\text{cut}} \leq \min\left(t_{\text{Max}}, t_{\text{conj}}\right).$$

Reflections ε^i in the Phase Cylinder of Pendulum and Maxwell Strata

- Action of reflection symmetries $\varepsilon^i : \delta \to \delta^i$ on trajectories of the pendulum



- Maxwell Set Max corresponding to reflections ε^i

$$\mathrm{MAX}^i = \left\{ \nu = (\lambda, t) \in N = C \times \mathbb{R}^+ \quad | \quad \lambda \neq \lambda^i, \quad \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(\lambda^i, t) \right\},$$

$$\mathrm{Max}^i = \mathrm{Exp}\left(\mathrm{MAX}^i\right) \subset M.$$

Description of Maxwell Strata

- Maxwell points are located in hypersurfaces

$$R_1 = y\cosh\frac{z}{2} - x\sinh\frac{z}{2} = 0, \qquad R_2 = x\cosh\frac{z}{2} - y\sinh\frac{z}{2} = 0, \qquad z = 0.$$

Theorem

First Maxwell time is bounded as:

$$\begin{aligned}
\lambda \in C_1 &\implies t_1^{MAX}(\lambda) = 4K(k),\\
\lambda \in C_2 &\implies t_1^{MAX}(\lambda) = 4kK(k),\\
\lambda \in C_3 \cup C_4 \cup C_5 &\implies t_1^{MAX}(\lambda) = +\infty.
\end{aligned}$$

Conjugate Locus

- Exponential mapping

$$\begin{aligned} \mathrm{Exp}: &\quad (\lambda, t) = (\gamma, c, t) \rightarrow q(t) \in M, \\ \mathrm{Exp}: &\quad N = C \times \mathbb{R}_{+} \rightarrow M. \end{aligned}$$

- $q_t = \mathrm{Exp}(\lambda, t)$ – conjugate point for $q_0 \Longleftrightarrow q$ –Critical value of Exp

$$d_\nu \mathrm{Exp} : T_\nu N \rightarrow T_{q_t} M \text{ is degenerate},$$

- Jacobian J of the exponential mapping i.e.,

$$J = \frac{\partial(x_t, y_t, z_t)}{\partial(\gamma, c, t)} = \begin{vmatrix} \frac{\partial x_t}{\partial \gamma} & \frac{\partial x_t}{\partial c} & \frac{\partial x_t}{\partial t} \\ \frac{\partial y_t}{\partial \gamma} & \frac{\partial y_t}{\partial c} & \frac{\partial y_t}{\partial t} \\ \frac{\partial z_t}{\partial \gamma} & \frac{\partial z_t}{\partial c} & \frac{\partial z_t}{\partial t} \end{vmatrix} = 0.$$

Conjugate Locus

Theorem

The n^{th} conjugate times are bounded as:

$$\lambda \in C_1 \quad \Longrightarrow \quad 4nK(k) \leq t^{\text{conj}}_{2n-1} \leq 2p^n_1(k),\ 2p^n_1(k) \leq t^{\text{conj}}_{2n} \leq 4(n+1)K(k),$$
$$\lambda \in C_2 \quad \Longrightarrow \quad 4nkK(k) \leq t^{\text{conj}}_{2n-1} \leq 2kp^n_1(k),\ 2kp^n_1(k) \leq t^{\text{conj}}_{2n} \leq 4(n+1)kK(k)$$
$$\lambda \in C_4 \quad \Longrightarrow \quad t^{\text{conj}}_1(\lambda) = 2\pi.$$

Conjecture

Generalized Rolle's Theorem

Between any two Maxwell points there is one conjugate point, along any geodesic.

A. A. Agrachev, V.M. Zakalyukin

- Conjecture holds for sub-Riemannian problem on Lie group $\mathrm{SH}(2)$

$$t_n^{\mathrm{Max}}(\lambda) \leq t_n^{\mathrm{conj}}(\lambda) \leq t_{n+1}^{\mathrm{Max}}(\lambda), \quad n \in \mathbb{N}.$$

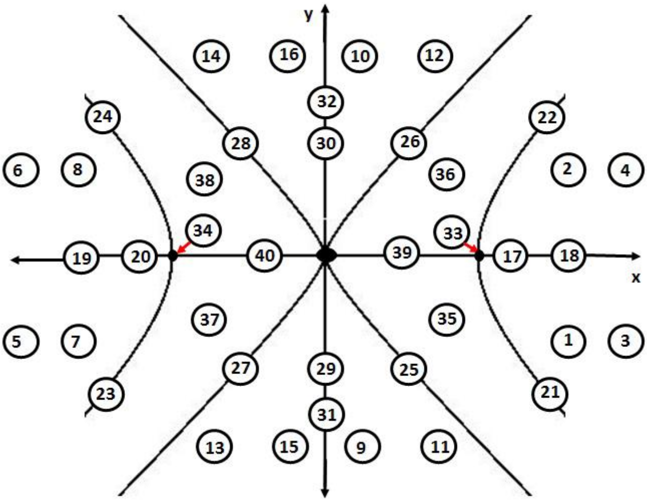
Stratifications in Preimage and Image of Exponential Mapping

- $\widehat{N} = \left\{ (\lambda, t) \in N \quad | \quad t \leq t_{\mathrm{Max}}(\lambda) = \cup_{i=1}^{2} D_i \cup \left(\cup_{i=1}^{40} N'_i \right) \right\}.$
- $\widehat{M} = M \backslash \{q_0\} = \cup_{i=1}^{2} M_i \cup \left(\cup_{i=1}^{40} M'_i \right)$

Theorem

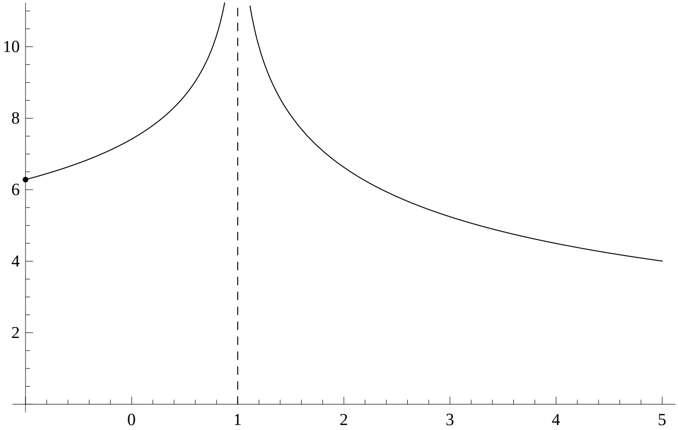
- $\mathrm{Exp} \; : \; D_i \to M_i, \quad i = 1, 2$ is a diffeomorphism
- $\mathrm{Exp} \; : \; N'_i \to M'_i, \quad i = 1, 40$ is a diffeomorphism

Stratification of Set $M' = \{z = 0\}$



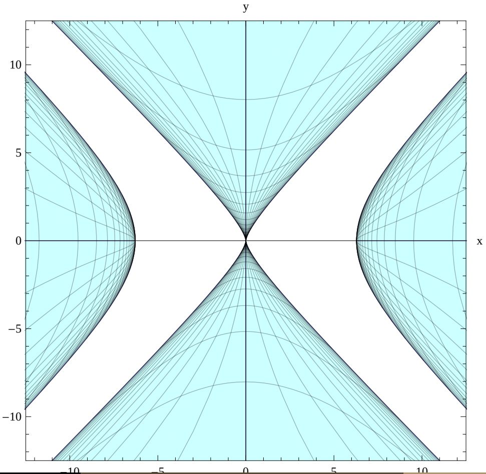
Cut Time

- $t_{\text{cut}}(\lambda) = t_{\text{Max}}(\lambda), \quad \lambda \in C,$
- $t_{\text{cut}} \circ \varepsilon^i = t_{\text{cut}}, \quad \varepsilon^i \in G$ where G is the group of all reflections
- $t_{\text{cut}} : C \to (0, +\infty]$ is a function of energy E of the pendulum



Global Structure of Cut Locus

- $\mathrm{Cut} = \mathrm{Max} \sqcup (\mathrm{Conj} \cap \mathrm{Cut}) = \mathrm{Cut}_{\mathrm{loc}} \cup \mathrm{Cut}_{\mathrm{glob}}$,
- $\mathrm{cl}(\mathrm{Cut}_{\mathrm{loc}}) \ni q_0, \quad d(q_0, \mathrm{Cut}_{\mathrm{loc}}) = 2\pi$,
- $\mathrm{Cut} \subset \{z = 0\}$.

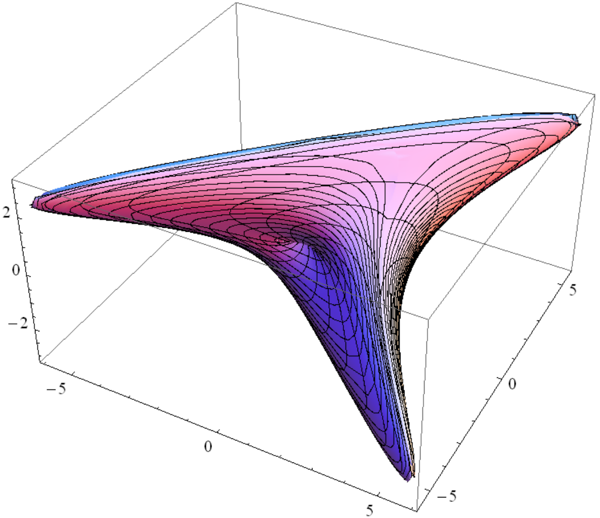


Sub-Riemannian Sphere

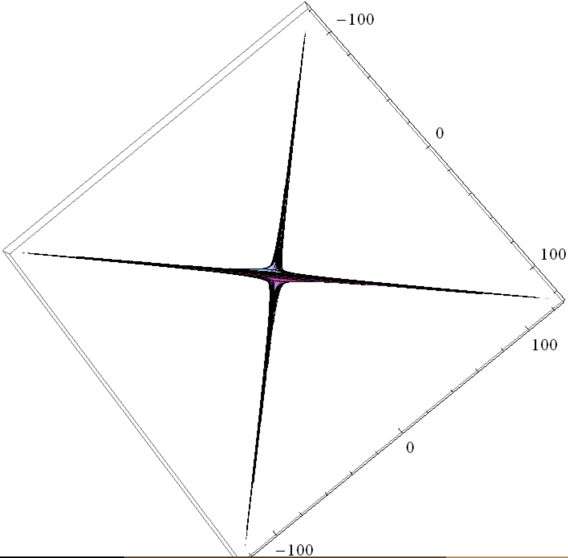
- $R > 0 \quad \Longrightarrow \quad S_R \cong S^2$,
- Sub-Riemannian sphere:

$$S_R = \{q = \mathrm{Exp}(\lambda, R) \in M | \lambda \in C, \quad t_{\mathrm{cut}}(\lambda) \geq R\}$$
$$= \{q \in M \quad | \quad d(q_0, q) = R\}.$$

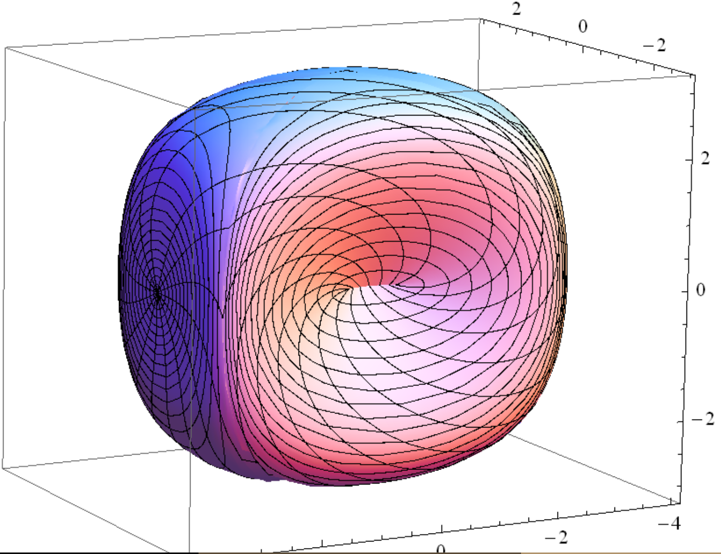
Sub-Riemannian Sphere with Radius $< 2\pi$ in Coordinates (x, y, z)



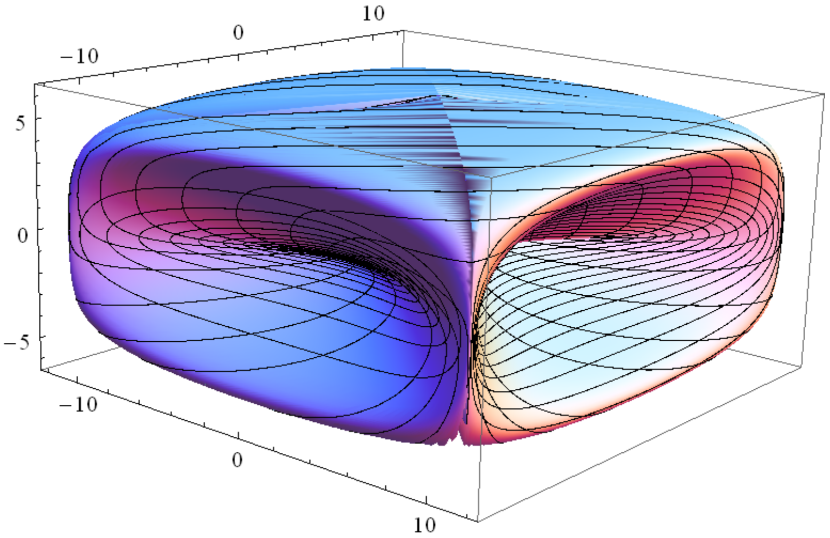
Sub-Riemannian Sphere with Radius $= 2\pi$ in Coordinates (x, y, z)



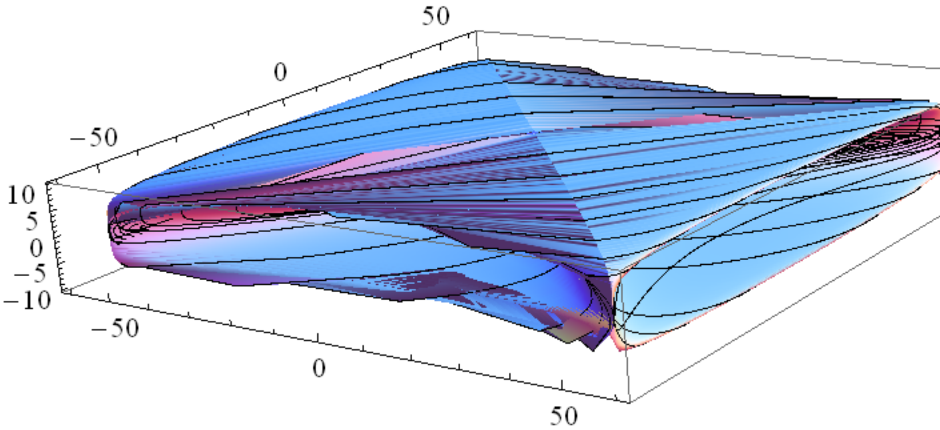
Sub-Riemannian Sphere with Radius $< 2\pi$ in Rectifying Coordinates (R_1, R_2, z)



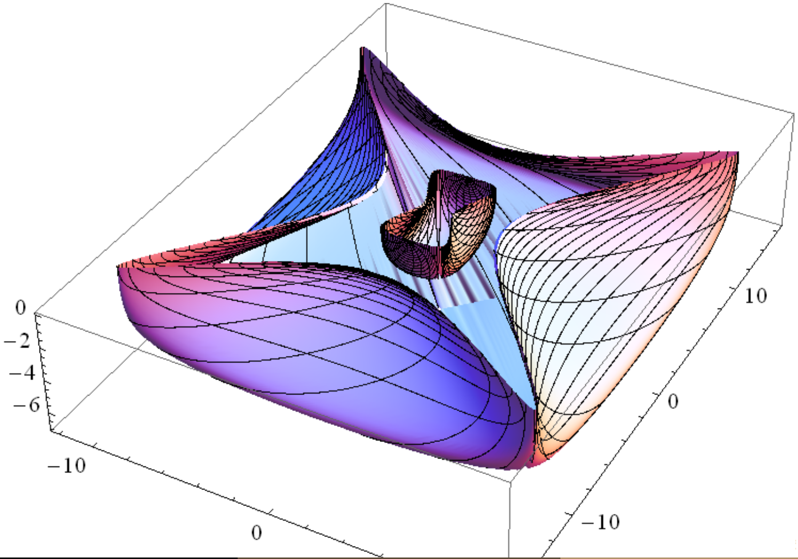
Sub-Riemannian Sphere with Radius $= 2\pi$ in Rectifying Coordinates (R_1, R_2, z)



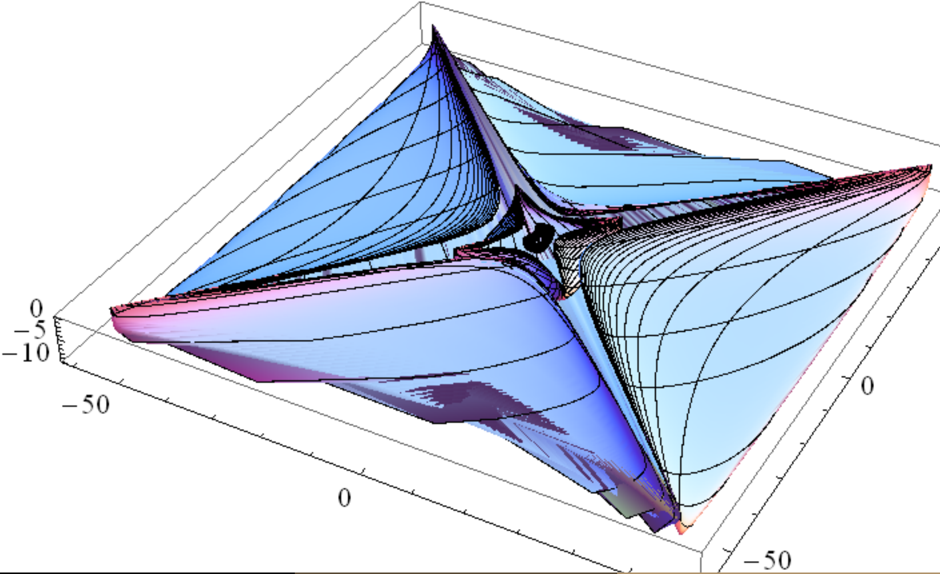
Sub-Riemannian Sphere with Radius $> 2\pi$ in Rectifying Coordinates (R_1, R_2, z)



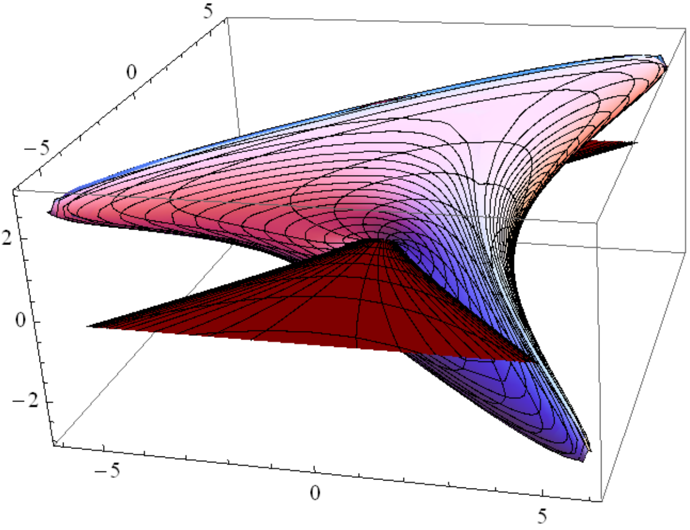
Matryoshka of Sub-Riemannian Wavefront with Radius π and 2π



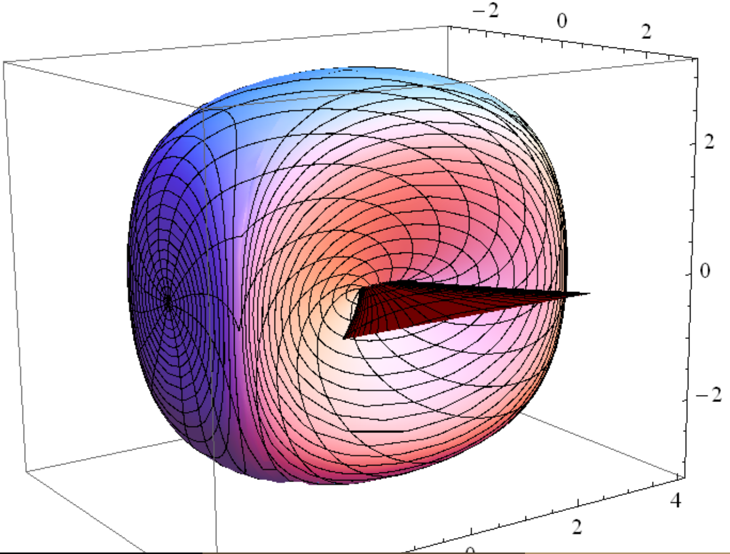
Matryoshka of Sub-Riemannian Wavefront with Radius π, 2π and 3π



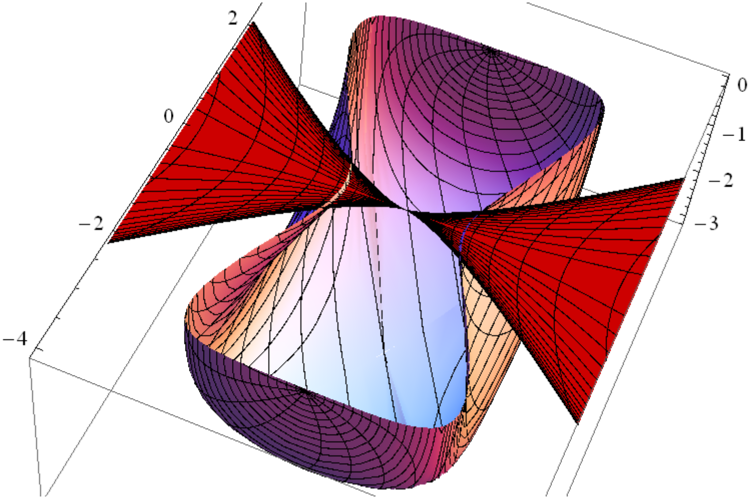
Sphere of Radius π and Intersection with Cut Locus in Coordinates (x, y, z)



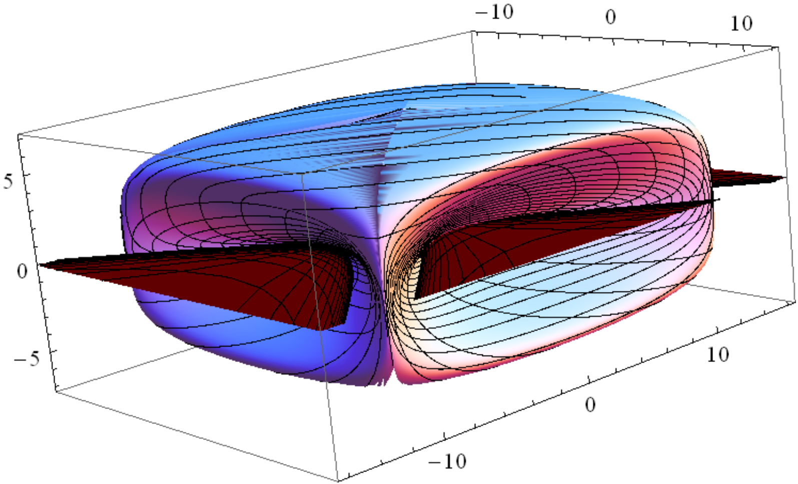
Sphere of Radius π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



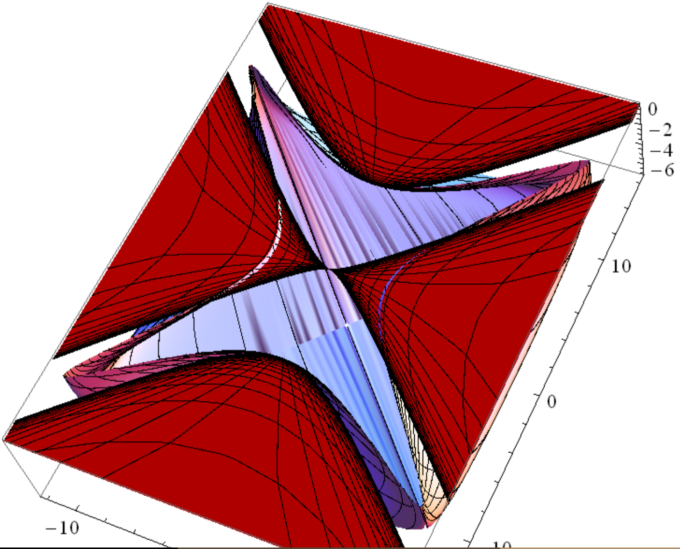
Wavefront of Radius π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



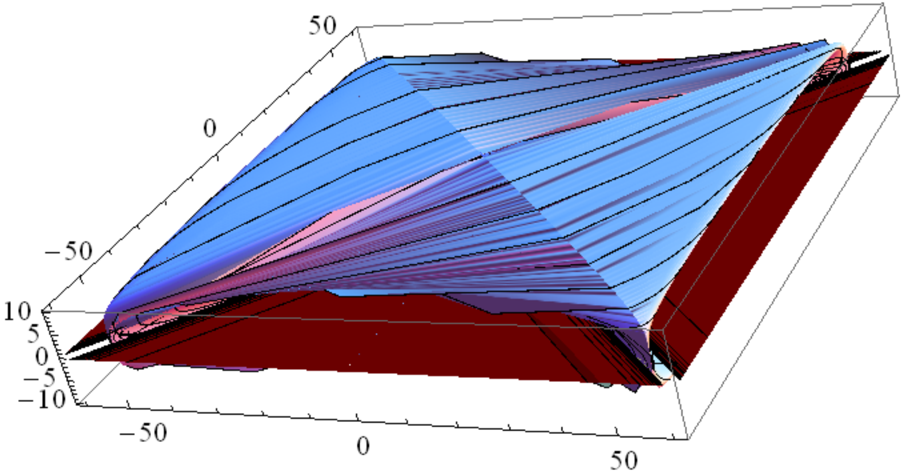
Sphere of Radius 2π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



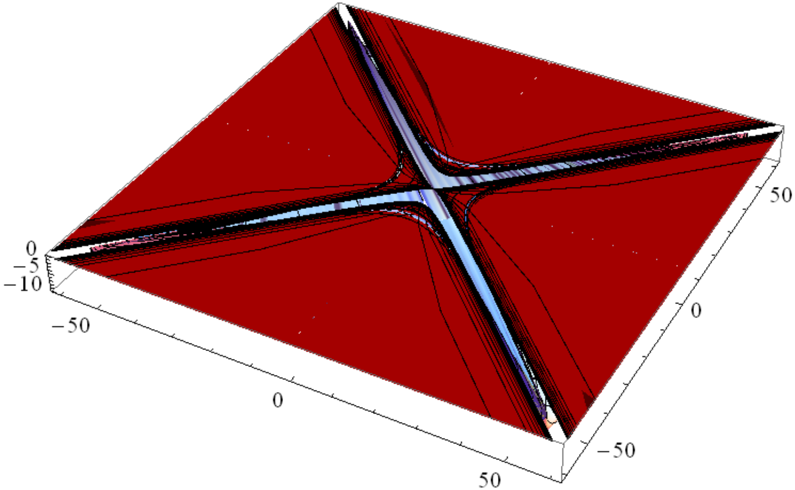
Wavefront of Radius 2π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



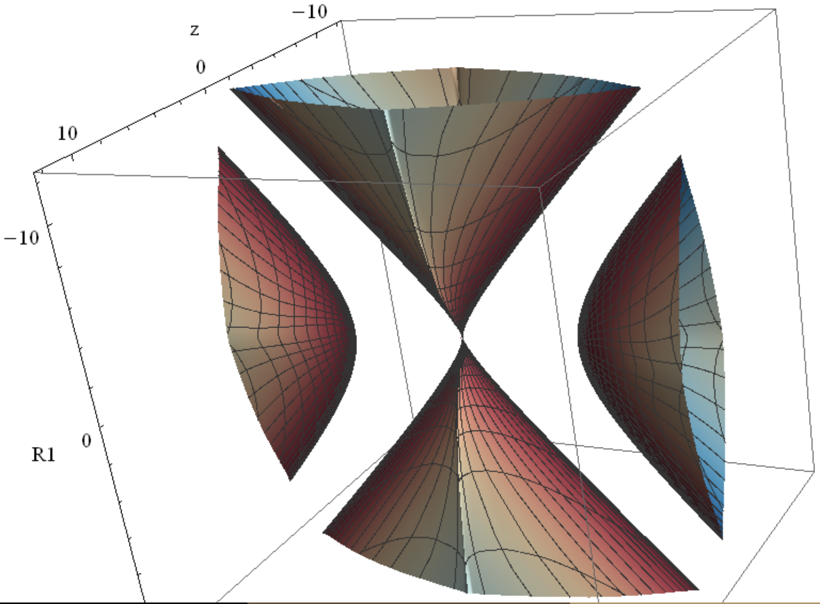
Sphere of Radius 3π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



Wavefront of Radius 3π and Intersection with Cut Locus in Rectifying Coordinates (R_1, R_2, z)



Sub-Riemannian Caustic



Sub-Riemannian Caustic - Local Component

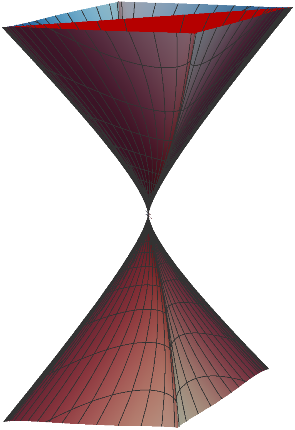
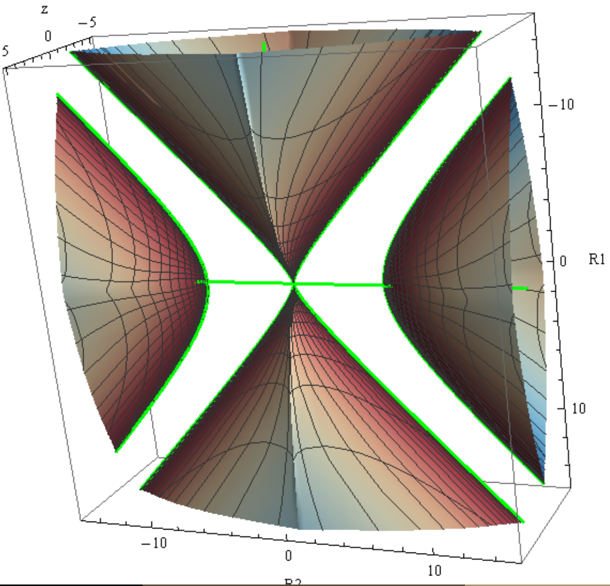
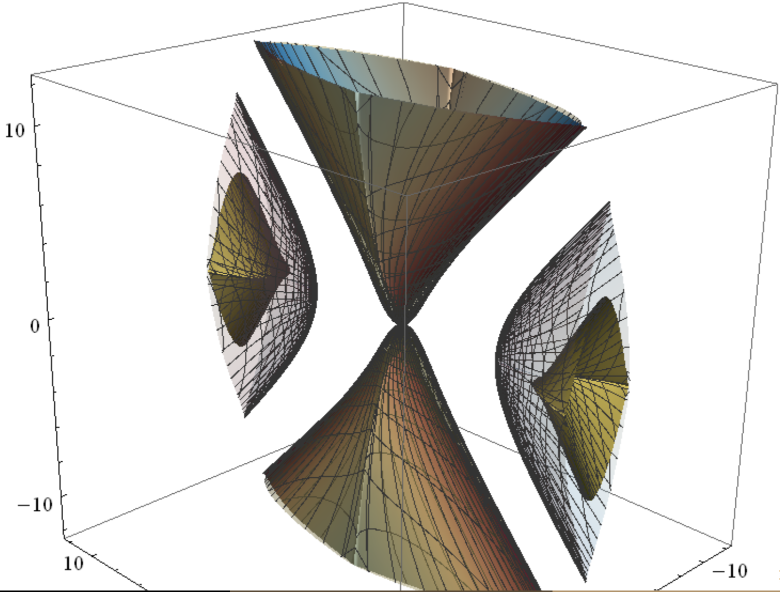


Figure: Local Component of the sub-Riemannian caustic

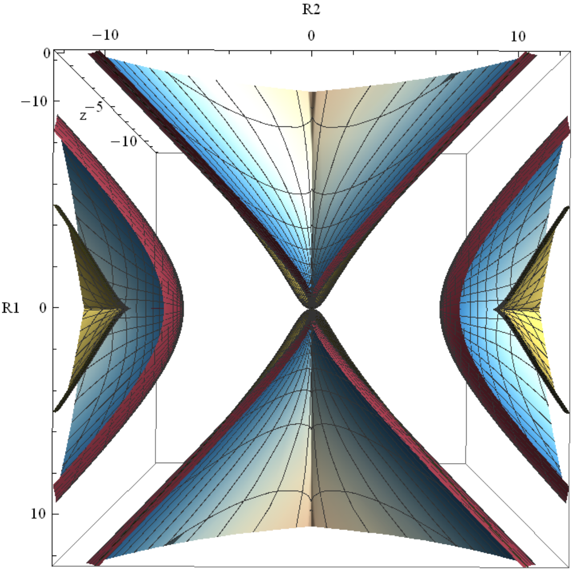
Sub-Riemannian Caustic with Boundary



Sub-Riemannian First and Second Caustic



Sub-Riemannian First, Second and Third Caustic



Publications

1. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Extremal Trajectories and Maxwell Strata in Sub-Riemannian Problem on Group of Motions of Pseudo-Euclidean Plane", Journal of Dynamical and Control Systems, Vol. 20 (2014), No. 3, 341–364
2. Y. A. Butt, A. I. Bhatti, Y. L. Sachkov, "Integrability by Quadratures in Optimal Control of a unicycle on a hyperbolic plane", American Control Conference, 1–3 Jul 2015, Chicago, Illinois
3. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Maxwell Strata and Conjugate Points in the SR Problem on the Lie Group SH(2)", Submitted
4. Y. A. Butt, Y. L. Sachkov, A. I. Bhatti, "Cut Locus and Optimal Synthesis in SR Problem on the Lie Group SH(2)", Submitted

Conclusion

- Parametrization of extremal trajectories
- Description of Maxwell strata and conjugate loci
- Generalized Rolle's theorem
- Description of the global component of the sub-Riemannian caustic
- Global explicit description of cut time and cut locus and optimal synthesis

And some concluding words

In continuation
of the yesterday banquet toasts . . .