

# Flat $(2, 3, 5, 8)$ Sub-Riemannian structure

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Conference on Control Theory and Mechanics  
Suzdal  
3–7 July 2015

# Part 1

## Introduction to sub-Riemannian geometry

- 1987 A.M. Vershik, V.Ya. Gershkovich, Nonholonomic Dynamical Systems. Geometry of distributions and variational problems.
- 1996 A. Bellaïche, The tangent space in sub-Riemannian geometry
- 1996 M. Gromov, Carnot-Carathéodory spaces seen from within
- 1997 V. Jurdjevic, Geometric Control Theory
- 2002 R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications.
- 2004 A.A. Agrachev, Yu. S., Control theory from the geometric viewpoint

## Sub-Riemannian minimizers and geodesics

- SR geometry  $(M, \Delta, g)$ :
  - smooth manifold  $M$ ,
  - vector distribution  $\Delta \subset TM$ ,
  - inner product  $g$  in  $\Delta$ .
- Horizontal curve  $q(\cdot)$ :  
 $q \in Lip([0, t_1], M)$ ,  $\dot{q}(t) \in \Delta_{q(t)}$  for almost all  $t \in [0, t_1]$ .
- Length of  $q(\cdot)$ :

$$l = \int_0^{t_1} \sqrt{g(\dot{q}(t), \dot{q}(t))} dt.$$

- SR distance between  $q_0, q_1 \in M$ :  
 $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ horizontal, } q(0) = q_0, q(t_1) = q_1\}$ .
- Minimizer  $q(\cdot)$ :  
horizontal curve such that  $l(q(\cdot)) = d(q(0), q(t_1))$ .
- Geodesic  $q(\cdot)$ :  
horizontal curve whose small subarcs are minimizers.

## 3D Heisenberg group

- $M = \mathbb{R}^3$ ,  $\Delta = \text{span}(X_1, X_2)$ ,  $g(X_i, X_j) = \delta_{ij}$ ,
- $X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$ ,  $X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$ .
- Left-invariant SR geometry  $\Rightarrow q_0 = \text{Id} = (0, 0, 0)$ .
- Geodesics:  $q_{\theta,c}(t)$ ,  $\theta \in S^1$ ,  $c \in \mathbb{R}$ ,  $t \in [0, +\infty)$ ,

$$\begin{aligned}x &= (\sin(\theta + ct) - \sin \theta)/c, \\y &= (\cos \theta - \cos(\theta + ct))/c, \\z &= (ct - \sin(ct))/(2c^2).\end{aligned}$$

- Minimizers:  $q_{\theta,c}(t)$ ,  $\theta \in S^1$ ,  $c \in \mathbb{R}$ ,  $t \in [0, 2\pi/|c|]$ .

- Distance between  $q_0$  and  $q_1 = (x_1, y_1, z_1)$ :

$$d(q_0, q_1) = \frac{\tau}{\sin \tau} r_1, \quad r_1 = \sqrt{x_1^2 + y_1^2},$$

$$\tau \in [0, \pi], \quad \frac{2\tau - \sin 2\tau}{\sin^2 \tau} = \frac{8|z_1|}{r_1^2}.$$

## Optimal control problem for SR minimizers

- $\Delta = \text{span}(X_1, \dots, X_k), \quad g(X_i, X_j) = \delta_{ij}$

$$\dot{q}(t) \in \Delta_{q(t)} \iff \dot{q}(t) = \sum_{i=1}^k u_i(t) X_i(q(t)),$$

$$l = \int_0^{t_1} \sqrt{g(\dot{q}, \dot{q})} dt = \int_0^{t_1} \sqrt{\sum_{i=1}^k u_i^2} dt \rightarrow \min$$

- Existence of a horizontal curve with  $q(0) = q_0, q_{t_1} = q_1$ :  
by Chow-Rashevsky Theorem: iff

$$\text{span}(X_i(q), [X_i, X_j](q), \dots) = T_q M \quad \forall q \in M.$$

- Existence of a minimizer with  $q(0) = q_0, q_{t_1} = q_1$ :  
by Filippov's theorem if one of the conditions hold:
  - $q_1$  close to  $q_0$ ,
  - SR balls  $B(q_0, R)$  are compact,
  - $\Delta, g$  left-invariant on a Lie group  $M$ .

# Hamiltonian vector field and its integrability

- Pontryagin maximum principle:

$q(\cdot)$  geodesic "  $\Rightarrow$  " it has a Hamiltonian lift  $\lambda(t) \in T_{q(t)}^*M$  such that

$$\dot{\lambda} = \vec{H}(\lambda), \quad \lambda \in T^*M,$$

$$H = \frac{1}{2} \sum_{i=1}^k h_i^2, \quad h_i(\lambda) = \langle \lambda, X_i(q) \rangle$$

(modulo abnormal geodesics).

- Liouville integrability of the Hamiltonian vector field  $\vec{H}$ :  
Find  $n$  independent integrals  $h_1, \dots, h_n$  in involution:

$$\{H, h_i\} = 0, \quad \{h_i, h_j\} = 0, \quad d_\lambda h_1, \dots, d_\lambda h_n \text{ independent a.e. on } T^*M$$

- Complicated for general SR structures  $\Rightarrow$  approximation?

# Linear approximation?

Let  $k = \dim \Delta_q = 2$ .

$$\dot{x} = u_1 X_1(x_0) + u_2 X_2(x_0), \quad x \in \mathbb{R}^n$$

- $n = 2 \Rightarrow$  Euclidean geometry approximates Riemannian one
- $n > 2 \Rightarrow$  no controllability
- $\text{Lie}(X_1(x_0), X_2(x_0))$  Abelian
- more sophisticated Lie algebras?

# Nilpotent approximation

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad q \in G \simeq \mathbb{R}^n$$

- $\text{Lie}(X_1, X_2)$  nilpotent:

$$[X_{i_{r+1}}, \dots, [X_{i_2}, X_{i_1}] \dots] = 0, \quad i_j \in \{1, 2\},$$

$r$  — step of Lie algebra  $\text{Lie}(X_1, X_2)$ .

- $G$  Lie group
- $X_i$  left-invariant vector fields on  $G$
- $G$  Carnot group: connected, simply connected, with graded Lie algebra

$$L = \bigoplus_{i=1}^r L_i, \quad [L_1, L_i] = L_{i+1}, \quad [L_i, L_j] \subset L_{i+j}, \quad L_1 = \text{span}\{X_1, X_2\}.$$

- Fundamental local approximation of generic SR structure
- E.g. left-invariant SR geometry on the 3D Heisenberg group approximates general contact 3D SR geometry (A.A.Agrachev, J.-P.Gauthier)



# Rank 2 free nilpotent sub-Riemannian geometry

- Growth vectors considered:  $(2, 3)$ ,  $(2, 3, 5)$ ,  $(2, 3, 5, 8)$ .
- Problems studied:
  - existence and smoothness of SR geodesics,
  - integrability of the normal Hamiltonian system of PMP,
  - regular and chaotic dynamics of the Hamiltonian system,
  - effective parametrization of exponential mapping,
  - discrete and continuous symmetries,
  - fixed points of the group of symmetries,
  - resulting study of global and local optimality of SR geodesics,
  - optimal synthesis (terminal point  $\mapsto$  optimal trajectory),
  - applications in robotics, mechanics, robotics,
  - effective software, including parallel for supercomputers<sup>1</sup>

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<sup>1</sup>see my affiliation :(((

# Rank 2 free nilpotent sub-Riemannian geometry

- Matryoshka of growth vectors:

$$(2, 3) \subset (2, 3, 5) \subset (2, 3, 5, 8).$$

- Problems studied:

- existence and smoothness of SR geodesics,
- integrability of the normal Hamiltonian system of PMP,
- regular and chaotic dynamics of the Hamiltonian system,
- effective parametrization of exponential mapping,
- discrete and continuous symmetries,
- fixed points of the group of symmetries,
- resulting study of global and local optimality of SR geodesics,
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- applications in robotics, mechanics, robotics,
- effective software, including parallel for supercomputers.

## Part 2

### Growth vector (2,3,5)

### Nilpotent SR geometry on the Cartan group

Originals:

1. Exponential mapping in the generalized Dido problem (in Russian), *Matem. Sbornik*, **194** (2003), 9: 63–90. English translation: *Sbornik: Mathematics* (2003), 194(9):1331–1360.
2. Symmetries of Flat Rank Two Distributions and Sub-Riemannian Structures, *Transactions of the American Mathematical Society*, **356** (2004), 2: 457–494.
3. Discrete symmetries in generalized Dido's problem, *Sbornik: Mathematics*, 197 (2006), 2: 235–257.
4. The Maxwell set in the generalized Dido problem, *Sbornik: Mathematics*, 197 (2006), 4: 595–621.
5. Complete description of the Maxwell strata in the generalized Dido problem, *Sbornik: Mathematics*, 197 (2006), 6: 901–950.

# Results and plans for the nilpotent (2, 3, 5) SR problem

## Results

- Sub-Riemannian geodesics were parametrized.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- Software for computing length minimizers for given boundary points was developed.
- Software for solving motion planning problem via nilpotent approximation was developed for general (2, 3, 5) systems, and tested for  $S^2$  rolling on  $\mathbb{R}^2$  without slipping or twisting, and a car with 2 off-hooked trailers.

## Plans

- Prove that the cut time is equal to the first Maxwell time.
- Describe the optimal synthesis.
- Describe the global structure of cut locus.
- Study singularities of SR sphere for the nilpotent and general (2, 3, 5) SR structures.
- Consider other (2, 3, 5) SR geometries.

Part 3 (Main)  
Growth vector  $(2,3,5,8)$   
Nilpotent SR geometry  
on the free nilpotent Carnot group  
with 2 generators, of step 4

Originals:

1. E.Sachkova, Yu.S., Abnormal geodesics in  $(2, 3, 5, 8)$  sub-Riemannian geometry, *work in progress*
2. L. Lokutsievsky, Yu.S., Regular and chaotic dynamics in  $(2, 3, 5, 8)$  sub-Riemannian geometry, *work in progress*

# Free nilpotent sub-Riemannian problem of rank 2, step $r$

$$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, \dots, X_n)$$

$$n = \sum_{k=1}^r \ell_2(k), \quad k\ell_2(k) = 2^k - \sum_{j|k} j\ell_d(j)$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad q \in G, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min.$$

Normal Hamiltonian system of Pontryagin Maximum Principle:

$$h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \lambda \in T^*G,$$

$$H(\lambda) = \frac{1}{2}(h_1^2(\lambda) + h_2^2(\lambda))$$

Is  $\dot{\lambda} = \vec{H}(\lambda)$  Liouville integrable?

# Integrals of the Hamiltonian system

Symmetries:

- right translations in  $G$   
 $Y_1, \dots, Y_n \in \text{Vec}(G), (R_g)_* Y_i = Y_i$
- Rotations in  $\Delta = \text{span}(X_1, X_2)$ :  
 $Z \in \text{so}(\Delta)$

Poisson algebra of integrals of the Hamiltonian field  $\vec{H}$ :

$$I = \text{span}(H; g_1, \dots, g_n; f; C_1, \dots, C_k)$$

- System Hamiltonian  $H$ ,
- Right-invariant Hamiltonians  $g_i(\lambda) = \langle \lambda, Y_i \rangle$ ,
- Hamiltonian of rotation:  $f(\lambda) = \langle \lambda, Z \rangle$ ,
- Casimir functions.  $C_j : L^* \rightarrow \mathbb{R}, \{C_j, h_i\} = 0 \forall i = 1, \dots, n$

1) Is  $\vec{H}$  Liouville integrable?

2) Does  $I$  contain  $n$  independent commuting integrals?

## Growth vector (2, 3)

$$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, X_2, X_3)$$

$$[X_1, X_2] = X_3$$

$$G = \mathbb{R}^3_{xyz}$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}$$

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \begin{cases} \dot{h}_1 = -h_2 h_3 \\ \dot{h}_2 = h_1 h_3 \\ \dot{h}_3 = 0 \\ \dot{q} = h_1 X_1 + h_2 X_2 \end{cases}$$

Integrability in trigonometric functions



## Liouville integrability in the (2, 3) case

- Right-invariant frame  $Y_1, Y_2, Y_3 \in \text{Vec}(G)$ ,  $Y_i(\text{Id}) = -X_i(\text{Id})$ :  
 $Y_1 = -\frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$ ,  $Y_2 = -\frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$ ,  $Y_3 = -\frac{\partial}{\partial z}$
- Rotation  $X_0 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$

Integrals of the field  $\vec{H}$ :

- System Hamiltonian  $H = (h_1^2 + h_2^2)/2$
- right-invariant Hamiltonians  
 $g_1 = -h_1 - yh_3$ ,  $g_2 = -h_2 + xh_3$ ,  $g_3 = -h_3$
- Hamiltonian of rotations  $h_0(\lambda) = \langle \lambda, X_0 \rangle$
- Casimir function  $h_3$

Algebra of integrals:  $I = \text{span}(H, g_1, g_2, g_3, h_0)$

Abelian subalgebra:  $\tilde{I} = \text{span}(H, g_2, g_3)$

## Growth vector (2, 3, 5)

$$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, X_2, X_3, X_4, X_5)$$

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5$$

$$G = \mathbb{R}_{xyzvw}^5$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{x^2+y^2}{2} \frac{\partial}{\partial w}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2+y^2}{2} \frac{\partial}{\partial v},$$
$$X_3 = \frac{\partial}{\partial z} + x \frac{\partial}{\partial v} + y \frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial v}, \quad X_5 = \frac{\partial}{\partial w}$$

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \begin{cases} \dot{h}_1 = -h_2 h_3 \\ \dot{h}_2 = h_1 h_3 \\ \dot{h}_3 = h_1 h_4 + h_2 h_5 \\ \dot{h}_4 = \dot{h}_5 = 0 \\ \dot{q} = h_1 X_1 + h_2 X_2 \end{cases}$$

## Integrability of the field $\vec{H}$ in the case (2, 3, 5)

$$2H = h_1^2 + h_2^2 = r^2 = \text{const}$$

$$h_1 = r \cos \theta, h_2 = r \sin \theta, h_3 = c, h_4 = \alpha \sin \beta, h_5 = -\alpha \cos \beta$$

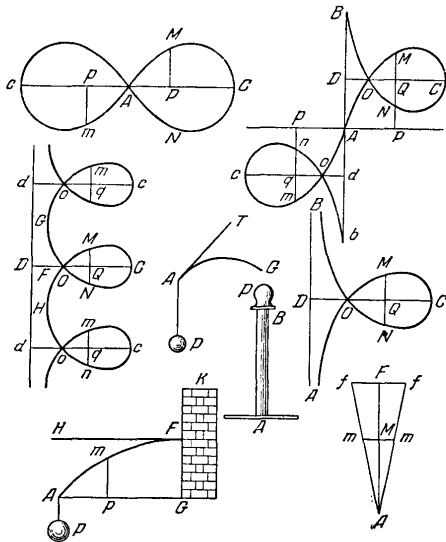
$$\dot{\theta} = c, \dot{c} = -\alpha r \sin(\theta - \beta), \dot{\alpha} = \dot{\beta} = 0$$

$$E = c^2/2 - \alpha r \cos(\theta - \beta) \text{ energy of pendulum}$$

Integrability in Jacobi's functions

$(x(t), y(t))$  Euler elasticae

# Description of elasticae by Leonhard Euler (1744)



## Integrals of the field $\vec{H}$ in the $(2, 3, 5)$ case

- System Hamiltonian  $H = (h_1^2 + h_2^2)/2$
- Right-invariant Hamiltonians  $g_1, g_2, g_3, g_4 = -h_4, g_5 = -h_5$
- Hamiltonian of rotation  $h_0(\lambda)$
- Casimir functions  $E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4, h_4, h_5$

Algebra of integrals:  $I = \text{span}(H, g_1, \dots, g_5, h_0, E)$

Abelian subalgebra:  $\tilde{I} = \text{span}(H, g_3, g_4, g_5, E)$

## Growth vector (2, 3, 5, 8)

$$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, \dots, X_8)$$

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5, [X_1, X_4] = X_6, \\ [X_1, X_5] = [X_2, X_4] = X_7, [X_2, X_5] = X_8.$$

$$G = \mathbb{R}_{x_1 \dots x_8}^8$$

Realization of nilpotent Lie algebras by polynomial vector fields:  
M.Grayson, R.Grossman, Nilpotent Lie algebras and vector fields  
(1989)

## Left-invariant frame in the case (2, 3, 5, 8)

$$\begin{aligned}X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4} \frac{\partial}{\partial x_7} - \frac{x_2^3}{6} \frac{\partial}{\partial x_8}, \\X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4} + \frac{x_1^3}{6} \frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4} \frac{\partial}{\partial x_7}, \\X_3 &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \frac{x_1^2}{2} \frac{\partial}{\partial x_6} + x_1 x_2 \frac{\partial}{\partial x_7} + \frac{x_2^2}{2} \frac{\partial}{\partial x_8}, \\X_4 &= \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7}, \\X_5 &= \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8}, \\X_6 &= \frac{\partial}{\partial x_6}, \quad X_7 = \frac{\partial}{\partial x_7}, \quad X_8 = \frac{\partial}{\partial x_8}.\end{aligned}$$

## Abnormal extremals

$$h_1 = h_2 = h_3 = 0,$$

$$\begin{pmatrix} \dot{h}_4 \\ \dot{h}_5 \end{pmatrix} = A \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}, \quad A = \begin{pmatrix} h_7 & -h_6 \\ h_8 & -h_7 \end{pmatrix} = \text{const}$$

$$\dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0,$$

$$\dot{q} = -h_5 X_1 + h_4 X_2.$$

Abnormal extremals are smooth

- $\det A < 0 \Rightarrow (x_1(t), x_2(t))$  hyperbolas,
- $\det A > 0 \Rightarrow (x_1(t), x_2(t))$  ellipses,
- $\det A = 0 \Rightarrow (x_1(t), x_2(t))$  parabolas or lines.

Hyperbolas and parabolas are not normal trajectories  $\Rightarrow$  they are strictly abnormal



# Normal Hamiltonian vector field in the case (2, 3, 5, 8)

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \left\{ \begin{array}{l} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = h_1 h_3, \\ \dot{h}_3 = h_1 h_4 + h_2 h_5, \\ \dot{h}_4 = h_1 h_6 + h_2 h_7, \\ \dot{h}_5 = h_1 h_7 + h_2 h_8, \\ \dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0, \\ \dot{q} = h_1 X_1(q) + h_2 X_2(q) \end{array} \right.$$

# Integrals of the field $\vec{H}$ in the case $(2, 3, 5, 8)$

- System Hamiltonian  $H = (h_1^2 + h_2^2)/2$
- Right-invariant Hamiltonians  $g_1, g_2, g_3, g_4, g_5, g_6 = -h_6,$   
 $g_7 = -h_7, g_8 = -h_8$
- Hamiltonian of rotation  $h_0$
- Casimir functions  
 $C = h_5^2 h_6 - 2h_4 h_5 h_7 + h_4^2 h_8 - 2h_3(h_6 h_8 - h_7^2) = C(g_3, \dots, g_8),$   
 $h_6, h_7, h_8$

Algebra of integrals:  $I = \text{span}(H, g_1, \dots, g_8, h_0)$

Abelian subalgebra:  $\tilde{I} = \text{span}(H, g_3, \dots, g_8)$

# Homogeneous integrals of $\vec{H}$ in the case $(2, 3, 5, 8)$

## Theorem

Let a homogeneous polynomial  $P_k = P_k(h_1, h_2)$ ,  $\deg P_k = k$ , be an integral of the field  $\vec{H}$ . Then:

1.  $P_1 = \sum_{i=6}^8 a_i h_i$ ,
2.  $P_2 = \sum_{i,j=6}^8 a_{ij} h_i h_j + bH$ ,
3.  $P_3 = \sum_{i,j,k=6}^8 a_{ijk} h_i h_j h_k + \sum_{i=6}^8 b_i h_i H + aC$ .

- Homogeneous integrals of order  $k \leq 3$  are expressed through Casimir functions  $h_6, h_7, h_8, C$  and system Hamiltonian  $H$ .
- Are there nontrivial homogeneous integrals of order  $k > 3$ ?
- 7 independent integrals in involution:  $H, g_3, \dots, g_8$ .
- Is  $\vec{H}$  Liouville integrable?

Reduction of the vertical (adjoint) subsystem  
for the system  $\dot{\lambda} = \vec{H}(\lambda)$

$$2H = h_1^2 + h_2^2 = 1$$

$$h_1 = \cos \theta, \quad h_2 = \sin \theta,$$

$$h_3 = c, \quad h_4 = a, \quad h_5 = b, \quad h_6 = m, \quad h_7 = p, \quad h_8 = n$$

$$\dot{\theta} = c,$$

$$\dot{c} = a \cos \theta + b \sin \theta,$$

$$\dot{a} = m \cos \theta + p \sin \theta,$$

$$\dot{b} = p \cos \theta + n \sin \theta,$$

$$m, \quad n, \quad p = \text{const.}$$

$$\delta = p^2 - mn \neq 0 \quad \Rightarrow$$

$$\dot{\theta} = (2pab - na^2 - mb^2)/(2\delta) + k,$$

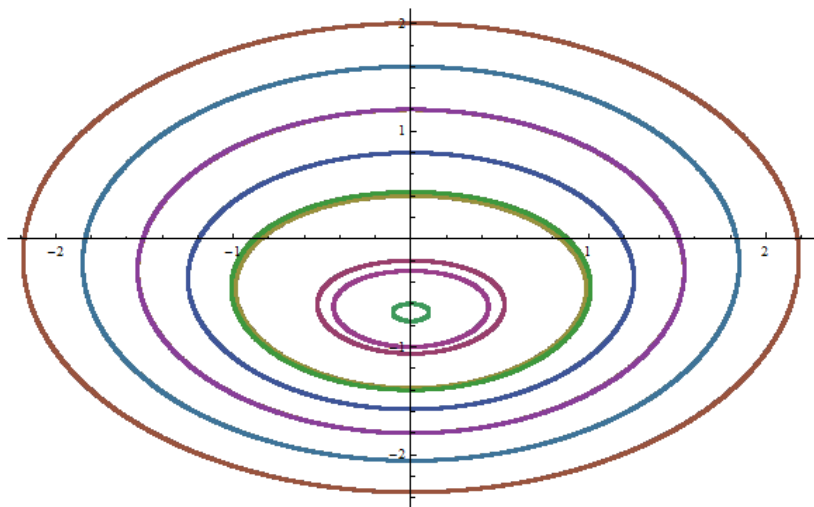
$$\dot{a} = m \cos \theta + p \sin \theta,$$

$$\dot{b} = p \cos \theta + n \sin \theta, \quad m, \quad n, \quad p, \quad k = \text{const.}$$

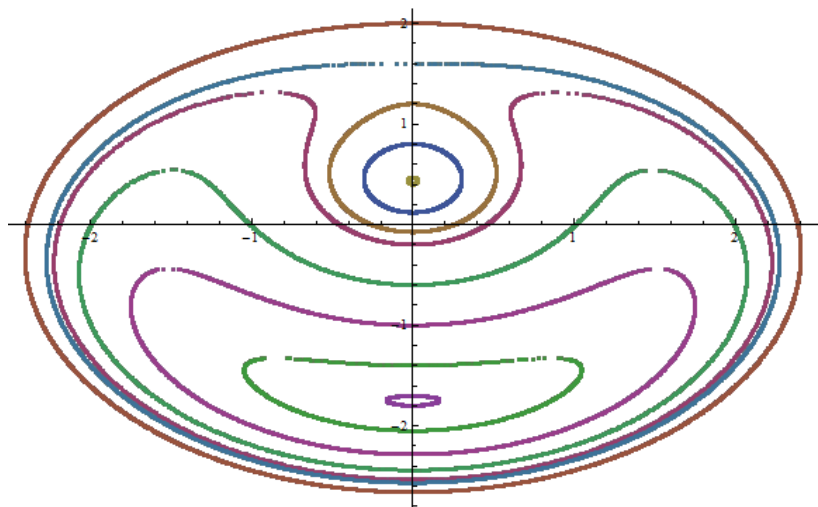
# Conclusion

- Free nilpotent sub-Riemannian problem with the growth vector  $(2, 3, 5, 8)$ ,
- Realization by polynomial vector fields in  $\mathbb{R}^8$ ,
- Symmetries of the Hamiltonian systems,
- Integrals of the normal Hamiltonian system:  
10 integrals, of which commute only 7,
- Casimir functions and co-adjoint orbits computed,
- 1-st, 2-nd, and 3-rd order (w.r.t.  $h_i$ ) integrals computed
- Mishchenko-Fomenko's noncommutative integrability theory fails
- (Non) commutative integrability remains an open question.
- Numerical evidence of Liouville non-integrability

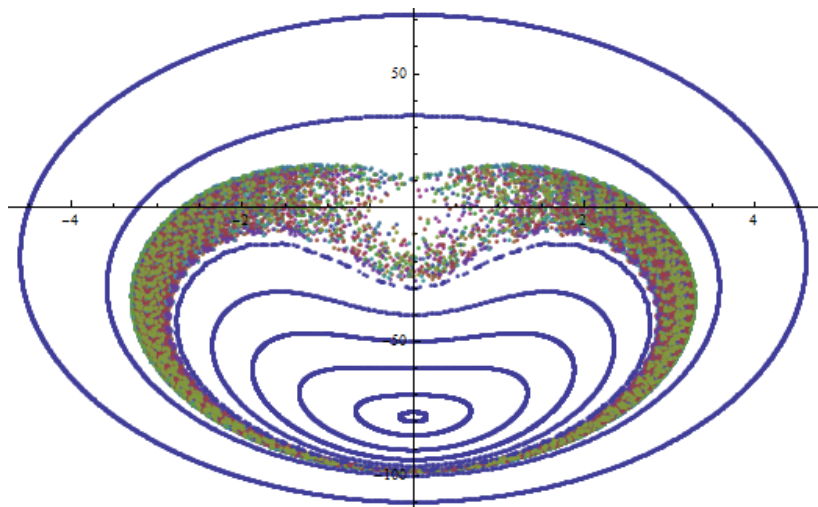
Orbits of Poincaré mapping:  
Integrable case with 1 periodic trajectory



Orbits of Poincaré mapping:  
Integrable case with 2 periodic trajectories

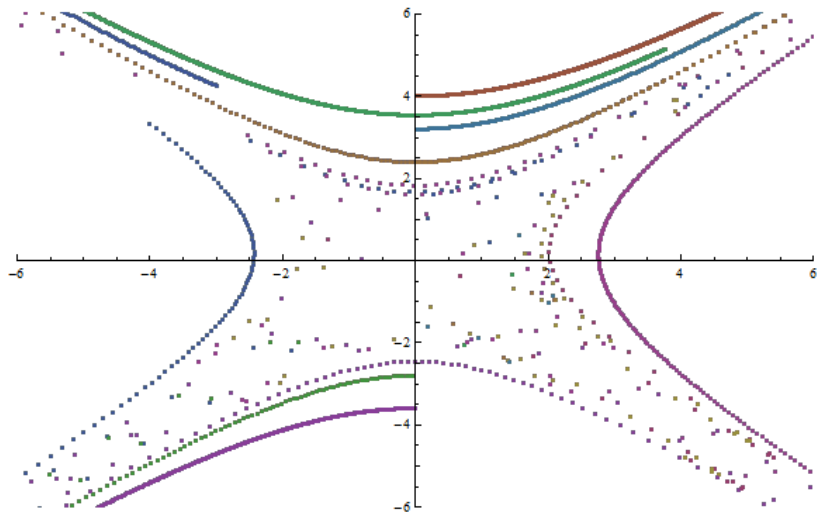


# Orbits of Poincaré mapping: Elliptic non-integrable case





# Orbits of Poincaré mapping: Hyperbolic non-integrable case



Two-parametric family of open problems:  
rank  $k$  free nilpotent sub-Riemannian problems of step  $N$ .