SYSTEMS OF QUADRATIC INEQUALITIES

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Abstract. We present a spectral sequence which efficiently computes Betti numbers of a closed semi-algebraic subset of $\mathbb{R}\mathrm{P}^n$ defined by a system of quadratic inequalities and the image of the homology homomorphism induced by the inclusion of this subset in $\mathbb{R}\mathrm{P}^n$. We do not restrict ourselves to the term E_2 of the spectral sequence and give a simple explicit formula for the differential d_2.

1. Introduction

In this paper we study closed semialgebraic subsets of $\mathbb{R}\mathrm{P}^n$ presented as the sets of solutions of systems of homogeneous quadratic inequalities. Systems are arbitrary: no regularity condition is required and systems of equations are included as special cases. Needless to say, standard Veronese map reduces any system of homogeneous polynomial inequalities to a system of quadratic ones (but the number of inequalities in the system increases). The nonhomogeneous affine case will be the subject of another publication.

To study a system of quadratic inequalities we focus on the dual object. Namely, we take the convex hull, in the space of all real quadratic forms on $\mathbb{R}^{n+1}$, of those quadratic forms involved in the system, and we try to recover the homology of the set of solutions from the arrangement of this convex hull with respect to the cone of degenerate forms. This approach allows to efficiently compute Betti numbers of the set of solutions for a very big number of variables n as long as the number of linearly independent inequalities is limited. Moreover, this approach works well for systems of integral quadratic inequalities (i. e. in the infinite dimension, far beyond the semi-algebraic context) as we plan to prove in another paper.

Let $p:\mathbb{R}^{n+1}\to\mathbb{R}^{k+1}$ be a homogeneous quadratic map and $K\subset\mathbb{R}^{k+1}$ a convex polyhedral cone in $\mathbb{R}^{k+1}$ (zero cone $K=\{0\}$ is permitted). We are going to study the semialgebraic set

$$X_p=\{[x_0,\ldots,x_n]\in\mathbb{R}\mathrm{P}^n \mid p(x_0,\ldots,x_n)\in K\}.$$

More precisely, we are going to compute the homology $H_*(X_p;\mathbb{Z}_2)$ and the image of the map $\iota_*:H_*(X_p;\mathbb{Z}_2)\to H_*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$, where $\iota:X_p\to\mathbb{R}\mathrm{P}^n$ is the inclusion.

In what follows, we use shortened notations $H_*(X_p;\mathbb{Z}_2)=H_*(X_p)$, $\mathbb{R}\mathrm{P}^n=\mathbb{P}^n$.

Let $\mathcal{Q}$ be the space of real quadratic forms on $\mathbb{R}^{n+1}$. Given $q\in\mathcal{Q}$, we denote by $\mathrm{i}^+(q)\in\mathbb{N}$ the positive inertia index of q that is the maximal dimension of a subspace of $\mathbb{R}^{n+1}$ where the form q is positive definite. Similarly, $\mathrm{i}^-(q)\doteq\mathrm{i}^+(-q)$ is the negative inertia index. We set:

$$\mathcal{Q}^j=\{q\in\mathcal{Q}:\mathrm{i}^+(q)\geq j\}.$$

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We denote by $\bar{p} : \mathbb{R}^{k+1^*} \to \mathcal{Q}$ the linear systems of quadratic forms associated to the map p. In coordinates:

$$p = \begin{pmatrix} p^0 \\ \vdots \\ p^k \end{pmatrix}, \quad p^i \in \mathcal{Q}, \qquad \bar{p}(\omega) = \omega p = \sum_{i=0}^{k} \omega_i p^i, \quad \forall\, \omega = (\omega_0, \ldots, \omega_k) \in \mathbb{R}^{k+1^*}.$$

More notations:

$$K^\circ = \{\omega \in \mathbb{R}^{k+1^*} : \langle \omega, y\rangle \leq 0,\ \forall y \in K\},\ \text{the dual cone to } K;$$
$$\Omega = K^\circ \cap S^k = \{\omega \in K^\circ : |\omega| = 1\};$$
$$C\Omega = K^\circ \cap B^{k+1} = \{\omega \in K^\circ : |\omega| \leq 1\};$$
$$\Omega^j = \{\omega \in \Omega : \mathrm{i}^+(\omega p) \geq j\}.$$

Theorem A. *There exists a first quadrant spectral sequence (E_r, d_r) converging to $H_{n-*}(X_p)$ such that $E_2^{ij} = H^i(C\Omega, \Omega^{j+1})$.*

We define $\mu \doteq \max_{\eta \in \Omega} \mathrm{i}^+(\eta)$. If $\mu = 0$ then $X_p = \mathbb{P}^n$; otherwise we can describe the term E_2 by the following table where cohomology groups are replaced with isomorphic ones according to the long exact sequence of the pair $(C\Omega, \Omega^{j+1})$.

$$\begin{array}{c|c|c|c|c|c|c|c|c|}
 & 0 & 0 & 0 & & & & & \\
n & \mathbb{Z}_2 & 0 & 0 & & & & & \\
 & \vdots & \vdots & \vdots & & & & & \\
\mu & \mathbb{Z}_2 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
 & 0 & H^0(\Omega^\mu)/\mathbb{Z}_2 & H^1(\Omega^\mu) & \cdots & H^i(\Omega^\mu) & \cdots & H^k(\Omega^\mu) & 0 \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & 0 & H^0(\Omega^{j+1})/\mathbb{Z}_2 & H^1(\Omega^{j+1}) & \cdots & H^i(\Omega^{j+1}) & \cdots & H^k(\Omega^{j+1}) & 0 \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & 0 & H^0(\Omega^1)/\mathbb{Z}_2 & H^1(\Omega^1) & \cdots & H^i(\Omega^1) & \cdots & H^k(\Omega^1) & 0 \\
\hline
\end{array}$$

Example 1. Let $n = k = 2,\ p(x_0, x_1, x_2) = \begin{pmatrix} x_0x_1 \\ x_0x_2 \\ x_1x_2 \end{pmatrix},\ K = \{0\}$. Then

$$\Omega = \Omega^1 = S^2, \quad \Omega^2 = \{\omega \in S^2 : \omega_0\omega_1\omega_2 < 0\}, \quad \Omega^3 = \emptyset.$$

The term E_2 has the form:

$$\begin{array}{|cccc}
\mathbb{Z}_2 & 0 & 0 & 0 \\
0 & (\mathbb{Z}_2)^3 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}_2 \\
\hline
\end{array}$$

In this case $d_2 : (\mathbb{Z}_2)^3 \to \mathbb{Z}_2$ is a non-vanishing differential and the set X_p consists of 3 points.

Let $\mathscr{G}_j = \left\{(V, q) \in Gr(j) \times \left(\mathcal{Q}^j \setminus \mathcal{Q}^{j+1}\right) : q\big|_V > 0\right\}$, where $Gr(j)$ is the Grassmannian of j-dimensional subspaces of $\mathbb{R}^{n+1}$. It is easy to see that the projection $\pi : \mathscr{G}_j \to \mathcal{Q}^j \setminus \mathcal{Q}^{j+1}$ defined by $(V, q) \mapsto q$ is a homotopy equivalence.

Let us consider the tautological vector bundle $\mathcal{V}_j$ over $\mathscr{G}_j$ whose fiber over $(V,q) \in \mathscr{G}_j$ is the space $V \subset \mathbb{R}^{n+1}$; we denote $w_1(\mathcal{V}_j) \in H^1(\mathscr{G}_j)$ the first Stiefel–Whitney class of this bundle. Recall that $w_1(\mathcal{V}_j)$ evaluated on a loop $f : S^1 \to \mathscr{G}_j$ vanishes if and only if $f^*\mathcal{V}_j$ is a trivial bundle. Moreover, the value of $w_1(\mathcal{V}_j)$ at f depends only on the curve $\pi \circ f$ in $\mathcal{Q}^j \setminus \mathcal{Q}^{j+1}$ and $w_1(\mathcal{V}_j) = \pi^*\nu_j$ for a well-defined class $\nu_j \in H^1\left(\mathcal{Q}^j \setminus \mathcal{Q}^{j+1}\right)$.

Proposition. *The differential d_2, of the spectral sequence (E_r, d_r) depends on the class $\bar{p}\big|^*_{\Omega^j \setminus \Omega^{j+1}}(\nu_j) \in H^1(\Omega^j \setminus \Omega^{j+1})$. If $\bar{p}\big|^*_{\Omega^j \setminus \Omega^{j+1}}(\nu_j) = 0,\ \forall\, j > 0$, then $E_3 = E_2$.*

The classes ν_j are defined without any use of the Euclidean structure on $\mathbb{R}^{n+1}$. This structure is however useful for the calculation of d_2. Given $q \in \mathcal{Q}$, let $\lambda_1(q) \geq \cdots \geq \lambda_{n+1}(q)$ be the eigenvalues of the symmetric operator Q on $\mathbb{R}^{n+1}$ defined by the formula $q(x) = \langle Qx, x\rangle,\ x \in \mathbb{R}^{n+1}$. Then $\mathcal{Q}^j = \{q \in \mathcal{Q} : \lambda_j > 0\}$. We set $\mathcal{D}_j = \{q \in \mathcal{Q} : \lambda_j(q) \neq \lambda_{j+1}(q)\}$ and denote by $\mathcal{L}_j^+$ the j-dimensional vector bundle over $\mathcal{D}_j$ whose fiber at a point $q \in \mathcal{D}_j$ equals $span\{x \in \mathbb{R}^{n+1} : Qx = \lambda_i(q)x,\ 1 \leq i \leq j\}$. Obviously, $\mathcal{Q}^j \setminus \mathcal{Q}^{j+1} \subset \mathcal{D}_j$ and $\nu_j = w_1\left(\mathcal{L}_j^+\right)\big|_{\mathcal{Q}^j \setminus \mathcal{Q}^{j+1}}$.

Now we set $\phi_j = \partial^* w_1(\mathcal{L}_j^+)$, where $\partial^* : H^1(\mathcal{D}_j) \to H^2(\mathcal{Q}, \mathcal{D}_j)$ is the connecting homomorphism in the exact sequence of the pair $(\mathcal{Q}, \mathcal{D}_j)$. Recall that

$$\mathcal{Q} \setminus \mathcal{D}_j = \{q \in \mathcal{Q} : \lambda_j(q) = \lambda_{j+1}(q)\}$$

is a codimension 2 algebraic subset of $\mathcal{Q}$ whose singular locus

$$sing\,(\mathcal{Q} \setminus \mathcal{D}_j) = \{q \in \mathcal{Q} : (\lambda_{j-1}(q) = \lambda_{j+1}(q)) \vee (\lambda_j(q) = \lambda_{j+2}(q))\}$$

has codimension 5 in $\mathcal{Q}$. Let $f : B^2 \to \mathcal{Q}$ be a continuous map defined on the disc B^2 and such that $f(\partial B^2) \subset \mathcal{D}_j$; the value of $\phi_j \in H^2(\mathcal{Q}, \mathcal{D}_j)$ at f equals the intersection number (modulo 2) of f and $\mathcal{Q} \setminus \mathcal{D}_j$ (this intersection number is well defined since the codimension of the singular locus is big enough).

Theorem B (the differential d_2)**.** *We have:*

$$d_2(x) = (x \smile \bar{p}^*\phi_j)\big|_{(C\Omega, \Omega^j)}, \quad \forall\, x \in H^*(C\Omega, \Omega^{j+1}),$$

where $\smile$ is the cohomological product.

Theorem C. *Let $(\iota_*)_a : H_a(X_p) \to H_a(\mathbb{P}^n),\ 0 \leq a \leq n$, be the homomorphism induced by the inclusion $\iota : X_p \to \mathbb{P}^n$. Then $\mathrm{rk}(\iota_*)_a = \dim E_\infty^{0,n-a}$.*

Next theorem about hyperplane sections is a step towards the understanding of functorial properties of the duality between the semi-algebraic sets X_p and the index functions $\mathrm{i}^+ \circ \bar{p}$.

Let V be a codimension one subspace of $\mathbb{R}^{n+1}$ and $\bar{V} \subset \mathbb{R}\mathrm{P}^n$ the projectivization of V. We define for $j > 0$ the following sets:

$$\Omega_V^j = \{\omega \in \Omega : \mathrm{i}^+\left(\omega p|_V\right) \geq j\}$$

Theorem D. *There exists a cohomology spectral sequence (G_r, d_r) of the first quadrant converging to $H_{n-*}(X_p, X_p \cap \bar{V})$ such that*

$$G_2^{i,j} = H^i(\Omega_V^j, \Omega^{j+1}),\ j > 0, \quad G_2^{i,0} = H^i(C\Omega, \Omega^1).$$

Theorem A is proved in Section 3, the differential d_2 is computed in Section 4, Theorem C on the imbedding to $\mathbb{R}\mathrm{P}^n$ is proved in Section 5, and Theorem D on the hyperplane sections in Section 6. In Section 7 we study the special case of the constant index function where higher differentials can be easily computed and consider some other examples.

Let us indicate the main general ideas these proofs are based on.

Regularization. There could be different definitions of regularization of a smooth inequality; for any reasonable one the inequality can be regularized without changing the homotopy type of the space of solutions. Indeed, given a polynomial a, the space of solutions of the inequality $a(x) \leq 0$ is a deformation retract of the space of solutions of the inequality $a(x) \leq \varepsilon$ for any sufficiently small $\varepsilon > 0$, and the inequality $a(x) \leq \varepsilon$ is regular for any ε from the complement of a discrete subset of $\mathbb{R}$. The regularization of the equation $a(x) = 0$ is a system of inequalities $\pm a(x) \leq \varepsilon$.

Duality. The pair $(\bar{p}, K^\circ)$, where $\bar{p} : K^\circ \to \mathcal{Q}$, can be considered as the *dual object* to X_p. Moreover, $\mathbb{P}^n \setminus X_p$ is homotopy equivalent to $B = \{(\omega, x) \in \Omega \times \mathbb{P}^n : (\omega p)(x) > 0\}$. For a regular system of quadratic inequalities, the spectral sequence (E_r, d_r) is the relative Leray spectral system of the map $(\omega, x) \mapsto \omega$ applied to the pair $(\Omega \times \mathbb{P}^n, B)$.

Localization. Setting $B(V) = B \cap (V \times \mathbb{P}^n)$ for $V \subset \Omega$, and given $\omega_0 \in \Omega$ we have: $B(O_{\omega_0}) \approx B(\omega_0) \approx \mathbb{P}^{\mathrm{i}^+(\omega_0 p)-1}$, where O_{ω_0} is any sufficiently small contractible neighborhood of ω_0 and $\approx$ is the homotopy equivalence. This fact allows to compute the member E_2 of the spectral sequence.

Regular homotopy. This is perhaps the most interesting tool which allows to compute the differential d_2. The notion of regular homotopy is based on the dual characterization for the regularity of a system of quadratic inequalities. We say that the system defined by the map p and cone K is regular if $p\big|_{\mathbb{R}^{n+1}\setminus\{0\}}$ is transversal to K: this amount to say $\mathrm{im}(D_x p) + K = \mathbb{R}^{k+1}$ for every $x \in \mathbb{R}^{n+1}\setminus\{0\}$ such that $p(x) \in K$.

The dual characterization of regularity concerns the linear map $\bar{p} : \Omega \to \mathcal{Q}$ but can be naturally extended to any smooth map $f : \Omega \to \mathcal{Q}$. Note that $\mathcal{Q}$ is the dual space to $\mathbb{R}^{n+1} \odot \mathbb{R}^{n+1}$, the symmetric square of $\mathbb{R}^{n+1}$. Let

$$\mathcal{Q}_0 = \{q \in \mathcal{Q} : \ker q \neq 0\},$$

the discriminant of the space of quadratic forms. Then $\mathcal{Q}_0$ is an algebraic hypersurface and

$$\mathrm{sing}\mathcal{Q}_0 = \{q \in \mathcal{Q}_0 : \dim \ker q > 1\}.$$

Given $q \in \mathcal{Q}_0 \setminus \mathrm{sing}\mathcal{Q}_0$ and $x \in \ker q \setminus 0$, the vector $x \odot x \in \mathcal{Q}^*$ is normal to the hypersurface $\mathcal{Q}_0$ at q. We define a co-orientation of $\mathcal{Q}_0 \setminus \mathrm{sing}\mathcal{Q}_0$ by the claim that $x \odot x$ is a positive normal. For any, maybe singular, $q \in \mathcal{Q}_0$ we define the *positive normal cone* as follows:

$$N_q^+ = \{x \odot x : x \in \ker q \setminus 0\}.$$

The cone N_q^+ consists of the limiting points of the sequences $N_{q_i}^+,\ i \in \mathbb{N}$, where $q_i \in \mathcal{Q}_0 \setminus \mathrm{sing}\mathcal{Q}_0$ and $q_i \to q$ as $i \to \infty$.

We say that $f : \Omega \to \mathcal{Q}$ is *not regular* (with respect to $\mathcal{Q}_0$) at $\omega \in \Omega$ if $f(\omega) \in \mathcal{Q}_0$ and there exists $y \in N_\omega^+$ such that $\langle D_\omega f v, y\rangle \leq 0,\ \forall\, v \in T_\omega\Omega$. The map f is regular if it is regular at any point. It is easy to check that the transversality of the quadratic map $p\big|_{\mathbb{R}^{n+1}\setminus\{0\}}$ to the cone K is equivalent to the regularity of the linear map $\bar{p} : \Omega \to \mathcal{Q}$ where, we remind, $\Omega = K^\circ \cap S^k$.

A homotopy $f_t : \Omega \to \mathcal{Q},\ 0 \le t \le 1$, is a *regular homotopy* if all f_t are regular maps. The following fundamental geometric fact somehow explains the results of this paper and gives a perspective for further research. If linear maps $\bar{p}_0, \bar{p}_1$ are regularly homotopic then the pairs $(\mathbb{P}^n, \mathbb{P}^n \setminus X_{p_0})$ and $(\mathbb{P}^n, \mathbb{P}^n \setminus X_{p_1})$ are homotopy equivalent. Note that the maps f_t in the homotopy connecting $\bar{p}_0$ and $\bar{p}_1$ are just smooth, not necessary linear. It is important that the cones $N_q^+,\ q \in \operatorname{sing}\mathcal{Q}_0$, are not convex. If N_q^+ would be convex then regular homotopy would preserve the term E_2 of our spectral sequence, the differentials $d_r,\ r \ge 2$, would vanish and E_2 would be equal to E_∞.

Regular homotopy was introduced in paper [2]. In the mentioned paper the author computed for a *regular* quadratic map p, the second term and the second differential of a spectral sequence converging to the cohomology of the double cover of X_p. Again in the case of a *regular* quadratic map, all the differentials of a spectral sequence (E'_r, d'_r) converging to $H^*(\mathbb{P}^n \setminus X_p)$ were announced (without proof) in [1]. We have to confess that, unfortunately, for the spectral sequence (E'_r, d'_r) only the differential d'_2 was computed correctly. Our differential d_2 and that d'_2 are closely related: indeed for regular systems they are defined, in a certain sense, by cup product with the same cohomology classes.
Universal upper bounds for the Betti numbers of the sets defined by systems of quadratic inequalities or equations were obtained in [3, 4, 5, 9].

Remark. An Hermitian quadratic form is a quadratic form $q : \mathbb{C}^{n+1} \to \mathbb{R}$ such that $q(iz) = q(z)$. Similarly, a "quaternionic" quadratic form is a quadratic form $q : \mathbb{H}^{n+1} \to \mathbb{R}$ such that $q(iw) = q(jw) = q(w)$. There are obvious Hermitian and "quaternionic" versions of the theory developed in this paper (for systems of Hermitian or "quaternionic" quadratic inequalities). You simply substitute $\mathbb{R}\mathrm{P}^n$ with $\mathbb{C}\mathrm{P}^n$ or $\mathbb{H}\mathrm{P}^n$, Stiefel–Whitney classes with Chern or Pontryagin classes, differentials d_r with differentials d_{2r-1} or d_{4r-3}, and compute homology with coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}_2$ (see also [1]).

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2. Preliminaries

2.1. **Semialgebraic geometry.** We recall here some useful facts from semialgebraic geometry. We assume the reader is familiar with the fact that semialgebraic *functions* can be triangulated and with Hardt's triviality theorem; for details the reader is referred to [6] or to [7].
If S is a semialgebraic set and $B \subset S$ is a compact semialgebraic set then, following [10], we say that $f : S \to [0, \infty)$ is a *rug function* for B in S if f is proper, continuous, semialgebraic and $f^{-1}(0) = B$. The following proposition can be found in [6] (pag. 229, Proposition 9.4.4).

Proposition 1. *Let $B \subset S$ be compact semialgebraic sets and f be a rug function for B in S. Then there are $\delta > 0$ and a continuous semialgebraic mapping $h : f^{-1}(\delta) \times [0, \delta] \to f^{-1}([0, \delta])$, such that $f(h(x, t)) = t$ for every $(x, t) \in f^{-1}(\delta) \times [0, \delta]$, $h(x, \delta) = x$ for every $x \in f^{-1}(\delta)$, and $h_{|f^{-1}(\delta) \times]0, \delta]}$ is a homeomorphism onto $f^{-1}(]0, \delta])$.*

Proof. By triangulating f we obtain a finite simplicial complex K and a semialgebraic homeomorphism $\phi : |K| \to S$, such that $f \circ \phi$ is affine on every simplex of K and B is union of images of simplices of K. Choose δ so small that for every vertex a of K such that $\phi(a) \notin B$, then $\delta < f(\phi(a))$. Let $x \in f^{-1}(\delta)$, $y = \phi^{-1}(x)$. The point y belongs to a simplex $\sigma = [a_0, \dots, a_d]$ of K. We may assume that $\phi(a_i) \in B$ for $i = 0, \dots, k$, and $\phi(a_i) \notin B$ for $i = k+1, \dots, d$. Let $(\lambda_0, \dots, \lambda_d)$ be the barycentric coordinates of y in σ. Note that since $f \circ \phi$ is affine on σ, then $\delta = f(x) = f(\phi(y)) = \sum_{i=0}^d \lambda_i f(\phi(a_i)) = \sum_{i=k+1}^d \lambda_i f(\phi(a_i))$. Hence, if we set $\alpha = \sum_{i=0}^k \lambda_i$, we have necessarily $0 < \alpha < 1$. For $t \in [0, \delta]$, we define $h(x,t)$ as the image by ϕ of the point of σ with barycentric coordinates $(\mu_0, \dots, \mu_d)$, where

$$\mu_i = \begin{cases} \frac{t\alpha+\delta-t}{\delta\alpha}\lambda_i & \text{for} \quad i = 0, \dots, k; \\ \frac{t}{\delta}\lambda_i & \text{for} \quad i = k+1, \dots, d. \end{cases}$$

Then h has the required properties.

□

Now we prove a result which describes the structure of some semialgebraic neighborhoods of a semialgebraic compact set.

Proposition 2. *Let $B \subset S$ be compact semialgebraic sets. Let f be a rug function for B in S. Then there exists δ_f such that for any $\delta' < \delta_f$ there is a semialgebraic retraction*

$$\pi : f^{-1}([0, \delta']) \to B.$$

Proof. First we show that there exists a semialgebraic retraction for small enough semialgebraic neighborhoods. Let $T_\delta = \alpha^{-1}([0, \delta])$ and choose $\delta_f = \delta$ and $\phi : |K| \to S$ as in Proposition 1. Given $x \in T_\delta$, let $y = \phi^{-1}(x)$. Then y belongs to some simplex $\sigma = [a_0, \dots, a_k]$; let $(\lambda_0, \dots, \lambda_k)$ be its barycentric coordinates with respect to σ. Since $f(x) \le \delta$ then there exist some vertices of σ belonging to $\phi^{-1}(B)$: let $a_0, \dots, a_k$ be these vertices. First notice that $\sum_{i=0}^k \lambda_i \neq 0$: if it were zero, then

$$f(x) = f(\phi(y)) = f(\phi(\sum_{i=k+1}^d \lambda_i a_i)) = \sum_{i=k+1}^d \lambda_i f(\phi(a_i)) > \delta$$

since $f \circ \phi$ is affine; but this contradicts $f(x) \le \delta$.
Now we define $p_\sigma : \phi^{-1}(T_\delta) \cap \sigma \to \phi^{-1}(B)$ by

$$p(x) = p_\sigma(\lambda_0, \dots, \lambda_d) = (\frac{\lambda_0}{\sum_{i=0}^k \lambda_i}, \dots, \frac{\lambda_k}{\sum_{i=0}^k \lambda_i}).$$

Then p_σ is continuous and semialgebraic and its restriction to $\phi^{-1}(B) \cap \sigma$ is the identity map. Defining $p_{\sigma'}$ in the same way as for p_σ for every simplex σ' we notice that since the $p_{\sigma'}$'s agree on the common faces, then they together define a semialgebraic continuous map $p : \phi^{-1}(T_\delta) \to \phi^{-1}(\delta)$.
Now put $\pi = \phi^{-1}_{|T_\delta} \circ p \circ \phi$: then π is a semialgebraic continuous retraction from T_δ to B; given $\delta' < \delta$ simply compose π with the inclusion $T_{\delta'} \subset T_\delta$ to obtain the required retraction.

□

In particular we derive the following corollary.

Corollary 3. *Let S be a semialgebraic set and $f : S \to [0,\infty)$ be a proper, continuous semialgebraic function. Then for $\epsilon > 0$ small enough the inclusions:*

$$\{f = 0\} \hookrightarrow \{f \leq \epsilon\} \quad \textit{and} \quad \{f > \epsilon\} \hookrightarrow \{f \geq \epsilon\} \hookrightarrow \{f > 0\}$$

are homotopy equivalences.

Proof. Let $T = f^{-1}([0,\delta])$ for δ small enough as given by propositions 1 and 2. Consider the function

$$g = \pi|_{f^{-1}(\delta)} : \{f = \delta\} \to \{f = 0\}$$

where π is the retraction defined in the proof of proposition 2. Then propositions 1 and 2 combined together prove that T is a mapping cylinder neighborhood of $\{f = 0\}$ in S, i.e. there is a homeomorphism

$$\psi : T \to M_g,$$

where M_g is the mapping cylinder of g, such that $\psi|_{\{f=\delta\}\cup\{f=0\}}$ is the identity map.
The conclusion follows from the structure of mapping cylinder neighborhoods. □

Finally we state the following corollary of Hardt's triviality.

Proposition 4. *Let A, B be semialgebraic sets and $g : A \to B$ be a semialgebraic, surjective map. Then g admits a semialgebraic section σ, i.e. a map $\sigma : B \to A$ such that $g(\sigma(b)) = b$ for every $b \in B$.*

Proof. By Hardt's triviality theorem there exists a finite partition

$$B = \coprod_{l=1}^{m} B_l,$$

semialgebraic sets F_l and semialgebraic homeomorphisms $\psi_l : B_l \times F_l \to g^{-1}(B_l)$ for $l = 1, \ldots, m$ such that $g(\psi_l(b, y)) = b$ or every $(b, y) \in B_l \times F_l$. For every $l = 1, \ldots, m$ let $a_l \in F_l$ and define

$$\sigma|_{B_l}(b) = \psi_l(b, a_l).$$

□

2.2. **Convexity properties.** We recall here some useful facts related to convex open sets of $\mathbb{R}^k$. We begin with the following; recall that for a given convex function a and $c \in \mathbb{R}$ the set $\{a < c\}$ is convex.

Lemma 5. *Let $a : \mathbb{R}^n \to [0,\infty)$ be a proper convex smooth function and $x_0 \in \mathbb{R}^n$ such that $a(x_0) = 0$, $da_{x_0} \equiv 0$ and the Hessian $He(a)_{x_0}$ of a at x_0 is positive definite. Let also $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then there exists $\overline{\epsilon} > 0$ such that for every $\epsilon < \overline{\epsilon}$*

$$\psi(\{a < \epsilon\}) \quad \textit{is convex.}$$

Proof. Let ϕ be the inverse of ψ, $y_0 = \psi(x_0)$ and $\hat{a} \doteq a \circ \phi$. Then the set $\psi(\{a < \epsilon\})$ equals $\{\hat{a} < \epsilon\}$. Since $da_{x_0} \equiv 0$, then

$$\mathrm{He}(\hat{a})_{y_0} = {}^t J\phi_{y_0} \mathrm{He}(a)_{x_0} J\phi_{y_0} > 0$$

and thus, by continuity of the map $y \mapsto \mathrm{He}(\hat{a})_y$, the function $\hat{a}$ is convex on $B(y_0, \epsilon')$ for sufficiently small ϵ'; hence for every $c > 0$ the set $\{\hat{a}_{|B(y_0,\epsilon')} < c\}$ is convex. Since a is proper, then there exists ϵ such that $\{y : a(\phi(y)) < \epsilon\} \subset B(y_0, \epsilon')$. Thus $\{\hat{a} < \epsilon\} = \{\hat{a}_{|B(y_0,\epsilon')} < \epsilon\}$ is convex. □

Consider a family of functions $a_w : x \mapsto a(x + x_0 - w), w \in W \subset \mathbb{R}^n$ with compact closure, with a satisfying the conditions of the previous lemma. Since $\mathrm{He}(a_w)_x = \mathrm{He}(a)_x$, then the exstimate on $\mathrm{He}(a_w)_w$ can be made uniform on W. In particular taking $a(x) = |x|^2$ we derive the following corollary.

Corollary 6. *Let U be an open subset of $\mathbb{R}^n$ and $\psi : U \to \mathbb{R}^n$ be a diffeomorphism onto its image. For every $x \in U$ there exists $\delta_c(x) > 0$ such that for every $B(y, r) \subset B(x, 3\delta_c(x))$ with $r < \delta_c(x)$*

$$\psi(B(y, r)) \quad \textit{is convex.}$$

Moreover if ψ is semialgebraic, then the function $x \mapsto \delta_c(x)$ can be chosen semialgebraic.

Proof. The first part follows immediately from Lemma 5 and the previous remark. In the case ψ is semialgebraic, then the condition for $\delta_c(x)$ to satisfy the requirements of the previous Corollary is a semialgebraic condition (according to Lemma 5 it is given by semialgebraic inequalities); thus the set $S = \{(x, \delta) \in U \times (0, \infty) : \delta \text{ satisfies the condition of Corollary 6}\}$ is semialgebraic. Consider thus the semialgebraic function $g : S \to U$ given by the restriction of the projection on the first factor. Then the first part of the proof tells that g is surjective; proposition 4 ensures there exists a semialgebraic section $x \mapsto (x, \delta_c(x))$ of g and the function δ_c is thus semialgebraic. □

We define now the tangent space to a convex set; this definition applies also in the case we have a set $\Omega \subset \mathbb{R}^{k+1}$ diffeomorphic to a convex set, using the diffeomorphism to define it.

Definition 7. *Let $K \subset \mathbb{R}^{k+1}$ be a convex set and $y \in K$. We define the tangent space to K at y by:*

$$T_y K = cone(K - y)$$

where $cone(K - y) = \{v \in \mathbb{R}^{k+1} : v = t(x - y) \text{ with } t > 0 \text{ and } x \in K\}$.

All the definitions concerning smooth maps can be extended to the case of convex sets (see [2]). For $\Omega = K^\circ \cap S^k$, with K° a convex cone, and $\omega \in \Omega$ we define:

$$T_\omega \Omega = T_\omega K \cap T_\omega S^k.$$

We will say that a map $K \to M$, where M is a smooth manifold, is a smooth map if it extends to a smooth map on an open neighborhood of K in $\mathbb{R}^{k+1}$; we will say that $f : \Omega \to M$ is smooth if it extends to a smooth map on K. In particular the restriction of a smooth map to Ω is clearly smooth.

2.3. **Space of quadratic forms.** Let V be a vector space; we denote by $\mathcal{Q}(V)$ the space of all quadratic forms on V. Notice that $\mathcal{Q}(V)$ is a vector space of dimension $d(d+1)/2$ where $d = \dim(V)$. We denote by $\mathcal{Q}_0(V)$ the set of degenerate quadratic forms, i.e. $\mathcal{Q}_0(V) = \{q \in \mathcal{Q}(V) : \ker(q) \neq 0\}$; we will write $\mathcal{Q}^+(V)$ for the set of positive definite quadratic forms.
In the case $V = \mathbb{R}^{n+1}$ we simply write $\mathcal{Q}, \mathcal{Q}_0$ and $\mathcal{Q}^+$ for $\mathcal{Q}(\mathbb{R}^{n+1}), \mathcal{Q}_0(\mathbb{R}^{n+1})$ and $\mathcal{Q}^+(\mathbb{R}^{n+1})$.
Suppose now that a scalar product in $\mathbb{R}^{n+1}$ has been fixed. Then we can identify each $q \in \mathcal{Q}$ with a symmetric $(n+1) \times (n+1)$ matrix Q by the rule:

$$q(x) = \langle x, Qx \rangle.$$

Now also the *value* of q at $x \in \mathbb{P}^n$ is defined: let S^n be the unit sphere (w.r.t. the fixed scalar product) in $\mathbb{R}^{n+1}$ and $p : S^n \to \mathbb{P}^n$ be the covering map; then, with a little abuse of notation, if $x = p(v)$ we will write $q(x)$ to abbreviate $q(v)$. The *eigenvalues* of q with respect to g are defined to be those of Q:

$$\lambda_1(q) \geq \cdots \geq \lambda_{n+1}(q).$$

Recall from the Introduction that we have defined

$$\mathcal{D}_j = \{q \in \mathcal{Q} \,:\, \lambda_j(q) \neq \lambda_{j+1}(q)\}.$$

Notice that $\mathcal{Q}^j \backslash \mathcal{Q}^{j+1} \subset \mathcal{D}_j$ for every possible choice of the scalar product in $\mathbb{R}^{n+1}$. On the space $\mathcal{D}_j$ is naturally defined the vector bundle:

$$\begin{tikzcd} \mathbb{R}^j \arrow[r] & \mathcal{L}_j^+ \arrow[d] \\ & \mathcal{D}_j \end{tikzcd}$$

whose fiber over the point $q \in \mathcal{D}_j$ is $(\mathcal{L}_j^+)_q = \mathrm{span}\{x \in \mathbb{R}^{n+1} \,:\, Qx = \lambda_i x, 1 \leq i \leq j\}$ and whose vector bundle structure is given by its inclusion in $\mathcal{D}_j \times \mathbb{R}^{n+1}$.
Similarly the vector bundle $\mathbb{R}^{n-j+1} \hookrightarrow \mathcal{L}_j^- \to \mathcal{D}_j$ has fiber over the point $q \in \mathcal{D}_j$ the vector space $(\mathcal{L}_j^-)_q = \mathrm{span}\{x \in \mathbb{R}^{n+1} \,:\, Qx = \lambda_i x, j+1 \leq i \leq n+1\}$ and vector bundle structure given by its inclusion in $\mathcal{D}_j \times \mathbb{R}^{n+1}$. Notice that

$$\mathcal{L}_j^+ \oplus \mathcal{L}_j^- = \mathcal{D}_j \times \mathbb{R}^{n+1}$$

and thus Whitney product formula holds for their total Stiefel-Whitney classes: $w(\mathcal{L}_j^+) \smile w(\mathcal{L}_j^-) = 1$. In particular:

$$w_1(\mathcal{L}_j^+) = w_1(\mathcal{L}_j^-).$$

We recall the following fact (see [12]).

Lemma 8. *Let $\mathbb{R}^{k+1} \to E \to X$ be a vector bundle and $\mathbb{P}^k \to P(E) \xrightarrow{\pi} X$ be its projectivization. Suppose $f : P(E) \to \mathbb{P}^n$ is a map which is a linear embedding on each fibre, let $x \in H^1(\mathbb{P}^n; \mathbb{Z}_2)$ be the generator and set $y = f^*x \in H^1(P(E); \mathbb{Z}_2)$. Then*

$$y^k = \sum_{i=1}^{k} \pi^* w_1(E) \smile y^{k-i}$$

Proof. Since f is a linear embedding on the fibres, then f^*x generates the cohomology of each fibre and by Leray-Hirsch theorem $H^*(P(E); \mathbb{Z}_2)$ is a $H^*(X; \mathbb{Z}_2)$-module with generators $\{1, y, \ldots, y^{k-1}\}$. The formula is the definition of Stiefel-Whitney classes. □

In the sequel we will need for $q \in \mathcal{D}_j$ the projective spaces:

$$P_j^+(q) \doteq \mathbb{P}(\mathcal{L}_j^+)_q \quad \text{and} \quad P_j^-(q) \doteq \mathbb{P}(\mathcal{L}_j^-)_q.$$

For a given $q \in \mathcal{Q}$ with $\mathrm{i}^-(q) = i$ (which implies $q \in D_{n+1-i}$) we will use the simplified notation

$$P^+(q) \doteq P_{n+1-i}^+(q) \quad \text{and} \quad P^-(q) \doteq P_{n+1-i}^-(q).$$

(even if $q \in \mathcal{D}_{n+1-i}$ for every metric still there is dependence on the metric for these spaces, but we omit it for brevity of notations; the reader should pay attention). Notice that $q|_{P^-(q)} < 0$ whereas $q|_{P^+(q)} \geq 0$, i.e. $P^+(q)$ contains also $\mathbb{P}(\ker q)$. The following picture may help the reader:

$$\underbrace{\lambda_1(q) \geq \cdots \geq \lambda_{n+1-\mathrm{i}^-(q)}(q)}_{P^+(q)} \geq 0 > \underbrace{\lambda_{n+2-\mathrm{i}^-(q)}(q) \geq \cdots \geq \lambda_{n+1}(q)}_{P^-(q)}$$

We recall the following result describing the local topology of the space of quadratic forms.

Proposition 9. *Let $q_0 \in \mathcal{Q}$ be a quadratic map and let V be its kernel. Then there exists a neighborhood U_{q_0} of q_0 and a smooth semialgebraic map $\phi : U_{q_0} \to \mathcal{Q}(V)$ such that: 1) $\phi(q_0) = 0$; 2) $\mathrm{i}^-(q) = \mathrm{i}^-(q_0) + \mathrm{i}^-(\phi(q))$; 3) $\dim\ker(q) = \dim\ker(\phi(q))$; 4) for every $p \in \mathcal{Q}$ we have $d\phi_{q_0}(p) = p|_V$.*

Proof. Let γ be a closed semialgebraic contour in the complex plane separating the non zero eigenvalues of q_0 from the origin. For any q such that the corresponding operator does not have eigenvalues on γ we define π_q to be the orthogonal projection onto the invariant subspace $V_\gamma(q)$ of the operator Q corresponding to the eigenvalues which lie inside the contour - formally speaking we have to consider the semialgebraic set S of the pairs (q, L) where L is a linear map from $\mathbb{R}^{n+1}$ to $V_\gamma(q)$ - and the correspondence $q \mapsto \pi_q$ is semialgebraic. Notice that in particular $\pi_{q_0}|_V = \mathrm{id}_V$. Then the correspondence $q \mapsto \Phi(q) = q \circ \pi_q|_V$ is semialgebraic and satisfyies the required properties. For details the reader is referred to [2]. □

2.4. **Nondegeneracy properties.** Consider the set $K = \{(x, q) \in \mathbb{R}^{n+1} \times \mathcal{Q} \,|\, x \in \ker q\}$ and the map $h : K \to \mathcal{Q}$ which is the restriction of the projection on the second factor. Let $\mathcal{Q}_0$ be the set of singular forms and

$$\mathcal{Q}_0 = \coprod Z_j$$

be a Nash stratification (i.e. smooth and semialgebraic) such that h trivializes over each Z_j (see [7]).
Given a quadratic form $q \in \mathcal{Q}$ we may abuse a little of notation and write $q(\cdot, \cdot)$ for the bilinear form obtained by polarizing q; no confusion will arise by distinguish the two from the number of their arguments.

Lemma 10. *Let r be a singular form and suppose $r \in Z_j$ for some stratum of $\mathcal{Q}_0$ as above. Then for every $q \in T_r Z_j$ and $x_0 \in ker(r)$ we have $q(x_0, x_0) = 0$.*

Proof. Let $r : I \to Z_j$ be a smooth curve such that $r(0) = r$ and $\dot{r}(0) = q$. By the triviality of p over Z_j it follows that there exists $x : I \to \mathbb{R}^{n+1}$ such that $x(0) = x_0$ and $x(t) \in \ker(r(t))$ for every $t \in I$. This implies $r(t)(x(t), x(t)) \equiv 0$ and deriving we get

$$0 = \dot{r}(0)(x(0), x(0)) + 2r(0)(x(0), \dot{x}(0)) = q(x_0, x_0).$$

□

We recall that for $K \subset \mathbb{R}^{k+1}$ we defined $\Omega = K^\circ \cap S^k$; it is diffeomorphic to a convex set and the notion of tangent space and of smooth map for it were introduced before.

Definition 11. *Let $f : \Omega \to \mathcal{Q}$ be a smooth map. We say that f is degenerate at $\omega_0 \in \Omega$ if there exists $x \in \ker(f(\omega_0))\backslash\{0\}$ such that for every $v \in T_{\omega_0}\Omega$ we have $(df_{\omega_0}v)(x,x) \leq 0$; in the contrary case we say that f is nondegenerate at ω_0. We say that f is nondegenerate if it is nondegenerate at each point $\omega \in \Omega$.*

Lemma 12. *Let $\Omega = \coprod V_i$ be a finite partiton with each V_i Nash and $f : \Omega \to \mathcal{Q}$ be a semialgebraic smooth map and $\mathcal{Q}_0 = \coprod Z_j$ as above. Suppose that for every V_i the map $f|_{V_i}$ is transversal to all strata of $\mathcal{Q}_0$. Then f is nondegenerate.*

Proof. Let $\omega_0 \in \Omega$ and $x \in \ker(f(\omega_0))\backslash\{0\}$; we must prove that there exists $v \in T_{\omega_0}\Omega$ such that $(df_{\omega_0}v)(x,x) > 0$. Let V_i such that $\omega_0 \in V_i$. Then $T_{\omega_0}V_i \subset T_{\omega_0}\Omega$; suppose $f(\omega_0) \in Z_j$. Since $f_{|V_i}$ is transversal to Z_j, then

$$\mathrm{im}(df_{|V_i})_{\omega_0} + T_{f(\omega_0)}Z_j = \mathcal{Q}.$$

Thus let $q^+ \in \mathcal{Q}$ be a positive definite form, $v \in T_{\omega_0}V_i$ and $\dot{r} \in T_{f(\omega_0)}Z_j$ such that

$$df_{\omega_0}v + \dot{r} = q^+.$$

Since $x \in \ker(f(\omega_0))\backslash\{0\}$, then the previous Fact implies $\dot{r}(x,x) = 0$, and plugging in the previous equation we get

$$(df_{\omega_0}v)(x,x) = (df_{\omega_0}v)(x,x) + \dot{r}(x,x) = q^+(x,x) > 0.$$

□

Lemma 13. *Let $f : \Omega \to \mathcal{Q}$ be a semialgebraic smooth map. Then there exists a definite positive form $q_0 \in \mathcal{Q}$ such that for every $\epsilon > 0$ sufficiently small the map $f_\epsilon : \Omega \to \mathcal{Q}$ defined by*

$$\omega \mapsto f(\omega) - \epsilon q_0$$

is nondegenerate.

Proof. Let $\Omega = \coprod V_i$ and $\mathcal{Q}_0 = \coprod Z_j$ be as above. For every V_i consider the map $F_i : V_i \times \mathcal{Q}^+ \to \mathcal{Q}$ defined by

$$(\omega, q_0) \mapsto f(\omega) - q_0.$$

Since $\mathcal{Q}^+$ is open in $\mathcal{Q}$, then F_i is a submersion and $F_i^{-1}(\mathcal{Q}_0)$ is Nash-stratified by $\coprod F_i^{-1}(Z_j)$. Then for $q_0 \in \mathcal{Q}^+$ the evaluation map $\omega \mapsto f(\omega) - q_0$ is transversal to all strata of $\mathcal{Q}_0$ if and only if q_0 is a regular value for the restriction of the second factor projection $\pi_i : V_i \times \mathcal{Q}^+ \to \mathcal{Q}^+$ to each stratum of $F_i^{-1}(\mathcal{Q}_0) = \coprod F_i^{-1}(Z_j)$. Thus let $\pi_{ij} = (\pi_i)_{|F_i^{-1}(Z_j)} : F_i^{-1}(Z_j) \to \mathcal{Q}^+$; since all datas are smooth semialgebraic, then by semialgebraic Sard's Lemma, the set $\Sigma_{ij} = \{\hat{q} \in \mathcal{Q}^+ \ : \ \hat{q} \text{ is a critical value of } \pi_{ij}\}$ is a semialgebraic subset of $\mathcal{Q}^+$ of dimension $\dim(\Sigma_{ij}) < \dim(\mathcal{Q}^+)$. Hence $\Sigma = \cup_{i,j}\Sigma_{ij}$ also is a semialgebraic subset of $\mathcal{Q}^+$ of dimension $\dim(\Sigma) < \dim(\mathcal{Q}^+)$ and for every $q_0 \in \mathcal{Q}^+\backslash\Sigma$ and for every i,j the restriction of $\omega \mapsto f(\omega) - q_0$ to V_i is transversal to Z_j. Thus by the previous Lemma $f - q_0$ is nondegenerate. Since Σ is semialgebraic of codimension at least one, then there exists $q_0 \in \mathcal{Q}^+\backslash\Sigma$ such that $\{tq_0\}_{t>0}$ intersects Σ in a finite number of points, i.e. for every $\epsilon > 0$ sufficiently small $\epsilon q_0 \in \mathcal{Q}^+\backslash\Sigma$. The conclusion follows. □

Let $f : \Omega \to \mathcal{Q}$ be a smooth map. We define, for every $V \subset \Omega$ the set

$$B_f(V) = \{(\omega, x) \in V \times \mathbb{P}^n \ : \ f(\omega)(x) > 0\}.$$

Lemma 14. *Let $f : \Omega \to \mathcal{Q}$ be a smooth* nondegenerate *map. Then there exists $\delta_1 : \Omega \to (0, +\infty)$ such that for every $\omega \in \Omega$, for every $V_1 \subset V_2$ closed convex neighborhoods of ω with diam$(V_2) < \delta_1(\omega)$ and for every $\eta \in V_1$ such that* $\mathrm{i}^-(f(\eta)) = \mathrm{i}^-(f(\omega))$ *and* $\det(f(\eta)) \neq 0$ *the inclusions*

$$(\eta, P^+(f(\eta))) \hookrightarrow B_f(V_1) \hookrightarrow B_f(V_2)$$

are homotopy equivalences.
Moreover in the case f is semialgebraic, then the function δ_1 can be chosen to be semialgebraic (but in general not continuous).

Proof. The existence of δ_1 is the statement of Lemma 8 of [2]. The fact that δ_1 can be chosen to be semialgebraic if f is semialgebraic follows directly from the proof of Lemma 7 of [2]: in fact the set S of pairs $(\omega, \delta) \in \Omega \times (0, \infty)$ such that δ_1 satisfies the requirement of the Lemma is semialgebraic (it is given by a formula with semialgebraic inequalities). Lemma 8 of [2] tells that the projection on the first factor $g|_S : S \to \Omega$ is surjective and, arguing as in Lemma 6, Proposition 4 gives the semialgebraicity. □

2.5. **Negativity properties.** Let now $f : \Omega \to \mathcal{Q}$ and $\omega \in \Omega$; let $M(\omega) < 0$ be such that

$$\lambda_{n+2-\mathrm{i}^-(f(\omega))}(f(\omega)) < M(\omega)$$

(notice that by definition $\lambda_{n+2-\mathrm{i}^-(\omega)}(f(\omega))$ is the biggest negative eigenvalue of $f(\omega)$). Then by continuity there exists $\delta_2''(\omega)$ such that for every neighborhood V of ω with $\mathrm{diam}(V) < \delta_2''(\omega)$ and for every $\eta \in V$

$$\lambda_{n+2-\mathrm{i}^-(f(\omega))}(f(\eta)) < M(\omega).$$

Thus for every neighborhood U of ω with $\mathrm{diam}(U) < \delta_2''(\omega)$ we define:

$$P^-(\omega, U) = \{x \in \mathbb{P}^n \,:\, \text{there exists } \eta \in U \text{ s.t. } x \in P^-_{n+1-\mathrm{i}^-(f(\omega))}(f(\eta))\}.$$

For $x, y \in \Omega$ we denote by $\mathrm{dist}(x, y)$ their euclidean distance and for $r > 0$ we set $B(x, r) = \{\omega \in \Omega \,:\, \mathrm{dist}(x, \omega) \leq r\}$. We claim the following.

Lemma 15. *For every $\omega \in \Omega$ there exists $0 < \delta_2'(\omega) < \delta_2''(\omega)$ such that for every neighborhood of ω with diam$(V) < \delta_2'(\omega)$*

$$\mathrm{Cl}(P^-(\omega, V)) \subseteq \mathbb{P}^n \backslash \{f(\omega)(x) \geq 0\}.$$

Proof. By absurd suppose for every $k \in \mathbb{N}$ the two sets $\mathrm{Cl}(P^-(\omega, B(\omega, 1/k)))$ and $\{f(\omega)(x) \geq 0\}$ intersect. Then for every $k \in \mathbb{N}$ there exists a sequence $x_k^l \to x_k$ such that for every x_k^l there exists $\omega_k^l \in B(\omega, 1/k)$ such that $x_k^l \in P^-_{n+1-\mathrm{i}^-(\omega)}(f(\omega_k^l))$ and $f(\omega)(x_k) \geq 0$.
Then it follows that $f(\omega_k^l)(x_k^l) < M(\omega)$ and, by extracting convergent subsequences, that

$$0 \leq \lim_{k\to\infty} f(\omega)(x_k) = \lim_{k\to\infty} f(\omega_k)(x_k) \leq M(\omega)$$

which is absurd since $M(\omega) < 0$ by definition. □

Lemma 16. *For every $\omega \in \Omega$ there exists $0 < \delta_2(\omega) < \delta_2''(\omega)$ such that for every neighborhood V of ω with diam$(V) < \delta_2(\omega)$ the following holds:*

$$\mathrm{Cl}(P^-(\omega, V)) \subset \mathbb{P}^n \backslash \beta_r(B_f(V)).$$

Moreover in the case f is semialgebraic, then $\omega \mapsto \delta_2(\omega)$ can be chosen semialgebraic.

Proof. Let W be a neighborhood of ω with $\mathrm{diam}(W) < \delta_2'(\omega)$. Then the two compact sets $\mathrm{Cl}(P^-(\omega, W))$ and $\{f(\omega)(x) \geq 0\}$ do not intersect by the previous Lemma. Consider the continuous function $a : \mathrm{Cl}(W) \times \mathbb{P}^n \to \mathbb{R}$ defined by $a(\eta, x) = f(\eta)(x)$ and a neighborhood U of $\{f(\omega)(x) \geq 0\}$ in $\mathbb{P}^n$ disjoint form $\mathrm{Cl}(P^-(\omega, W))$. Then $\beta_r^{-1}(U) \cap \{a \geq 0\}$ is an open neighborhood of $\{\omega\} \times \{f(\omega)(x) \geq 0\}$ in $\{a \geq 0\}$. Consider now $b : \{a \geq 0\} \to \mathbb{R}$ defined by $(\eta, x) \mapsto d(\eta, \omega)$. Then, since $\{a \geq 0\}$ is compact, the family $\{b^{-1}[0, \delta)\}_{\delta > 0}$ is a fundamental system of neighborhoods of $b^{-1}(0) = \{\omega\} \times \{f(\omega)(x) \geq 0\}$ in $\{a \geq 0\}$. Thus there exists $\overline{\delta}$ such that $b^{-1}[0, \overline{\delta}) \subset \beta_r^{-1}(U) \cap \{a \geq 0\}$. Hence any $\delta_2(\omega)$ such that $B(\omega, 3\delta_2(\omega)) \subset B(\omega, \overline{\delta}) \cap W$ satisfies the requirement, since every neighborhood V of ω with $\mathrm{diam}(V) < \delta_2(\omega)$ is contained in $B(\omega, 3\delta_2(\omega))$ and

$$\begin{aligned} \mathrm{Cl}(P^-(\omega, B(\omega, 3\delta_2(\omega)))) &\subset \mathrm{Cl}(P^-(\omega, W)) \\ &\subset \mathbb{P}^n \backslash \beta_r(\{a \geq 0\}) \subset \mathbb{P}^n \backslash \beta_r(B_f(B(\omega, 3\delta_2(\omega)))). \end{aligned}$$

Suppose now that f is semialgebraic. Then the set $S = \{(\omega, \delta) \in \Omega \times (0, \infty) \,|\, \forall r < 2\delta,\ \forall x \in \mathrm{Cl}(P^-(\omega, B(\omega, r)))\ :\ x \in \mathbb{P}^n \backslash \beta_r(B_f(B(\omega, r)))\}$ is semialgebraic too. Let $g : S \to \Omega$ be the restriction of the projection on the first factor; then g is semialgebraic and by the previous part of the proof it is surjective (for every $\omega \in \Omega$ there exists a δ satisfying the query). Proposition 4 implies that g has a semialgebraic section $\omega \mapsto (\omega, \delta_2(\omega))$ and δ_2 is the required semialgebraic function. □

2.6. **Spectral sequences.** Here we fix some notations and make some remarks concerning spectral sequences which will be useful in the sequel. We always make use of $\mathbb{Z}_2$ coefficients, in order to avoid sign problems; the following results still hold for $\mathbb{Z}$ coefficients, but sign must be put appropriately. All the introductory material we present here is covered (up to some small modifications) in [8], to which the reader is referred for more precise details.
We begin with the following.

Lemma 17. *Let (C_*, ∂_*) be an acyclic free chain complex and (D_*, ∂_*^D) be an acyclic subcomplex. Then there exists a chain homotopy*

$$K_* : C_* \to C_{*+1}$$

such that $\partial_{*+1} K_* + K_{*-1} \partial_* = I_*$ and $K_*(D_*) \subset (D_{*+1})$.

Proof. By taking a right inverse s_{q-1}^D of ∂_q^D, which exists since D_{q-1} and hence Z_{q-1}^D are free, a chain contraction K_q^D for D is defined by: $K_q^D = s_q^D(I_q - s_{q-1}^D \partial_q^D)$. Since Z_q is free, then it is possible to extend s_{q-1}^D to a right inverse s_{q-1} of ∂_q:

$$s_{q-1} : Z_{q-1} = B_{q-1} \to C_q.$$

Then by setting

$$K_q = s_q(I_q - s_{q-1} \partial_q)$$

we obtain a chain contraction for the complex (C_*, ∂_*) which restricts to a chain contraction for the subcomplex (D_*, ∂_*^D). □

Let now X be a topological space and Y be a subspace. Consider an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ for X; we assume A to be ordered. For every $\alpha_0, \ldots, \alpha_p \in A$ we define $V_{\alpha_0 \cdots \alpha_p}$ to be $V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}$ (sometimes we will use the shortened notations

$\overline{\alpha}$ for $(\alpha_0, \ldots, \alpha_p)$ and $V_{\overline{\alpha}}$ for $V_{\alpha_0 \cdots \alpha_p}$). The Mayer-Vietoris bicomplex $E_0^{*,*}(Y)$ for the pair (X, Y) relative to the cover $\mathcal{U}$ is defined by

$$E_0^{p,q}(Y,\mathcal{U}) = \check{C}^p(\mathcal{U}, \mathcal{U} \cap Y; C^q) = \prod_{\alpha_0 < \cdots < \alpha_p} C^q(V_{\alpha_0 \cdots \alpha_p}, V_{\alpha_0 \cdots \alpha_p} \cap Y).$$

This bicomplex is endowed with two differentials: $d : E_0^{p,q}(Y,\mathcal{U}) \to E_0^{p,q+1}(Y,\mathcal{U})$ and $\delta : E_0^{p,q}(Y,\mathcal{U}) \to E_0^{p+1,q}(Y,\mathcal{U})$ defined for $\eta = (\eta_{\alpha_0 \cdots \alpha_p}) \in E_0^{p,q}(Y,\mathcal{U})$ by:

$$(d\eta)_{\alpha_0 \cdots \alpha_p} = d\eta_{\alpha_0 \cdots \alpha_p} \quad \text{and} \quad (\delta\eta)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p} \eta_{\alpha_0 \cdots \check{\alpha}_i \cdots \alpha_p}|_{V_{\alpha_0 \cdots \alpha_p}}.$$

By the Mayer-Vietoris principle, each row of the augmented chain complex of $E_0^{*,*}(Y,\mathcal{U})$ is exact, i.e. for each $q \geq 0$ the chain complex

$$0 \to C^q_{\mathcal{U}}(X, Y) \to \check{C}^0(\mathcal{U}, \mathcal{U} \cap Y, C^q) \to \cdots$$

is acyclic - we recall that $(C^*_{\mathcal{U}}(X, Y), d)$ is defined to be the complex of $\mathcal{U}$-small singular cochains and that the following isomorphism holds:

$$H_d(C^*_{\mathcal{U}}(X, Y)) \simeq H^*(X, Y).$$

From this it follows that

$$H^*(X, Y) \simeq H^*_D(E_0^{*,*}(Y,\mathcal{U}))$$

where $H^*_D(E_0^{*,*}(Y,\mathcal{U}))$ is the cohomology of the complex $E_0^{*,*}(Y,\mathcal{U})$ with differential $D = d + \delta$. We also recall that

$$r^* : C^*_{\mathcal{U}}(X, Y) \to \check{C}^*(\mathcal{U}, \mathcal{U} \cap Y, C^*)$$

induces isomorphisms on cohomologies; if we take a chain contraction K for the Mayer-Vietoris rows of the pair (X, Y), then we can define a homotopy inverse f to r^* by the following procedure. If $c = \sum_{i=0}^{n} c_i$ and $Dc = \sum_{i=0}^{n+1} b_i$ then we set

$$f(c) = \sum_{i=0}^{n} (dK)^i c_i + \sum_{i=0}^{n+1} K(dK)^{i-1} b_i.$$

We define now $E_1^{*,*}(Y,\mathcal{U}) = H_d(E_0(Y,\mathcal{U}))^{*,*}$. The bicomplex $E_1^{*,*}(Y,\mathcal{U})$ is naturally endowed with a differential $d_1(\mathcal{U})$ defined in the following way: let $\eta \in E_0^{p,q}(Y,\mathcal{U})$ be such that $d\eta = 0$, i.e. η defines a class denoted by $[\eta]_1$ in $E_1^{p,q}(Y,\mathcal{U})$; then $d_1(\mathcal{U})[\eta_1]$ is defined to be $[\delta\eta]_1 \in E_1^{p+1,q}(Y,\mathcal{U})$.
In general we say that an element $\eta \in E_0^{p,q}(Y,\mathcal{U})$ such that $d\eta = 0$ can be extended to a *zig-zag* of lenght r if there exist $\eta_i \in E_0^{p+i,q-i}(Y,\mathcal{U})$ for for $i = 0, \ldots, r-1$ such that $\eta_0 = \eta$ and $\delta\eta_i = d\eta_{i+1}$ for every $i = 0, \ldots, r-2$ (notice that this is a necessary condition for η to define a class in $H_D(E_0(Y,\mathcal{U}))$).
Thus $\eta \in E_0^{p,q}(Y,\mathcal{U})$ can be extended to a zig-zag of lenght 1 if and only if $d\eta = 0$, i.e. η defines a class $[\eta]_1 \in E_1^{p,q}(Y,\mathcal{U})$. We define inductively $E_r(Y,\mathcal{U})$ from $E_{r-1}(Y,\mathcal{U})$ in the following way:

$$E_r(Y,\mathcal{U}) = H_{d_{r-1}(\mathcal{U})}(E_{r-1}(Y,\mathcal{U}))$$

and if $\eta \in E_0^{p,q}(Y,\mathcal{U})$ is such that its class is defined in $E_r^{p,q}(Y,\mathcal{U})$ we denote it by $[\eta]_r$; moreover we define the differential $d_r(\mathcal{U}) : E_r(Y,\mathcal{U}) \to E_r(Y,\mathcal{U})$ by the formula:

$$d_r(\mathcal{U})[\eta]_r = [\delta\eta_{r-1}]_r,$$

where $\eta_0, \ldots, \eta_{r-1}$ is a zig-zag of lenght r extending η (the fact that this zig-zag exists is ensured by the fact that the class $[\eta]_r$ is defined and similarly for the fact that $[\delta\eta_{r-1}]_r$ is defined). If $\eta \in E_0^{p,q}(Y,\mathcal{U})$ defines a class $[\eta]_r \in E_r^{p,q}(Y,\mathcal{U})$, then it is said to *survive to* $E_r(Y,\mathcal{U})$. If this inductive procedure stabilize, i.e. if we have $E_r(Y,\mathcal{U}) = E_{r+l}(Y,\mathcal{U})$ for some $r \geq 0$ and for every $l \geq 0$, then we denote by $E_\infty(Y,\mathcal{U})$ this stable value. It is a remarkable fact that in this case, setting $E_r^*(Y,\mathcal{U}) = \oplus_{p+q=*}E_r^{p,q}(Y,\mathcal{U})$, we have for every $l \in \mathbb{Z}$:

$$E_\infty^l(Y,\mathcal{U}) \simeq H_D^l(E_0^{*,*}(Y,\mathcal{U})) \simeq H^l(X,Y)$$

but this isomorphisms are not canonical, i.e in general they only tell that the dimensions of the previous vector spaces coincide.
The sequence of vector spaces with differentials $(E_r(Y,\mathcal{U}), d_r(\mathcal{U}))_{r\geq 0}$ is called the *Mayer-Vietoris spectral sequence associated to* $\mathcal{U}$ and the fact the previous isomorphism holds translates the sentence that *the spectral sequence converges to* $H_D^*(E_0^{*,*}(Y,\mathcal{U}))$.
We recall also that the bicomplex $E_0^{*,*}(Y,\mathcal{U})$ is endowed with a $\mathbb{Z}_2$-bilinear product $E_0^{p,q}(Y,\mathcal{U}) \times E_0^{r,s}(Y,\mathcal{U}) \to E_0^{p+r,q+s}(Y,\mathcal{U})$ defined for $\eta \in E_0^{p,q}(Y,\mathcal{U}), \psi \in E_0^{r,s}(Y,\mathcal{U})$ by:

$$(\eta \cdot \psi)_{\alpha_0\cdots\alpha_{p+r}} = \eta_{\alpha_0\cdots\alpha_p}|_{V_{\alpha_0\cdots\alpha_{p+r}}} \smile \psi_{\alpha_p\cdots\alpha_{p+r}}|_{V_{\alpha_0\cdots\alpha_{p+r}}},$$

where on the right hand side we perform the usual cup product. The differentials D, d, δ are derivations with respect to this product (we are using $\mathbb{Z}_2$ coefficients and no signs are appearing) and each $E_r(Y,\mathcal{U})$ inherits a product structure from $E_{r-1}(Y,\mathcal{U})$; it is worth noticing that the product structure of $E_\infty^*(Y,\mathcal{U})$ is different from that of $H_D^*(E_0^{*,*}(Y,\mathcal{U}))$. In the case we have a continuous map $f : X \to \Omega$ and an open cover $\mathcal{W}$ of Ω we have that $f^{-1}\mathcal{W}$ is an open cover of X. Setting $\mathcal{U} = f^{-1}\mathcal{W}$ in the previous constuction, the corresponding spectral sequence is named the *relative Leray's spectral sequence of* f *with respect to the cover* $\mathcal{W}$ (the case $Y = \emptyset$ correspond to the usual Leray's construction as presented in [8].)
If we take the direct limit over all the open covers of Ω (with the natural restriction homomorphisms) we get what is called the *(relative) Leray's spectral sequence of the map* f:

$$(E_r(Y), d_r) = \varinjlim_{\mathcal{U}=f^{-1}\mathcal{W}} \{(E_r(Y,\mathcal{U}), d_r(\mathcal{U}))\}$$

The following result can be stated in much more generality (see [11]), but for our purpose the following version is sufficient.

Theorem 18. *Let* $Y \subset X$ *and* Ω *be semialgebraic sets and* $f : X \to \Omega$ *be a continuous, semialgebraic map. Then the Leray's spectral sequence of* f *converges to* $H^*(X,Y)$ *and the following holds:*

$$E_2^{p,q}(Y) \simeq \check{H}^p(\Omega, \mathcal{F}^q)$$

where $\mathcal{F}^q$ *is the sheaf generated by the presheaf* $V \mapsto H^q(f^{-1}(V), f^{-1}(V) \cap Y)$.

If we let $Z \subset Y$ be a subspace, then $E_0^{*,*}(Y)$ is naturally included in the Mayer-Vietoris bicomplex $E_0^{*,*}(Z)$ for the pair (X,Z) relative to the cover $\mathcal{U}$ (here we omit to write the $\mathcal{U}$ to avoid heavy notations):

$$i_0 : E_0^{*,*}(Y) \hookrightarrow E_0^{*,*}(Z).$$

Since i_0 obviously commutes with the total differentials, then it induces a morphism of spectral sequence, and thus a map

$$i_0^* : H_D^*(E_0(Y)) \to H_D^*(E_0(Z)).$$

At the same time the inclusion $j : (X, Z) \hookrightarrow (X, Y)$ induces a map

$$j^* : H^*(X, Y) \to H^*(X, Z).$$

With the previous notations we prove the following lemma.

Lemma 19. *There are isomorphisms* $f_Y^* : H_D^*(E_0(Y)) \to H^*(X, Y)$ *and* $f_Z^* : H_D^*(E_0(Z)) \to H^*(X, Z)$ *such that the following diagram is commutative:*

$$\begin{tikzcd} H_D^*(E_0(Y)) \arrow[r, "i_0^*"] \arrow[d, "f_Y^*"'] & H_D^*(E_0(Z)) \arrow[d, "f_Z^*"] \\ H^*(X,Y) \arrow[r, "j^*"] & H^*(X,Z) \end{tikzcd}$$

Proof. The augmented Mayer-Vietoris complex for the pair (X, Y) relative to $\mathcal{U}$ is a subcomplex of the augmented Mayer-Vietoris complex for the pair (X, Z) relative to $\mathcal{U}$. Thus by Lemma 17 for every $q \geq 0$ there exists a chain contraction K_Z for the complex

$$0 \to C_{\mathcal{U}}^q(X, Z) \to \check{C}^0(\mathcal{U}, \mathcal{U} \cap Z, C^q) \to \cdots$$

which restricts to a chain contraction K_Y for the complex

$$0 \to C_{\mathcal{U}}^q(X, Y) \to \check{C}^0(\mathcal{U}, \mathcal{U} \cap Y, C^q) \to \cdots$$

We define f_Y and f_Z with the above construction and we take f_Y^* and f_Z^* to be the induced maps in cohomology. Then f_Z restricted to $E_0^{*,*}(Y)$ coincides with f_Y and since j^* is induced by the inclusion $j^\natural : C_{\mathcal{U}}^q(X, Y) \to C_{\mathcal{U}}^q(X, Z)$, then the conclusion follows. □

Remark 1. Notice that $i_0 : E_0^{*,*}(Y) \to E_0^{*,*}(Z)$ induces maps of spectral sequences respecting the bigradings $(i_r)_{a,b} : E_r^{a,b}(Y) \to E_r^{a,b}(Z)$ and thus also a map $i_\infty : E_\infty(Y) \to E_\infty(Z)$. Even tough $E_\infty(Y) \simeq H^*(X, Y)$ and $E_\infty(Z) \simeq H^*(X, Z)$, in general i_∞ does not equal j_* (neither their ranks do); the same considerations hold for the more general case of a map of pairs $f : (X, Y) \to (X', Y')$.

We recall also the following fact. Given a first quadrant bicomplex $E_0^{*,*}$ with total differential $D = d + \delta$ and associated convergent spectral sequence $(E_r, d_r)_{r \geq 0}$, then

$$E_\infty^* \simeq H_D^*(E_0)$$

and there is a *canonical* homomorphism

$$p_E : H_D^*(E_0) \to E_\infty^{0,*}$$

constructed as follows. Let $[\psi]_D \in H_D^k(E_0)$; then there exists $\psi_i \in E_0^{i,k-i}$ for $i = 0, \ldots, k$ such that $D(\psi_0 + \cdots + \psi_k) = 0$ and

$$[\psi]_D = [\psi_0 + \cdots + \psi_k]_D.$$

By definition of the differentials $d_r, r \geq 0$, the element ψ_0 survives to E_∞. We check that the correspondence

$$p_E : [\psi]_D \mapsto [\psi_0]_\infty$$

is well defined: since $\psi_0 \in E_0^{0,k}$ and $E_0^{i,j} = 0$ for $i < 0$, then $[\psi_0]_\infty = [\psi_0']_\infty$ if and only if ψ_0 and ψ_0' survive to E_∞ and $[\psi_0]_1 = [\psi_0']_1$; if $\psi = \psi' + D\phi$, then $\psi_0 = \psi_0' + d\phi_0$ and thus $[\psi_0]_1 = [\psi_0']_1$.

2.7. **Construction of regular covers.** The spectral sequence we are going to introduce will be a kind of Leray's spectral sequence, hence defined using direct limits over open covers. The aim of this section is to detect a family of covers of Ω, cofinal in the family of all covers, for which the direct limit map on spectral sequences will be an isomorphism and such that they will be pratical for computations.

Lemma 20. *Let $f : \Omega \to \mathcal{Q}$ be a smooth map transversal to all strata of $\mathcal{Q}_0 = \coprod Z_j$. For every $\omega \in \Omega$ let $U_{f(\omega)}$ and $\phi : U_{f(\omega)} \to \mathcal{Q}(ker(f(\omega))$ be defined by setting $q_0 = f(\omega)$ in proposition 9. Then there exists $\delta_3'(\omega) > 0$ and $\psi : B(\omega, \delta_3'(\omega)) \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$, where $l + dim(\mathcal{Q}(ker(f(\omega))) = dim(\Omega)$, such that ψ is a diffeomorphism onto its image and the following diagram is commutative:*

$$\begin{tikzcd}
B(\omega, \delta_3'(\omega)) \arrow[rr, "\psi"] \arrow[dr, "\phi \circ f"'] & & \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l \arrow[dl, "p_1"] \\
& \mathcal{Q}(\ker f(\omega)) &
\end{tikzcd}$$

Moreover if f is semialgebraic then $\omega \mapsto \delta_3'(\omega)$ can be chosen to be semialgebraic.

Proof. If $\det(f(\omega)) \neq 0$ then let $\delta_3'(\omega) > 0$ be such that $f(B(\omega, \delta_3'(\omega))) \cap \mathcal{Q}_0 = \emptyset$; in the contrary case let $f(\omega) \in Z_j$ for some j. Consider $\phi : U_{f(\omega)} \to \mathcal{Q}(\ker f(\omega))$ the map given by the previous proposition. Since $d\phi_{f(\omega)}p = p_{|\ker f(\omega)}$ then $d\phi_{f(\omega)}$ is surjective. On the other hand by transversality of f to Z_j we have:

$$\mathrm{im}(df_\omega) + T_{f(\omega)}Z_j = \mathcal{Q}$$

Since $\phi(Z_j) = \{0\}$ (notice that this condition implies $(d\phi_{f(\omega)})|_{T_{f(\omega)}Z_j} = 0$) then

$$\mathcal{Q}(\ker f(\omega)) = \mathrm{im}(d\phi_{f(\omega)}) = \mathrm{im}(d(\phi \circ f)_\omega)$$

which tells $\phi \circ f$ is a submersion at ω. Thus by the rank theorem there exists U_ω and a diffeomorphism onto its image $\psi : U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$ such that $p_1 \circ \psi = \phi \circ f$. Taking $\delta_3'(\omega) > 0$ such that $B(\omega, \delta_3'(\omega)) \subset U_\omega$ concludes the proof.
In the case f is semialgebraic, then the set

$$S = \{(\omega, \delta) \in \Omega \times (0, \infty) \,:\, \psi|_{B(\omega,\delta)} \text{ is a diffeomorphism}\}$$

is semialgebraic too (by semialgebraic rank theorem ψ is semialgebraic (see [7]) and the condition to be a diffeomorphism is a semialgebraic condition on its Jacobian). By the previous part of the proof we have that the restriction $g|_S$ of the projection on the first factor is surjective and the semialgebraic choice for δ_3' follows (as in the proof of lemma 16) from proposition 4. □

Corollary 21. *Under the assumption of lemma 20, for every $\omega \in \Omega$ there exists $\delta_3(\omega) > 0$ such that for every $B(\omega', r) \subset B(\omega, 3\delta_3(\omega))$ with $r < \delta_3(\omega)$ then*

$$\psi(B(\omega', r)) \quad \textit{is convex.}$$

In particular if $\omega \in B(\omega_k, r_k)$ *for some* $\omega_0, \ldots, \omega_i \in \Omega$ *and* $r_0, \ldots, r_i < \delta_3(\omega)$*, then for every* $j \in \mathbb{N}$ *the space*

$$\{\eta \in \Omega \,:\, \mathrm{i}^-(f(\eta)) \leq n-j\} \cap (\bigcap_{k=0}^{i} B(\omega_k, r_k)) \quad \textit{is acyclic.}$$

Moreover if f *is semialgebraic, then* δ_3 *can be chosen semialgebraic.*

Proof. The first part of the statement follows by applying lemma 20 and corollary 6 to $\psi : U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$.
For the second part notice that by Proposition 9 we have for every $\eta \in U_\omega$ (using the above notations):

$$\mathrm{i}^-(f(\eta)) = \mathrm{i}^-(f(\omega)) + \mathrm{i}^-(p_1(\psi(\eta))).$$

This implies that, setting as above $\Omega_{n-j}(f) \doteq \{\eta \in \Omega \,:\, \mathrm{i}^-(f(\eta)) \leq n-j\}$,

$$\psi(U_\omega \cap \Omega_{n-j}(f)) \subseteq \mathcal{Q}_{n-j}(\ker f(\omega)) \times \mathbb{R}^l,$$

where $\mathcal{Q}_{n-j}(\ker(f(\omega))) = \{q \in \mathcal{Q}(\ker f(\omega)) \,:\, \mathrm{i}^-(q) \leq n-j\}$. Since for each $k = 0, \ldots, i$ the set $\psi(B(\omega_k, r_k))$ is convex, then

$$\bigcap_{k=0}^{i} \psi(B(\omega_k, r_k)) \quad \text{is convex}$$

and by hypothesis it contains $\psi(\omega)$. Since $\mathcal{Q}_{n-j}(\ker f(\omega)) \times \mathbb{R}^l$ (if nonempty) has linear conical structure with respect to $\psi(\omega)$, then

$$\psi(\Omega_{n-j}(f)) \cap \bigcap_{k=0}^{i} \psi(B(\omega_k, r_k)) \quad \text{is acyclic}$$

and since $\psi : \bigcap_k B(\omega_k, r_k) \subset U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$ is a homeomorphism onto its image the conclusion follows.
In the case f is semialgebraic, then ψ is semialgebraic and we let δ_3' and δ_c be given by Lemma refcomm and Corollary 6 respectively. As both δ_3' and δ_c can be chosen semialgebraic, then the same holds true for $\delta_3 = \min\{\delta_3', \delta_c\}$.

□

Let now $f : \Omega \to \mathcal{Q}$ be smooth, semialgebraic and transversal to all strata of $\mathcal{Q}_0 = \coprod Z_j$. Then we define $\delta : \Omega \to (0, \infty)$ by

$$\delta(\omega) = \min\{\delta_1(\omega), \delta_2(\omega), \delta_3(\omega)\}.$$

By construction δ can be chosen to be semialgebraic. Under this assumption we prove the following.

Lemma 22. *Let* $\mathcal{W}$ *be an open cover of* Ω *and* f *and* δ *as above. Then there exists a locally finite refinement* $\mathcal{U} = \{V_\alpha = B(x_\alpha, \delta_\alpha), x_\alpha \in \Omega\}_{\alpha \in A}$ *satisfying the following conditions: (i) for every multi-index* $\bar{\alpha} = (\alpha_0 \cdots \alpha_i)$ *with* $V_{\bar{\alpha}} \neq \emptyset$ *there exists* $\omega_{\bar{\alpha}} \in V_{\bar{\alpha}}$ *such that for every* $k = 0, \ldots, i$ *the following holds:*

$$B(x_{\alpha_k}, \delta_{\alpha_k}) \subset B(\omega_{\bar{\alpha}}, \delta(\omega_{\bar{\alpha}}));$$

for every $\bar{\alpha}$ *multi-index we let* $n_{\bar{\alpha}}$ *be the minimum of* $\mathrm{i}^- \circ f$ *over* $V_{\bar{\alpha}} \neq \emptyset$*, then the cover* $\mathcal{U}$ *can be chosen as to satisfy (ii):*

$$n_{\alpha_0 \cdots \alpha_i} = \max\{n_{\alpha_0}, \ldots, n_{\alpha_i}\}.$$

Proof. We first set some notations. Let $\mathcal{N}=\coprod_{i=1}^{l} N_i \subset \Omega$ be a finite family of disjoint smooth submanifold such that $\delta_{|\mathcal{N}}$ is continuous. For $i=1,\ldots,l$ let also $N_i' \subset N_i$ be a compact subset and define $\mathcal{N}'=\coprod N_i'$.
Then there exists $\epsilon(\mathcal{N},\mathcal{N}')>0$ such that for $i\neq j$ the two sets $\{x\in\Omega\,:\,d(x,N_i')<\epsilon(\mathcal{N},\mathcal{N}')\}$ and $\{x\in\Omega\,:\,d(x,N_j')<\epsilon(\mathcal{N},\mathcal{N}')\}$ are disjoint.
Let $\mathcal{W}_{\mathcal{N}'}$ be the cover $\{W\cap\mathcal{N}'\,:\,W\in\mathcal{W}\}$ and $\lambda_{\mathcal{N}'}>0$ be its Lebesgue number. Finally let $\delta'_{\mathcal{N}'}=\min_{\eta\in\mathcal{N}'}3\delta(\eta)>0$ which exists since $\delta_{|\mathcal{N}}$ is continuos and $\mathcal{N}'$ is compact.
We define $\delta(\mathcal{N},\mathcal{N}')>0$ to be any number such that

$$\delta(\mathcal{N},\mathcal{N}')<\min\{\epsilon(\mathcal{N},\mathcal{N}'),\lambda_{\mathcal{N}'},\delta'_{\mathcal{N}'}\}.$$

We construct now the desired cover. Let $h:|K|\to\Omega$ be a smooth semialgebraic triangulation (i.e. smooth one each simplex) of Ω respecting the semialgebraic sets $\{\omega\in\Omega\,|\,\mathrm{i}^-(f(\omega))=k\}_{k\in\mathbb{N}}$ and such that δ is continuous on each simplex (see [7]). Thus $\Omega=\coprod S_i$, where $i=0,\ldots,k$ and S_i is the image under h of the i-th skeleton of the complex K.
Let $S_0=\{x_0,\ldots,x_v\}$ and define

$$\mathcal{U}_0\doteq\{B(x_i,\delta(S_0,S_0)),i=0,\ldots,v\}$$

and $T_0=\cup_i B(x_i,\delta(S_0,S_0))$.
Now proceed inductively: first set $S_i=\coprod_{\sigma_{i,j}\in K_i}h(\sigma_{i,j})$ and $S_i'=\coprod h(\sigma_{i,j})\backslash T_{i-1}$. Then let $\mathcal{U}_i=\{B(x_i^j,\delta_i)\,:\,x_i^j\in S_i'\,\text{and}\,\delta_i<\delta(S_i,S_i')\}$ be such that $\mathcal{U}_i$ and $\mathcal{U}_i\cap S_i'$ have the same combinatorics; let also T_i be defined by

$$T_i=\cup_{V\in\mathcal{U}_i}V.$$

With the previous settings we finally define

$$\mathcal{U}\doteq\mathcal{U}_0\cup\cdots\cup\mathcal{U}_k.$$

Then $\mathcal{U}$ verifies by construction the requirements and this concludes the proof. $\square$

Definition 23. *Let $f:\Omega\to\mathcal{Q}$ be a smooth semialgebraic map transverse to all strata of $\mathcal{Q}_0=\coprod Z_j$ and δ the semialgebraic function min$\{\delta_1,\delta_2,\delta_3\}$ where δ_1,δ_2 and δ_3 are given by Lemma 16, Lemma 14 and Corollary 21. Let $\mathcal{W}$ the open cover of Ω defined by*

$$\mathcal{W}=\{V_\alpha=B(x_\alpha,\delta_\alpha)\}_{\alpha\in A}$$

for certain $x_\alpha\in\Omega$ and $\delta_\alpha>0$, $\alpha\in A$. Then $\mathcal{W}$ will be called an f-regular cover of Ω if it satisfies conditions (i) and (ii) of Lemma 22.

In particular Lemma 22 tells that the set of f-regular covers is cofinal in the set of all covers of Ω.

3. The spectral sequence

Using the above notations for $\Omega=K^\circ\cap S^k\subset(\mathbb{R}^{k+1})^*$ we define

$$B=\{(\omega,x)\in\Omega\times\mathbb{P}^n\,:\,(\omega p)(x)>0\}.$$

The following lemma relates the topology of B to that of X. We have that $B\subset\Omega\times\mathbb{P}^n$ and we call β_l and β_r the restrictions to B of the projection on the first and the second factor.

Lemma 24. *The projection β_r on the second factor defines a homotopy equivalence between B and $\mathbb{P}^n\backslash X=\beta_r(B)$.*

Proof. The equality $\beta_r(B)=\mathbb{P}^n\backslash X$ follows from $(K^\circ)^\circ=K$. For every $x\in\mathbb{P}^n$ the set $\beta_r^{-1}(x)$ is the intersection of the set $\Omega\times\{x\}$ with an open half space in $(\mathbb{R}^{k+1})^*\times\{x\}$. Let (ω_x,x) be the center of gravity of the set $\beta_r^{-1}(x)$. It is easy to see that ω_x depends continuosly on $x\in\beta_r(B)$. Further it follows form convexity considerations that $(\omega_x/\|\omega_x\|,x)\in B$ and for any $(\omega,x)\in B$ the arc $(\frac{t\omega_x+(1-t)\omega_x}{\|t\omega_x+(1-t)\omega_x\|},x)$, $0\leq t\leq 1$ lies entirely in B. It is clear that $x\mapsto(\omega_x/\|\omega_x\|,x)$, $x\in\beta_r(B)$ is a homotopy inverse to β_r. $\square$

We first construct a slightly more general spectral sequence (F_r,d_r) converging to $H^*(\Omega\times\mathbb{P}^n,B)$ which in general is *not* isomorphic to $H_{n-*}(X)$. The required spectral sequence (E_r,d_r) arises by applying the following Theorem to a modification $(\hat{q},\hat{K})$ of the pair (q,K) such that $H^*(\hat{\Omega}\times\mathbb{P}^n,\hat{B})\simeq H_{n-*}(X)$.

Theorem 25. *There exists a cohomology spectral sequence of the first quadrant (F_r,d_r) converging to $H^*(\Omega\times\mathbb{P}^n,B;\mathbb{Z}_2)$ such that for every $i,j\geq 0$*

$$F_2^{i,j}=H^i(\Omega,\Omega^{j+1};\mathbb{Z}_2).$$

Proof. Fix a positive definite form and consider the well defined function $\alpha:\Omega\times\mathbb{P}^n\to\mathbb{R}$ defined by $(\omega,x)\mapsto(\omega p)(x)$. The function α is continuos, proper ($\Omega\times\mathbb{P}^n$ is compact), semialgebraic and $B=\{\alpha>0\}$. By corollary 3, there exists $\epsilon>0$ such that the inclusion:

$$B(\epsilon)=\{\alpha>\epsilon\}\hookrightarrow B$$

is a homotopy equivalence.
Consider the projection $\beta_l(\epsilon):B(\epsilon)\to\Omega$ on the first factor; then by theorem 18 there exists a cohomology spectral sequence $(F_r(\epsilon),d_r(\epsilon))$ converging to the cohomology group $H^*(\Omega\times\mathbb{P}^n,B(\epsilon);\mathbb{Z}_2)\simeq H^*(\Omega\times\mathbb{P}^n,B;\mathbb{Z}_2)$ such that:

$$F_2^{i,j}(\epsilon)=\check{H}^i(\Omega,\mathcal{F}^j(\epsilon))$$

where $\mathcal{F}^j(\epsilon)$ si the sheaf generated by the presheaf $V\mapsto H^j(V\times\mathbb{P}^n,\beta_l(\epsilon)^{-1}(V);\mathbb{Z}_2)$. Let now ω be in Ω; then for the stalk $(\mathcal{F}_j(\epsilon))_\omega=\varinjlim_{\omega\in V}\mathcal{F}_j(\epsilon)(V)$ we have from Lemma 14

$$(\mathcal{F}_j(\epsilon))_\omega\simeq H^j(\mathbb{P}^n,\mathbb{P}^{n-\mathrm{ind}^-(\omega p-\epsilon q_0)})$$

Hence if we set $\mathrm{i}^-(\epsilon)$ for the function $\omega\mapsto\mathrm{ind}^-(\omega p-\epsilon q_0)$, the following holds:

$$(\mathcal{F}^j(\epsilon))_\omega=\begin{cases}\mathbb{Z}_2 & \text{if } \mathrm{i}^-(\epsilon)(\omega)>n-j;\\ 0 & \text{otherwise}\end{cases}$$

Thus the sheaf $\mathcal{F}^j(\epsilon)$ is zero on the closed set $\Omega_{n-j}(\epsilon)=\{\mathrm{i}^-(\epsilon)\leq n-j\}$ and is locally constant with stalk $\mathbb{Z}_2$ on its complement; hence:

$$F_2^{i,j}(\epsilon)=\check{H}^i(\Omega,\mathcal{F}^j(\epsilon))=\check{H}^i(\Omega,\Omega_{n-j}(\epsilon);\mathbb{Z}_2).$$

Consider now, for $\epsilon>0$, the complex $(F_0(\epsilon),D(\epsilon)=d+\delta)$. Then for $\epsilon_1<\epsilon_2$ the inclusion $C(\epsilon_2)\hookrightarrow C(\epsilon_1)$ defines a morphism of filtered differential graded modules $i_0(\epsilon_1,\epsilon_2):(F_0(\epsilon_1),D(\epsilon_1))\to(F_0(\epsilon_2),D(\epsilon_2))$ turning $\{(F_0(\epsilon),D(\epsilon))\}_{\epsilon>0}$ into an inverse system and thus $\{(F_r(\epsilon),d_r(\epsilon))\}_{\epsilon>0}$ into an inverse system of spectral sequences. We define

$$(F_r,d_r)=\varprojlim_{\epsilon}\{(F_r(\epsilon),d_r(\epsilon))\}.$$

We examine $i_2(\epsilon_1,\epsilon_2):F_2^{i,j}(\epsilon_1)\to F_2^{i,j}(\epsilon_2)$; it is readily verified that for $i,j\geq 0$ the map $i_2(\epsilon_1,\epsilon_2)_{i,j}:F_2^{i,j}(\epsilon_1)\to F_2^{i,j}(\epsilon_2)$ equals the map

$$i^*(\epsilon_1,\epsilon_2):\check{H}^i(\Omega,\Omega_{n-j}(\epsilon_1))\to\check{H}^i(\Omega,\Omega_{n-j}(\epsilon_2))$$

given by the inclusion of pairs $(\Omega,\Omega_{n-j}(\epsilon_2))\hookrightarrow(\Omega,\Omega_{n-j}(\epsilon_1))$. By semialgebraicity $i^*(\epsilon_1,\epsilon_2)$ is an isomorphism for small ϵ_1,ϵ_2, hence $i_2(\epsilon_1,\epsilon_2)$ is definitely an isomorphism and thus $i_\infty(\epsilon_1,\epsilon_2)$ and $i_0^*(\epsilon_1,\epsilon_2):H_D^*(F_0(\epsilon_1))\to H_D^*(F_0(\epsilon_2))$ are definitely isomorphisms. Thus we have

$$F_2^{i,j}\simeq\varprojlim\{H^i(\Omega,\Omega_{n-j}(\epsilon);\mathbb{Z}_2)\}.$$

The following lemma 26 gives $\varinjlim\{H_*(\Omega,\Omega_{n-j}(\epsilon))\}\simeq H_*(\Omega,\Omega^{j+1})$ (using the long exact sequences of pairs). The chain of isomorphisms $\varprojlim\{H^i(\Omega,\Omega_{n-j}(\epsilon);\mathbb{Z}_2)\}\simeq(\varinjlim\{H_i(\Omega,\Omega_{n-j}(\epsilon);\mathbb{Z}_2)\})^*=(H_i(\Omega,\Omega^{j+1};\mathbb{Z}_2))^*$ finally gives

$$F_2^{i,j}=H^i(\Omega,\Omega^{j+1};\mathbb{Z}_2).$$

□

Lemma 26. *For every $j\in\mathbb{N}$ we have $\Omega^{j+1}=\bigcup_{\epsilon>0}\Omega_{n-j}(\epsilon)$; moreover every compact subset of Ω^{j+1} is contained in some $\Omega_{n-j}(\epsilon)$ and in particular*

$$\varinjlim_{\epsilon}\{H_*(\Omega_{n-j}(\epsilon))\}=H_*(\Omega^{j+1}).$$

Proof. Let $\omega\in\bigcup_{\epsilon>0}\Omega_{n-j}(\epsilon)$; then there exists $\bar{\epsilon}$ such that $\omega\in\Omega_{n-j}(\epsilon)$ for every $\epsilon<\bar{\epsilon}$. Since for ϵ small enough

$$\mathrm{i}^-(\epsilon)(\omega)=\mathrm{i}^-(\omega)+\dim(\ker\omega p)$$

then it follows that

$$\mathrm{i}^+(\omega)=n+1-\mathrm{i}^-(\omega)-\dim(\ker\omega p)\geq j+1.$$

Viceversa if $\omega\in\Omega^{j+1}$ the previous inequality proves $\omega\in\Omega_{n-j}(\epsilon)$ for ϵ small enough, i.e. $\omega\in\bigcup_{\epsilon>0}\Omega_{n-j}(\epsilon)$.
Moreover if $\omega\in\Omega_{n-j}(\epsilon)$ then, eventually choosing a smaller ϵ, we may assume ϵ properly separates the spectrum of ωp and thus, by algebraicity of the map $\omega\mapsto\omega p$, there exists U open neighborhood of ω such that ϵ properly separates also the spectrum of $\omega p'$ for every $\omega'\in U$ (see [13]). hence $\omega'\in\Omega_{n-j}(\epsilon)$ for every $\omega'\in U$. From this consideration it easily follows that each compact set in Ω^{j+1} is contained in some $\Omega_{n-j}(\epsilon)$ and thus

$$\varinjlim_{\epsilon}\{H_*(\Omega_{n-j}(\epsilon))\}=H_*(\Omega^{j+1}).$$

□

Remark 2. Lemma 14 is not really needed to construct the spectral sequence. If we consider $C(\epsilon)=\{(\omega,x)\in\Omega\times\mathbb{P}^n\ :\ (\omega p)(x)\geq 0)\}$ then by lemma 3 the inclusion $C(\epsilon)\hookrightarrow B$ is a homotopy equivalence for ϵ small enough. Consider the projection $\beta_l(\epsilon):C(\epsilon)\to\Omega$ on the first factor; then by theorem 18 there exists a cohomology spectral sequence $(F_r(\epsilon),d_r(\epsilon))$ converging to the cohomology group $H^*(\Omega\times\mathbb{P}^n,C(\epsilon);\mathbb{Z}_2)\simeq H^*(\Omega\times\mathbb{P}^n,B;\mathbb{Z}_2)$ such that:

$$F_2^{i,j}(\epsilon)=\check{H}^i(\Omega,\mathcal{F}^j(\epsilon))$$

where $\mathcal{F}^j(\epsilon)$ si the sheaf generated by the presheaf $V \mapsto H^j(V\times\mathbb{P}^n, \beta_l(\epsilon)^{-1}(V);\mathbb{Z}_2)$. Since $C(\epsilon)$ and Ω are locally compact and $\beta_l(\epsilon)$ is proper ($C(\epsilon)$ is compact), then the following isomorphism holds for the stalk of $\mathcal{F}^j(\epsilon)$ at each $\omega\in\Omega$ (see [11], Remark 4.17.1, p. 202):

$$(\mathcal{F}^j(\epsilon))_\omega \simeq H^j(\{\omega\}\times\mathbb{P}^n, \beta_l(\epsilon)^{-1}(\omega);\mathbb{Z}_2).$$

The set $\beta_l(\epsilon)^{-1}(\omega)=\{x\in\mathbb{P}^n : (\omega p)(x)\geq\epsilon\}=\{x\in\mathbb{P}^n : (\omega p-\epsilon q_0)(x)\geq 0\}$ has the homotopy type of a projective space of dimension $n-\mathrm{ind}^-(\omega p-\epsilon q_0)$ and it follows that, as above, $F_2^{i,j}(\epsilon)\simeq H^i(\Omega,\Omega_{n-j}(\epsilon))$. Letting ϵ be small enough, Lemma 26 gives as before

$$\varprojlim_{\epsilon}\{F_2^{i,j}(\epsilon)\}=H^i(\Omega,\Omega^{j+1}).$$

It is possible to show that actually the two spectral sequences agree, but we prefer the previous approach because it is more practical for computations.

Remark 3. In the case $K\neq -K$, i.e. $\Omega\neq S^l$, then (E_r,d_r) converges to $H_{n-*}(X,\mathbb{Z}_2)$. This follows by comparing the two cohomology long exact sequences of the pairs $(\Omega\times\mathbb{P}^n, B)$ and $(\mathbb{P}^n,\mathbb{P}^n\backslash X)$ via the map β_r. In this case $\beta_r:\Omega\times\mathbb{P}^n\to\mathbb{P}^n$ is a homotopy equivalence and the Five Lemma and Lemma 24 together give

$$H^*(\Omega\times\mathbb{P}^n,B)\simeq H^*(\mathbb{P}^n,\mathbb{P}^n\backslash X)\simeq H_{n-*}(X)$$

the last isomorphism being given by Alexander-Pontryagin Duality.

Theorem A. *There exists a cohomology spectral sequence of the first quadrant (E_r,d_r) converging to $H_{n-*}(X;\mathbb{Z}_2)$ such that*

$$E_2^{i,j}=H^i(C\Omega,\Omega^{j+1};\mathbb{Z}_2).$$

Proof. Keeping in mind the previous remark, we work the general case (i.e. also the case $K=\{0\}$). We replace K with $\hat{K}=(-\infty,0]\times K$, the map p with the map $\hat{p}:\mathbb{R}^{n+1}\to\mathbb{R}^{k+2}$ defined by $\hat{p}=(-q_0,p)$, where q_0 is a positive definite form and Ω with

$$\hat{\Omega}=\hat{K}^\circ\cap S^{k+1}.$$

We also define

$$\hat{\Omega}^{j+1}=\{(\eta,\omega)\in\hat{\Omega} : \mathrm{ind}^+(\omega p-\eta q_0)\geq j+1\}.$$

Then, by construction,

$$\hat{p}^{-1}(\hat{K})=p^{-1}(K)=X.$$

Applying Theorem 25 to the pair $(\hat{p},\hat{K})$, with the previous remark in mind, we get a spectral sequence $(\hat{E}_r,\hat{d}_r)$ converging to $H_{n-*}(X;\mathbb{Z}_2)$ with

$$\hat{E}_2^{i,j}=H^i(\hat{\Omega},\hat{\Omega}^{j+1};\mathbb{Z}_2).$$

We identify Ω^{j+1} with $\hat{\Omega}^{j+1}\cap\{\eta=0\}$ and we claim that the inclusion of pairs $(\hat{\Omega},\Omega^{j+1})\hookrightarrow(\hat{\Omega},\hat{\Omega}^{j+1})$ induces an isomorphism in cohomology. This follows from the fact that $\hat{\Omega}^{j+1}$ deformation retracts onto Ω^{j+1} along the meridians (the deformation retraction is defined since $j\geq 0$ and $\mathrm{i}^+(1,0,\ldots,0)=0$, thus the "north pole" of S^{k+1} does not belong to any of the $\hat{\Omega}^{j+1}$). If $\eta_1\leq\eta_2$ then $\mathrm{ind}^+(\omega p-\eta_1 q_0)\geq \mathrm{ind}^+(\omega p-\eta_2 q_0)$: thus if $(\eta,\omega)\in\hat{\Omega}^{j+1}$ then all the points on the meridian arc connecting (η,ω) with $\Omega=\hat{\Omega}\cap\{\eta=0\}$ belong to $\hat{\Omega}^{j+1}$.
Noticing that $(\hat{\Omega},\Omega^{j+1})\approx(C\Omega,\Omega^{j+1})$, where $C\Omega$ stands for the topological space cone of Ω, concludes the proof. □

Corollary 27. *Let $\mu = \max_{\eta\in\Omega} \mathrm{i}^+(\eta)$, and $0 \le b \le n-\mu-k$ then*

$$H_b(X) = \mathbb{Z}_2.$$

In particular if $n \ge \mu + k$ then X is nonempty.

Proof. Simply observe that the group $E_2^{0,n-b}$ equals $\mathbb{Z}_2$ for $0 \le b \le n-\mu-k$ and that all the differentials $d_r : E_r^{0,n-b} \to E_r^{r,n-b+r-1}$ for $r \ge 0$ are zero, since they take values in zero elements. Hence

$$\mathbb{Z}_2 = E_\infty^{0,n-b} = H_b(X).$$

□

4. The second differential

Suppose that a scalar product on $\mathbb{R}^{n+1}$ has been fixed and let $w_1(\mathcal{L}_j^+) \in H^1(\mathcal{D}_j)$ be the first Stiefel-Whitney class of $\mathcal{L}_j^+ \to \mathcal{D}_j$ (the definition of the previous bundle depends on the fixed scalar product).
Notice that given $x \in H^i(\Omega, \Omega^{j+1})$ then the product

$$x \smile \partial^* \bar{p}^* w_1(\mathcal{L}_j^+) \in H^{i+2}(\Omega, \Omega^{j+1} \cup D_j)$$

and since $\Omega^j \subset \Omega^{j+1} \cup D_j$ we can consider the restriction $(x \smile \partial^* \bar{p}^* w_1(\mathcal{L}_j^+))|_{(\Omega,\Omega^j)} \in H^{i+2}(\Omega, \Omega^j)$. With the previous notations we prove the following theorem which describes the second differential for the spectral sequence of Theorem 25.

Theorem 28. *Let $\partial^* : H^1(D_j) \to H^2(\Omega, D_j)$ be the connecting homomorphism. Then for every $i, j \ge 0$ the differential $d_2 : F_2^{i,j} \to F_2^{i+2,j-1}$ is given by:*

$$d_2(x) = (x \smile \partial^* \bar{p}^* w_1(\mathcal{L}_j^+))|_{(\Omega,\Omega^j)}.$$

Proof. We fix at the very beginning a scalar product g; for this proof we will use in the notations for the various objects their dependence on g.
Recall from Theorem 25 that we have defined (F_r, d_r) by:

$$(F_r, d_r) = \varprojlim_{\epsilon} \varinjlim_{\beta_l^{-1}\mathcal{W}} \{(F_r(\epsilon, \mathcal{U}), d_r(\epsilon, \mathcal{U}))\}$$

where the $(\epsilon, \mathcal{U})$-pair is the relative Leray's spectral sequence for the pair $(\Omega \times \mathbb{P}^n, B(\epsilon))$, the map β_l and the cover $\mathcal{U} = \beta_l^{-1}\mathcal{W}$ (the direct limit ranges over all covers of Ω). The set $B(\epsilon)$ was defined using the function $\alpha : \Omega \times \mathbb{P}^n \to \mathbb{R}$, $\alpha(\omega, x) = (\omega p)(x)/q_0(x)$, where q_0 is a positive definite form, as $B(\epsilon) = \{\alpha > \epsilon\}$. By Lemma 13 we may assume q_0 is such that the map:

$$f_\epsilon : \omega \mapsto \omega p - \epsilon q_0$$

is nondegenerate (and also can be made transversal to $\mathcal{Q}_0$ and $\mathcal{Q}\backslash\mathcal{D}^g$, where $\mathcal{D}^g = \cap_j \mathcal{D}_j^g$). In this way Lemma 22 ensures the existence of an f_ϵ-regular cover of Ω:

$$\mathcal{W} = \{V_\alpha = B(x_\alpha, \delta_\alpha)\}_{\alpha\in A}$$

Thus the proof develops as follows. (i) We compute for ϵ small and $\mathcal{U} = \beta_l^{-1}\mathcal{W}$ the differential $d_2(\epsilon, \mathcal{U})$. Since $\mathcal{W}$ is f_ϵ-regular, then by Lemma 21, it is acyclic for each $\mathcal{F}^j(\epsilon)$ and thus the limit map gives for every $i, j \in \mathbb{Z}$ isomorphisms:

$$F_2^{i,j}(\epsilon, \mathcal{U}) \simeq F_2^{i,j}(\epsilon).$$

Under this isomorphism the differential $d_2(\epsilon,\mathcal{U})$ happens to be given by:

$$x \mapsto (x \smile f_\epsilon^* \partial^* w_1(\mathcal{L}_j^+)^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$$

Thus under the limit map the second differential is given by the previous formula for every f_ϵ-regular cover; since the set of such covers is cofinal in all covers of Ω, then the previous is actually the expression for $d_2(\epsilon)$. (ii) We perform the ϵ-limit and get the expression for $d_2(\epsilon)$.
We stress that the definition of our spectral sequence using direct and inverse limits is somehow formal: both limits are attained for ϵ small enough and $\mathcal{W}$ a f_ϵ-regular cover.
We introduce now some auxiliary material. Let $K_0^{*,*} = K_0^{*,*}(\mathcal{U})$ be the Kunneth bicomplex associated to the map $\beta_l : \Omega \times \mathbb{P}^n \to \Omega$ with respect to $\mathcal{U}$. Notice that $F_0^{*,*}(\epsilon,\mathcal{U})$ is a subcomplex of $K_0^{*,*}$ and we denote by δ_F, d_F and δ_K, d_K the respective bicomplex differentials (the first two are the restriction to $F_0^{*,*}$ of the second two).
For every $\omega \in \Omega$ and $\epsilon > 0$ we let $\mathrm{i}^-(\epsilon)(\omega) = \mathrm{ind}^-(\omega p - \epsilon q_0)$ and for every multi-index $\bar{\alpha} = (\alpha_0, \ldots, \alpha_i)$ such that $V_{\bar{\alpha}} \neq \emptyset$ we let $n_{\bar{\alpha}}$ be the minimum of $\mathrm{i}^-(\epsilon)$ over $V_{\bar{\alpha}}$. We take an order on the index set A such that

$$\alpha \leq \beta \quad \text{implies} \quad n_\alpha \leq n_\beta.$$

In this way, by Lemma 22, for every multi-index $\bar{\alpha} = (\alpha_0, \ldots, \alpha_i)$ such that $V_{\bar{\alpha}} \neq \emptyset$ we have that $n_{\bar{\alpha}} = n_{\alpha_i}$. For every multi-index $\bar{\alpha}$ such that $V_{\bar{\alpha}} \neq \emptyset$ let $\omega_{\bar{\alpha}}$ be given by Lemma 22, $\mathrm{i}^-(\epsilon)(\omega_{\bar{\alpha}}) = n_{\bar{\alpha}}$, and we let $\eta_{\bar{\alpha}} \in V_{\bar{\alpha}}$ be such that $\det(f_\epsilon(\eta_{\bar{\alpha}})) \neq 0$, $\mathrm{i}^-(\epsilon)(\eta_{\bar{\alpha}}) = n_{\bar{\alpha}}$ and $f_\epsilon(\eta_{\bar{\alpha}}) \in \mathcal{D}^g$ (such $\eta_{\bar{\alpha}}$ always exists, and by transversality of the map f_ϵ to $\mathcal{Q}_0$ and to $\mathcal{Q}\backslash\mathcal{D}^g$, which have respectively codimension one and two, there are plenty of them).
For every $0 \leq j \leq n$ and $\alpha \in A$ we define

$$N(\alpha, j) = (P_j^-)^g(f_\epsilon(\eta_\alpha))$$

where the g on $(P_j^-)^g$ denotes the dependence on the fixed scalar product. Moreover we let $\nu(\alpha, j) \in C^j(\mathbb{P}^n)$ be the cochain defined by the intersection number with $N(\alpha, j)$; since $N(\alpha, j)$ is a $\mathbb{Z}_2$-cycle, then $\nu(\alpha, j)$ is well defined and it represents the Poincaré dual of $N(\alpha, j)$. Then we define a cochain $\psi^{0,j} \in K_0^{0,j}$ by

$$\psi^{0,j}(\alpha) = \beta_r^* \nu(\alpha, j).$$

Notice that if $n - n_\alpha + 1 \leq j \leq n$ then , by Lemma 16, $N(\alpha, j) \subset \mathbb{P}^n \backslash \beta_r(B_\alpha(\epsilon))$ and thus $\nu(\alpha, j) \in C^j(\mathbb{P}^n, \beta_r(B_\alpha(\epsilon))$. Hence

$$(1) \qquad n - n_\alpha + 1 \leq j \leq n \quad \text{implies} \quad \psi^{0,j}(\alpha) \in C^j(V_\alpha \times \mathbb{P}^n, B_\alpha(\epsilon))$$

Moreover $N(\alpha, n - n_\alpha + 1)$ is a $(n_\alpha - 1)$-dimensional projective space contained in $\mathbb{P}^n \backslash \beta_r(B_{\alpha_0 \ldots \alpha_i \alpha}(\epsilon))$ for every $(\alpha_0, \ldots, \alpha_i)$; thus by Lemma 14 if $n - n_\alpha + 1 \leq j \leq n$ then the cohomology class of $\nu(\alpha, j)$ generates $H^j(\mathbb{P}^n, \beta_r(B_\alpha(\epsilon)))$. Hence it follows that for every $\overline{\alpha} = (\alpha_0 \cdots \alpha_i \alpha)$ such that $V_{\overline{\alpha}} \neq \emptyset$

$$(2) \quad n - n_\alpha + 1 \leq j \leq n \quad \text{implies} \quad [\psi^{0,j}(\alpha)_{|\overline{\alpha}}] \text{ generates } H^j(V_{\overline{\alpha}} \times \mathbb{P}^n, B_{\overline{\alpha}}(\epsilon)) = \mathbb{Z}_2$$

For every $\alpha_0, \alpha_1 \in A$ such that $V_{\alpha_0\alpha_1} \neq \emptyset$ we consider a curve $c_{\alpha_0\alpha_1} : I \to V_{\alpha_0} \cup V_{\alpha_1}$ such that $c_{\alpha_0\alpha_1}(i) = \eta_{\alpha_i}$, $i = 0, 1$; since $\Omega \backslash f_\epsilon^{-1}(\mathcal{D}^g)$ has codimension two in Ω, then we may choose $c_{\alpha_0\alpha_1}$ such that for every $t \in I$ we have $f_\epsilon(c_{\alpha_0\alpha_1}(t)) \in \mathcal{D}^g$. Consider the $\mathbb{R}^{n-j+1}$-bundle $L_j^g(\alpha_0\alpha_1) = c^*_{\alpha_0\alpha_1} f^*_\epsilon(\mathcal{L}_j^-)^g$ over I and its projectivization $P(L_j^g(\alpha_0\alpha_1))$. Then the natural map

$$P(L_j^g(\alpha_0\alpha_1)) \to \mathbb{P}^n$$

defines a $(n-j+1)$-chain $T(\alpha_0\alpha_1, j-1)$ in $\mathbb{P}^n$. Let $\tau(\alpha_0\alpha_1, j-1)$ be the $j-1$-cochain defined by the intersection number with $T(\alpha_0\alpha_1, j-1)$. This cochain is defined only on singular chains that are transverse to $T(\alpha_0\alpha_1, j-1)$, but since such chains define the same homology groups as the singular ones we may restrict to them. Slightly abusing of notations we define $\theta^{1,j-1} \in K_0^{1,j-1}$ by setting for every α_0, α_1 with $V_{\alpha_0\alpha_1} \neq \emptyset$

$$\theta^{1,j-1}(\alpha_0\alpha_1) = \beta_r^* \tau(\alpha_0\alpha_1, j-1).$$

The reader that feels uncomfortable with this assumption may prefer to use from the very beginning triangulations of all the topological spaces we introduced (everything is semialgebraic) and a bicomplex with simplicial cochains instead of singular cochains; then using dual cell decompositions the above cochains happen to be everywhere defined. This procedure will end up with an isomorphic spectral sequence, but it is remarkably more cumbersome.

Notice that $\partial T(\alpha_0\alpha_1, j-1) = N(\alpha_0, j) + N(\alpha_1, j)$, hence $d\tau(\alpha_0\alpha_1, j-1) = \nu(\alpha_0, j) + \nu(\alpha_1, j)$; it follows that

$$\delta_K \psi^{0,j} = d_K \theta^{1,j-1}. \tag{3}$$

Moreover by construction if $n - n_{\alpha_0} + 1 \leq j \leq n$ and $n - n_{\alpha_1} + 1 \leq j \leq n$, which implies $n - n_{\alpha_0\alpha_1} + 1 \leq j \leq n$, then

$$\theta^{1,j-1}(\alpha_0\alpha_1) \in C^{j-1}(V_{\alpha_0\alpha_1} \times \mathbb{P}^n, B_{\alpha_0\alpha_1}(\epsilon)). \tag{4}$$

We compute now $\delta_K \theta^{1,j-1}$. Let $(\alpha_0\alpha_1\alpha_2) = \bar{\alpha}$ be such that $V_{\bar{\alpha}} \neq \emptyset$. Then the curves $c_{\alpha_0\alpha_1}, c_{\alpha_1\alpha_2}$ and $c_{\alpha_2\alpha_0}$ define a map $\sigma_{\alpha_0\alpha_1\alpha_2} : S^1 \to \Omega$ and we have the bundle $L_j^g(\alpha_0\alpha_1\alpha_2) = \sigma^*_{\alpha_0\alpha_1\alpha_2} f^*_\epsilon(\mathcal{L}_j^-)^g$ and its projectivization $P(L_j^g(\alpha_0\alpha_1\alpha_2))$ over S^1. The natural map

$$P(L_j^g(\alpha_0\alpha_1\alpha_2)) \to \mathbb{P}^n$$

defines a $(n-j+1)$-cochain whose pullback under β_r^* by construction equals $\delta_K \theta^{1,j-1}(\alpha_0\alpha_1\alpha_2)$. Thus by Lemma 8 we have:

$$\delta_K \theta^{1,j-1}(\alpha_0\alpha_1\alpha_2) = w_1(\partial(\alpha_0\alpha_1\alpha_2))(\psi^{0,j-1}(\alpha_2)_{|\alpha_0\alpha_1\alpha_2}) + dr^{2,j-1}(\alpha_0\alpha_1\alpha_2) \tag{5}$$

where $w_1(\partial(\alpha_0\alpha_1\alpha_2)) = w_1(L_j^g(\alpha_0\alpha_1\alpha_2))$. Let now $\xi^i \in F_1^{i,j}(\epsilon, \mathcal{U})$; we define $\xi^{i,0} \in K_0^{i,0}$ by

$$\xi^{i,0}(\alpha_0 \dots \alpha_i) \equiv \xi^i(\alpha_0 \dots \alpha_i)$$

i.e. the values of $\xi^{i,0}(\alpha_0 \dots \alpha_i)$ on every 0-chain equals $\xi^i(\alpha_0 \dots \alpha_i) \in \mathbb{Z}_2$. Notice that by construction $d_K \xi^{i,0} = 0$ and that

$$d_1 \xi^i = 0 \quad \text{implies} \quad \delta_K \xi^{i,0} = 0. \tag{6}$$

Everything is ready now for the part (i) of the proof.

Pick $x \in F_2^{i,j}(\epsilon, \mathcal{U}) \simeq F_2^{i,j}(\epsilon)$ and $\xi^i \in F_1^{i,j}(\epsilon, \mathcal{U})$ such that $d_1 \xi^1 = 0$ and $x = [\xi^i]_2$.

According to the definition of $d_2(\epsilon,\mathcal{U})$, to compute it on x we must find in $F_0(\epsilon,\mathcal{U})$ a zig-zag:

$$\begin{tikzcd} 0 & \\ \eta_0 \arrow[u, "d_F"] \arrow[r, "\delta_F"] & \cdot \\ & \eta_1 \arrow[u, "d_F"'] \end{tikzcd}$$

such that $[\eta_0]_2 = x$. This will give

$$d_2(\epsilon,\mathcal{U})x = [\delta\eta_1]_2.$$

We claim that $\eta_0 = \xi^{i,0}\cdot\psi^{0,j}$, $\eta_1 = \xi^{i,0}\cdot\theta^{1,j-1}$ is such a zig-zag. First notice that since $\xi^i \in F_1^{i,j}(\epsilon,\mathcal{U})$, then (1) implies $\xi^{i,0}\cdot\psi^{0,j} \in F_0^{i,j}(\epsilon,\mathcal{U})$. Moreover by (2) it follows that $[\xi^{i,0}\cdots\psi^{0,j}]_1 = \xi^i$ and thus

$$[\xi^{i,0}\cdot\psi^{0,j}]_2 = x.$$

We calculate now:

$$\begin{aligned}\delta_F(\xi^{i,0}\cdot\psi^{0,j}) &= \delta_K(\xi^{i,0}\cdot\psi^{0,j}) = \xi^{i,0}\cdot\delta_K\psi^{0,j} = \xi^{i,0}\cdot d_K\theta^{1,j-1}\\ &= d_K(\xi^{i,0}\cdot\theta^{1,j-1}) = d_F(\xi^{i,0}\cdot\theta^{1,j-1}).\end{aligned}$$

The first equality comes from $F_0^{i,j}(\epsilon,\mathcal{U})\subset K_0^{i,j}$; the second from $d_1\xi^i = 0$; the third from (3); the fourth from (6); the last by $\xi^{i,0}\cdot\theta^{1,j-1}\in F_0^{i+1,j-1}(\epsilon,\mathcal{U})$, which is a direct consequence of (4). Thus the chosen pair is such a required zig-zag and we can finally compute $d_2(\epsilon,\mathcal{U})(x) = [\delta_F(\xi^{i,0}\cdot\theta^{1,j-1})]_2$. We have:

$$\delta_F(\xi^{i,0}\cdot\theta^{1,j-1}) = \delta_K(\xi^{i,0}\cdot\theta^{1,j-1}) = \xi^{i,0}\cdot\delta_K\theta^{1,j-1}$$

and thus by (5) we derive:

$$[\delta_F(\xi^{i,0}\cdot\theta^{1,j-1})]_1(\alpha_0\cdots\alpha_{i+2}) = \xi^i(\alpha_0\cdots\alpha_i)w_1(\partial(\alpha_i\alpha_{i+1}\alpha_{i+2})).$$

This gives the description of $[\delta_F(\xi^{i,0}\cdot\theta^{1,j-1})]_1$ as the cochain representing (using $F_2(\epsilon,\mathcal{U})\simeq F_2(\epsilon)$) the cohomology class $(x\smile f_\epsilon^*\partial^* w_1(\mathcal{L}_j^+)^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$ (here we are using the fact that $w_1(\mathcal{L}_j^+) = w_1(\mathcal{L}_j^-)$). This proves part (i).
We proceed now with part (ii). Define $\Gamma_{1,j}\in H^2(\mathcal{Q},\mathcal{D}_j)$ by

$$\Gamma_{1,j}^g \doteq \partial^* w_1(\mathcal{L}_j^+)^g.$$

Consider now the following sequences of maps:

$$H^2(\mathcal{Q},\mathcal{D}_j^g)\xrightarrow{f_\epsilon^*} H^2(\Omega, D_j^g(\epsilon))\xrightarrow{r_\epsilon^*} H^2(\Omega,\Omega_{n-j+1}(\epsilon)\backslash\Omega_{n-j}(\epsilon)).$$

Notice that $r_\epsilon^* f_\epsilon^*\Gamma_{1,j}^g$ does not depend on g and thus the differential $d_2(\epsilon)$ is given by

$$x\mapsto (x\smile f_\epsilon^*\Gamma_{1,j}^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$$

for *any* g. Let now $g = q_0$; then in this case $D_j^{q_0} = D_j^{q_0}(\epsilon)$ and $f^* = f_\epsilon^*$. Consider the following commutative diagram of inclusions:

$$\begin{tikzcd}
(\Omega, \Omega^j) \arrow[r, "\iota"] & (\Omega, \Omega^{j+1} \cup D_j^{q_0}) \\
(\Omega, \Omega_{n-j+1}(\epsilon)) \arrow[u, "\rho(\epsilon)"] \arrow[r, "\iota(\epsilon)"] & (\Omega, \Omega_{n-j}(\epsilon) \cup D_j^{q_0}) \arrow[u, "\hat{\rho}(\epsilon)"']
\end{tikzcd}$$

Then, using $\rho(\epsilon)$ also for the inclusion $(\Omega, \Omega_{n-j}(\epsilon)) \hookrightarrow (\Omega, \Omega^{j+1})$ and setting $\gamma_{1,j} = f^* \Gamma_{1,j}^{q_0}$, we have for $x \in H^i(\Omega, \Omega^{j+1})$ the following chain of equalities:

$$\begin{aligned}
\rho(\epsilon)^*((x \smile \gamma_{1,j})|_{(\Omega,\Omega^j)}) &= \rho(\epsilon)^* \iota^*(x \smile f^* \Gamma_{1,j}^{q_0}) = \iota(\epsilon)^* \hat{\rho}(\epsilon)^*(x \smile f^* \Gamma_{1,j}^{q_0}) \\
&= \iota(\epsilon)^*(\rho(\epsilon)^* x \smile f_\epsilon^* \Gamma_{1,j}^{q_0}) = d_2(\epsilon)(\rho(\epsilon)^* x).
\end{aligned}$$

This proves that the following diagram is commutative:

$$\begin{tikzcd}
H^i(\Omega, \Omega^{j+1}) \arrow[r, "(\cdot \smile \gamma_j)|_{(\Omega,\Omega^j)}"] \arrow[d, "\rho(\epsilon)^*"'] & H^{i+2}(\Omega, \Omega^j) \arrow[d, "\rho(\epsilon)^*"] \\
H^i(\Omega, \Omega_{n-j}(\epsilon)) \arrow[r, "d_2(\epsilon)"] & H^{i+2}(\Omega, \Omega_{n-j+1}(\epsilon))
\end{tikzcd}$$

From this the conclusion follows. □

We are now ready to prove the statement concerning the second differential of the spectral sequence of Theorem A.
We let $\partial^* : H^1(D_j) \to H^2(C\Omega, D_j)$ be the connecting homomorphism and we define $\gamma_{1,j} \in H^2(C\Omega, D_j)$ by

$$\gamma_{1,j} = \partial^* \bar{p}^* w_1(\mathcal{L}_j^+)$$

(notice that all the previous objects are those associated to q_0 and that $\gamma_{1,j} = \bar{p}^* \phi_j$ as defined in the Introduction).

Theorem B. *For every $i, j \geq 0$ the differential $d_2 : E_2^{i,j} \to E_2^{i+2,j-1}$ is given by:*

$$d_2(x) = (x \smile \gamma_{1,j})|_{(C\Omega,\Omega^j)}.$$

Proof. We replace now K with $\hat{K} = (-\infty, 0] \times K$, the map p with the map $\hat{p} = (q_0, p) : \mathbb{R}^{n+1} \to \mathbb{R}^{k+2}$, where $q_0 \in \mathcal{Q}^+$, and we apply the previoius Theorem to $(\hat{p}, \hat{K})$. As for Teorem A we use the deformation retraction $(\hat{\Omega}, \hat{\Omega}^{j+1}) \to (\hat{\Omega}, \Omega^{j+1}) = (C\Omega, \Omega^{j+1})$. Notice that we have also the deformation retraction

$$r : (\hat{\Omega}, \hat{D}_j) \to (\hat{\Omega}, D_j)$$

where D_j is identified with $\hat{D}_j \cap \{\eta = 0\}$: by definition $\omega \in D_j$ if and only if $(\eta, \omega) \in \hat{D}_j$ and for every $0 < j < n+1$ we have $(1, 0, \ldots, 0) \notin D_j$ since all the eigenvalues of $\langle (1, 0, \ldots, 0), \hat{p} \rangle = -q_0$ with respect to q_0 coincide. Then by construction

$$r^* \gamma_{1,j} = \partial^* \bar{p}^* w_1(\mathcal{L}_j^+)$$

and by naturality the conclusion follows.

□

5. Projective inclusion

In this section we study the image of the homology of X under the inclusion map

$$\iota : X \to \mathbb{P}^n.$$

Using the above notations, we define $\hat{B} = \{(\hat{\omega}, x) \in \hat{\Omega} \times \mathbb{P}^n \,:\, (\hat{\omega}\hat{p})(x) > 0\}$ and we call (E_r, d_r) the spectral sequence of Theorem A converging to $H^*(\hat{\Omega} \times \mathbb{P}^n, \hat{B})$. Moreover we let $K_0^{*,*}$ be the Leray bicomplex for the map $\hat{\Omega} \times \mathbb{P}^n \to \hat{\Omega}$ (it equals the Kunneth bicomplex for $\hat{\Omega} \times \mathbb{P}^n$). Thus there is a morphism of spectral sequence $(i_r : E_r \to K_r)_{r \geq 0}$ induced by the inclusion $j : (\hat{\Omega} \times \mathbb{P}^n, \emptyset) \to (\hat{\Omega} \times \mathbb{P}^n, B)$. With the above notations we prove the following theorem which gives the rank of the homomorphism

$$\iota_* : H_*(X) \to H_*(\mathbb{P}^n).$$

Theorem C. *For every $b \in \mathbb{Z}$ the following holds:*

$$\mathrm{rk}(\iota_*)_b = \mathrm{rk}(i_\infty)_{0,n-b}.$$

Moreover the map $(i_\infty)_{0,n-b} : E_\infty^{0,n-b} \to K_\infty^{0,n-b} = \mathbb{Z}_2$ is an isomorphism onto its image.

Proof. First we look at the following commutative diagram of maps

$$\begin{array}{ccc}
H^{n-b}(\hat{\Omega} \times \mathbb{P}^n, \hat{B}) & \xrightarrow{(j^*)_{n-b}} & H^{n-b}(\hat{\Omega} \times \mathbb{P}^n) \\
\uparrow \beta_l^* & & \uparrow \beta_l^* \\
H^{n-b}(\mathbb{P}^n, \mathbb{P}^n \backslash X) & \xrightarrow{(j'^*)_{n-b}} & H^{n-b}(\mathbb{P}^n) \\
\uparrow P^* & & \uparrow P^* \\
H_b(X) & \xrightarrow{(\iota_*)_b} & H_b(\mathbb{P}^n)
\end{array}$$

where the maps ι_*, j^* and j'^* are those induced by inclusions and the P^*'s are Poincaré duality isomorphisms; commutativity follows from naturality of Poincaré duality. Since $\hat{\Omega} \approx C\Omega$, then it is contractible and $\beta_l : (\hat{\Omega} \times \mathbb{P}^n, \hat{B}) \to (\mathbb{P}^n, \mathbb{P}^n \backslash X)$ is a homotopy equivalence; hence all the vertical arrows are isomorphisms. Thus we identify $(\iota_*)_b$ with $(j^*)_{n-b}$.
Let now $\epsilon > 0$ be such that $\hat{B}(\epsilon) \hookrightarrow \hat{B}$ is a homotopy equivalence, where $\hat{B}(\epsilon) = \{(\hat{\omega}, x) \in \hat{\Omega} \times \mathbb{P}^n \,:\, (\hat{\omega}\hat{p})(x) > \epsilon\}$ (such ϵ exists by Lemma 3). Then the inclusion of pairs

$$(\hat{\Omega} \times \mathbb{P}^n, \hat{B}(\epsilon)) \xrightarrow{\hat{j}(\epsilon)} (\hat{\Omega} \times \mathbb{P}^n, \hat{B})$$

also is a homotopy equivalence and the inclusion $(\hat{\Omega} \times \mathbb{P}^n, \emptyset) \xrightarrow{j} (\hat{\Omega} \times \mathbb{P}^n, \hat{B})$ factors trough:

$$\begin{tikzcd}
(\hat{\Omega}\times\mathbb{P}^n,\emptyset) \arrow[rr, "j"] \arrow[dr, "j(\epsilon)"'] & & (\hat{\Omega}\times\mathbb{P}^n,\hat{B}) \\
& (\hat{\Omega}\times\mathbb{P}^n,\hat{B}(\epsilon)) \arrow[ur, "\hat{j}(\epsilon)"'] &
\end{tikzcd}$$

Since $\hat{j}(\epsilon)$ is a homotopy equivalence, it follows that:

$$\mathrm{rk}(j^*)_{n-b} = \mathrm{rk}(j(\epsilon)^*)_{n-b}.$$

Let now $\mathcal{W}$ be any cover of $\hat{\Omega}$ and $\mathcal{U} = \beta_l^{-1}\mathcal{W}$. Consider the Leray-Mayer-Vietoris bicomplexes $\hat{F}^{*,*}(\epsilon,\mathcal{U})$ and $K_0^{*,*}(\mathcal{U})$ with their respective associated spectral sequences; since $i_0(\epsilon,\mathcal{U}) : \hat{F}_0^{*,*}(\epsilon,\mathcal{U}) \hookrightarrow K_0^{*,*}(\mathcal{U})$ there is a morphism of respective spectral sequences. Moreover by Mayer-Vietoris argument, the spectral sequence $(\hat{F}_r(\epsilon,\mathcal{U}),\hat{d}_r(\epsilon,\mathcal{U}))_{r\geq 0}$ converges to $H^*(\hat{\Omega}\times\mathbb{P}^n,\hat{B}(\epsilon))$ and $(K_r(\mathcal{U}),d_r(\mathcal{U}))_{r\geq 0}$ converges to $H^*(\hat{\Omega}\times\mathbb{P}^n,\emptyset)$. We look now at the following commutative diagram:

$$\begin{tikzcd}
E_\infty^{0,n-b}(\epsilon,\mathcal{U}) \arrow[r, "(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}"] & K_\infty^{0,n-b}(\mathcal{U}) \\
H_D^{n-b}(E_0(\epsilon,\mathcal{U})) \arrow[u, "p_E(\epsilon,\mathcal{U})"] \arrow[r, "(i_0^*(\epsilon,\mathcal{U}))_{n-b}"] & H_D^{n-b}(K_0(\mathcal{U})) \arrow[u, "p_K"'] \\
H^{n-b}(\hat{\Omega}\times\mathbb{P}^n,\hat{B}) \arrow[u, "(f_E^*)^{-1}"] \arrow[r, "(j^*)_{n-b}"] & H^{n-b}(\hat{\Omega}\times\mathbb{P}^n) \arrow[u, "(f_K^*)^{-1}"']
\end{tikzcd}$$

The upper square is commutative, since if we let $\psi = \psi_0 + \cdots + \psi_{n-b} \in E_0^{n-b}$ with $D\psi = 0$, then (avoiding the $(\epsilon,\mathcal{U})$-notations, but only for the next formula):

$$p_K(i_0^*)_{n-b}[\psi]_E = p_K[\psi]_K = [\psi_0]_{\infty,K} = (i_\infty)_{0,n-b}[\psi_0]_{\infty,E} = (i_\infty)_{0,n-b}p_E[\psi]_E.$$

The lower square is the one coming from Lemma 19 with the vertical arrows inverted, hence it is commutative.
Since $K_\infty(\mathcal{U}) = K_2(\mathcal{U})$ has only one column (the first), then $p_K(\mathcal{U}) : H_D^{n-b}(K_0(\mathcal{U})) \to K_\infty^{0,n-b}(\mathcal{U})$ is an isomorphism, hence for $0 \leq b \leq n$ and using the above identifications we can identify the map $(j^*)_{n-b}$ with

$$(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}(p_E(\epsilon,\mathcal{U}))_{n-b} : H_D^{n-b}(E_0(\epsilon,\mathcal{U})) \to \mathbb{Z}_2.$$

Since $(p_E(\epsilon,\mathcal{U}))_{n-b}$ is surjective, then:

$$\mathrm{rk}(j^*)_{n-b} = \mathrm{rk}(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}.$$

By Corollary 21 and Lemma 22 there exists a family $\mathcal{C}$ of covers which is cofinal in the family of all covers such that for every $\mathcal{U} \in \mathcal{C}$ the natural map $\hat{F}_2^{i,j}(\epsilon,\mathcal{U}) \to$

$\hat{F}_2^{i,j}(\epsilon)$ is an isomorphism. It follows that $\mathrm{rk}(i_\infty(\epsilon,\mathcal{U})_{0,n-b}) = \mathrm{rk}(i_\infty(\epsilon))_{0,n-b}$, and thus by semialgebraicity we have

$$\mathrm{rk}(i_\infty(\epsilon))_{0,n-b} = \mathrm{rk}(i_\infty)_{0,n-b}.$$

It remains to study the map $(i_\infty)_{0,n-b} : E_\infty^{0,n-b} \to K_\infty^{0,n-b} = K_2^{0,n-b}$.
If $E_\infty^{0,n-b}$ is zero, then $(i_\infty)_{0,n-b}$ is obviously an isomorphism onto its image.
If $E_\infty^{0,n-b}$ is not zero then, since $E_2^{0,n-b} = H^0(C\Omega, \Omega^{n-b+1})$, it must be $\Omega^{n-b+1} = \emptyset$ and thus $\hat{\Omega}^{n-b+1} = \emptyset$ and

$$E_\infty^{0,n-b} = E_2^{0,n-b} = \mathbb{Z}_2.$$

From this it follows that

$$i_\infty^{0,n-b} = i_2^{0,n-b}.$$

By the definition of the two spectral sequences as direct limits, for ϵ sufficiently small and $\mathcal{U}$ an f_ϵ regular cover, we see that $i_2(\epsilon,\mathcal{U})^{0,n-b}$ is the identity and thus also $i_2^{0,n-b} : H^0(\hat{\Omega}, \emptyset) \to H^0(\hat{\Omega}) \otimes H^{n-b}(\mathbb{P}^n)$ is the identity and then the conclusion follows. □

Remark 4. Since here we do not need the cover to be convex, the existence of the family $\mathcal{C}$ follows from easier consideration. Let $h : \hat{\Omega} \to |K| \subset \mathbb{R}^N$ be a triangulation respecting the filtration $\{\hat{\Omega}_j\}_{j=0}^{n+2}$, and $\mathcal{W}$ be a cover of $\hat{\Omega}$. Let $\mathcal{V}'$ be a *convex* cover of $|K|$ refining $h(\mathcal{W})$ and such that for every $U' \in \mathcal{V}'$ the intersection $h(\hat{\Omega}_j) \cap U'$ is contractible for every j (the existence of such a $\mathcal{V}'$ follows from the fact that $h(\hat{\Omega}_j)$ is a subcomplex of $|K|$). Then the cover $\mathcal{V} = h^{-1}(\mathcal{V}')$ refines $\mathcal{W}$ and since for every j and $U \in \mathcal{V}$ the intersection $\hat{\Omega}_j \cap U$ is contractible, then the natural map $\hat{F}_2^{i,j}(\epsilon, \beta_l^{-1}\mathcal{V}) \to \hat{F}_2^{i,j}(\epsilon)$ is an isomorphism.

We can immediately derive the following elementary corollary

Corollary 29. *If $b > n - \mu$ then $(j_*)_b = 0$.*

Proof. Since $n - b < \mu$ then $\Omega^{n-b+1} \neq \emptyset$. This gives $E_2^{0,n-b} = 0$ and thus applying the previous theorem the conclusion follows. □

6. Hyperplane section

We consider here the following problem: given $X \subset \mathbb{P}^n$ defined by quadratic inequalities and V a codimension one subspace of $\mathbb{R}^{n+1}$ with projectivization $\bar{V} \subset \mathbb{P}^n$, determine the homology of $(X, X \cap \bar{V})$.
Thus let $p : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} \supseteq K$ be homogeneous quadratic and $X = p^{-1}(K) \subset \mathbb{P}^n$.
Let h be a degree one homogeneous polynomial such that

$$V = \{h = 0\} = \{h^2 = 0\}.$$

We can consider the function $\mathrm{i}_V^+ : \Omega \to \mathbb{N}$ defined by

$$\mathrm{i}_V^+(\omega) = \mathrm{i}^+(\omega p|_V)$$

and we try describe the homology of $(X, X \cap \bar{V})$ only in terms of i^+ and i_V^+.
We introduce the quadratic map $p_h : \mathbb{R}^{n+1} \to \mathbb{R}^{k+2}$ defined by

$$p_h \doteq (p, h^2).$$

Then we have the following equalities:

$$X = p_h^{-1}(K \times \mathbb{R}) \quad \text{and} \quad X \cap \bar{V} = p_h^{-1}(K \times (-\infty, 0]).$$

We consider $\hat{\Omega} = (K \times (-\infty, 0])^\circ \cap S^{k+1}$, and the function $\mathrm{i}_h^+ : \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{N}$ defined by

$$\mathrm{i}_h^+(\omega, t) = \mathrm{i}^+(\bar{p}_h(\omega, t)) = \mathrm{i}^+(\omega p + th^2), \quad (\omega, t) \in \mathbb{R}^{k+1} \times \mathbb{R}.$$

For the moment we define, for $j \in \mathbb{Z}$ the set

$$\hat{\Omega}^{j+1} = \{\eta \in \hat{\Omega} \, : \, \mathrm{i}_h^+(\eta) \geq j+1\}$$

and we identify Ω with $\{(\omega, t) \in \hat{\Omega} \, : \, t = 0\}$.
With the previous notations we prove the following.

Lemma 30. *There exists a cohomology spectral sequence (G_r, d_r) of the first quadrant converging to $H_{n-*}(X, X \cap \bar{V})$ such that*

$$G_2^{i,j} = H^i(\hat{\Omega}^{j+1}, \Omega^{j+1}).$$

Proof. Consider for $\epsilon > 0$ the sets $C_h(\epsilon) = \{(\eta, x) \in \hat{\Omega} \times \mathbb{P}^n \, : \, (\eta p_h)(x) \geq \epsilon\}$ and $C(\epsilon) = C_h(\epsilon) \cap \Omega \times \mathbb{P}^n$. By Lemma 3 for small ϵ the inclusion

$$(C_h(\epsilon), C(\epsilon)) \hookrightarrow (B_h, B)$$

is a homotopy equivalence (here B_h stands for $\{(\eta, x) \in \hat{\Omega} \times \mathbb{P}^n \, : \, (\eta p_h)(x) > 0\}$ and B for $B_h \cap \Omega \times \mathbb{P}^n$).
Consider the projection $\beta_r : \hat{\Omega} \times \mathbb{P}^n \to \mathbb{P}^n$; then $\beta_r(B_h) = \mathbb{P}^n \backslash (X \cap H)$ and $\beta_r(B) = \mathbb{P}^n \backslash X$; moreover by Lemma 24 the previous are homotopy equivalences. Hence it follows:

$$H^*(C_h(\epsilon), C(\epsilon)) \simeq H^*(B_h, B) \simeq H^*(\mathbb{P}^n \backslash (X \cap H), \mathbb{P}^n \backslash X) \simeq H_{n-*}(X, X \cap H)$$

where the last isomorphism is given by Alexander-Pontryagin Duality. Consider now $\beta_l : C_h(\epsilon) \to \hat{\Omega}$. Then by Theorem 18 there is a cohomology spectral sequence $(G_r(\epsilon), d_r(\epsilon))$ converging to $H^*(C_h(\epsilon), C(\epsilon))$ such that

$$G_2^{i,j} = \check{H}^i(\hat{\Omega}, \mathcal{G}^j(\epsilon))$$

where $\mathcal{G}^j(\epsilon)$ is a sheaf such that for $\eta \in \hat{\Omega}$

$$(\mathcal{G}^j(\epsilon))_\eta = H^j(\beta_l^{-1}(\eta) \cap C_h(\epsilon), \beta_l^{-1}(\eta) \cap C(\epsilon))$$

(here, reasoning as in Remark 2, we are using the fact that both $C_h(\epsilon)$ and $C(\epsilon)$ are compact). We use now $\mathrm{i}_h^-(\epsilon) : \hat{\Omega} \to \mathbb{N}$ for the function $\eta \mapsto \mathrm{i}^-(\eta p_h - \epsilon g)$ where g is an arbitrary positive definite form, and we set $\hat{\Omega}_{n-j}(\epsilon) = \{\mathrm{i}_h^-(\epsilon) \leq n-j\}$. If $\eta \notin \Omega$, then $(\beta_l^{-1}(\eta) \cap C_h(\epsilon), \beta_l^{-1}(\eta) \cap C(\epsilon)) \simeq (\mathbb{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)}, \emptyset)$; on the contrary if $\eta \in \Omega$ then $(\beta_l^{-1}(\eta) \cap C_h(\epsilon), \beta_l^{-1}(\eta) \cap C(\epsilon)) = (\mathbb{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)}, \mathbb{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)})$. Since Ω is closed in $\hat{\Omega}$, it follows that

$$G_2^{j,j}(\epsilon) = \check{H}^i(\hat{\Omega}_{n-j}(\epsilon), \Omega_{n-j}(\epsilon)).$$

We define now

$$(G_r, d_r) = \varprojlim_{\epsilon} \{(G_r(\epsilon), d_r(\epsilon))\}$$

and Lemma 26 finally gives

$$G_2^{i,j} = H^i(\hat{\Omega}^{j+1}, \Omega^{j+1}).$$

□

We are ready now for the proof of Theorem D; we define for $j>0$ the following set:

$$\Omega_V^j=\{\omega\in\Omega:\mathrm{i}^+(\omega p|_V)\geq j\}.$$

Theorem D. *There exists a cohomology spectral sequence (G_r,d_r) of the first quadrant converging to $H_{n-*}(X_p,X_p\cap\bar V)$ such that*

$$G_2^{i,j}=H^i(\Omega_V^j,\Omega^{j+1}),\ j>0,\quad G_2^{i,0}=H^i(C\Omega,\Omega^1).$$

Proof. Take the spectral sequence (G_r,d_r) to be that of lemma 30; then it remains to prove that $G_2^{i,j}$ is isomorphic to the group described in the statement.
In the case $j=0$ we have that $\hat\Omega^1$ contains $(0,\ldots,0,1)$ and, since $t_1\leq t_2$ implies $\mathrm{i}_h(\omega,t_1)\leq\mathrm{i}_h^+(\omega,t_2)$, the set $\hat\Omega^1$ is contractible. Thus, using the long exact sequences of the pairs, we see that for every $i\geq 0$ the following holds:

$$G_2^{i,0}=H^i(\hat\Omega^1,\Omega^1)\simeq H^i(C\Omega,\Omega^1).$$

We study now the case $j>0$.
We identify $\hat\Omega\backslash\{(0,\ldots,0,1)\}$ with $\Omega\times[0,\infty)$ via the index preserving homeomorphism

$$(\omega,t)\mapsto(\omega,t)/\|\omega\|.$$

Thus, under the above identification, we have for $j>0$

$$\hat\Omega^{j+1}=\{(\omega,t)\in\Omega\times[0,\infty)\ :\ \mathrm{i}_h^+(\omega,t)\geq j+1\}$$

and letting $\pi:\Omega\times[0,\infty)$ be the projection onto the first factor, we see that

$$\pi(\hat\Omega^{j+1})=\{\omega\ :\ \exists t>0\ \text{ s.t. }\ \mathrm{i}_h^+(\omega,t)\geq j+1\}.$$

We prove that $\pi:\hat\Omega^{j+1}\to\pi(\hat\Omega^{j+1})$ is a homotopy equivalence. Let $\omega\in\pi(\hat\Omega^{j+1})$, then there exists $t_\omega>0$ such that $(\omega,t_\omega)\in\hat\Omega^{j+1}$. Since $\hat\Omega^{j+1}$ is open, then there exists an open neighboroud $U_\omega\times(t_1,t_2)$ of (ω,t) in $\hat\Omega^{j+1}$; in particular for every $\eta\in U_\omega$ we have $(\eta,t_\omega)\in\hat\Omega^{j+1}$ and $\sigma_\omega:\eta\mapsto(\eta,t_\omega)$ is a section of π over U_ω. Collating together the different σ_ω for $\omega\in\pi(\hat\Omega^{j+1})$, with the help of a partition of unity, we get a section $\sigma:\pi(\hat\Omega^{j+1})\to\hat\Omega^{j+1}$ of π. Since for every $\omega\in\pi(\hat\Omega^{j+1})$ the set $\{t\geq 0\ :\ (\omega,t)\in\hat\Omega^{j+1}\}$ is an interval, a straight line homotopy gives the homotopy between $\sigma\circ\pi$ and the identity on $\hat\Omega^{j+1}$. This implies $\pi:\hat\Omega^{j+1}\to\pi(\hat\Omega^{j+1})$ is a homotopy equivalence. Using the five lemma and the naturality of the commutative diagrams of the long exact sequences of pairs given by $\pi:(\hat\Omega^{j+1},\Omega^{j+1})\to(\pi(\hat\Omega^{j+1}),\Omega^{j+1})$ we get ($\pi_{|\Omega^{j+1}}=\mathrm{Id}_{|\Omega^{j+1}}$):

$$G_2^{i,j}=H^i(\hat\Omega^{j+1},\Omega^{j+1})\simeq H^i(\pi(\hat\Omega^{j+1}),\Omega^{j+1}).$$

It remains to prove that for $j>0$

$$\pi(\hat\Omega^{j+1})=\Omega_V^j.$$

First suppose that $(\omega,t)\in\hat\Omega^{j+1}$. Then there exists a subspace W^{j+1} of dimension at least $j+1$ such that $\bar p(\omega,t)|_{W^{j+1}}>0$. Then

$$\omega p|_{W^{j+1}\cap V}=\bar p(\omega,t)|_{W^{j+1}\cap V}>0$$

and by Grassmann formula

$$\dim(W^{j+1}\cap V)=\dim(W^{j+1})+\dim(V)-\dim(W^{j+1}+V)\geq j$$

which implies $\mathrm{i}^+_V(\omega) \geq j$, i.e. $\pi(\omega,t) \in \Omega^j_V$. Thus

$$\pi(\hat{\Omega}^{j+1}) \subset \Omega^j_V.$$

Now let ω be in Ω^j_V; we prove that there exists $t > 0$ such that $\mathrm{i}^+_h(\omega,t) \geq j+1$. Since $\omega \in \Omega^j_V$ then there exists a subspace $V^j \subset V$ of dimension at least j such that

$$\omega p|_{V^j} > 0.$$

Fix a scalar product on $\mathbb{R}^{n+1}$ and let $e \in \mathbb{R}^{n+1}$ be such that $V^\perp = span\{e\}$; consider the space $W = \{\lambda e\}_{\lambda \in \mathbb{R}} + V^j$, whose dimension is at least $j+1$ since $e \perp V^j \subset V$. Then the matrix for $\bar{p}_h(\omega,t)|_W$ with respect to the fixed scalar product has the form:

$$Q_W(\omega,t) = \begin{pmatrix} \omega a_0 + t & {}^t\omega a \\ \omega a & \omega Q_{V^j} \end{pmatrix}$$

where ωQ_{V^j} is the matrix for $\bar{p}(\omega,t)|_{V^j} = \omega p|_{V^j}$. Since $\omega p|_{V^j} > 0$ we have that for $t > 0$ big enough $\det(Q_W(\omega,t)) = t\det(\omega Q_{V^j}) + \det\left(\begin{smallmatrix} \omega a_0 & \omega a \\ \omega a & \omega Q_{V^j} \end{smallmatrix}\right)$ has the same sign of $\det(\omega Q_{V^j}) > 0$. For such a t we have

$$\bar{p}_h(\omega,t)|_W > 0$$

and since $\dim(W) \geq j+1$ this implies $(\omega,t) \in \hat{\Omega}^{j+1}$ and $\omega \in \pi(\hat{\Omega}^{j+1})$. Thus

$$\Omega^j_V \subset \pi(\hat{\Omega}^{j+1})$$

and this concludes the proof.

□

7. Remarks on higher differentials and examples

Let $X \subset \mathbb{P}^n$ be a compact, locally contractible subset and consider the two inclusions:

$$X \xrightarrow{j} \mathbb{P}^n \quad \text{and} \quad \mathbb{P}^n \backslash X \xrightarrow{c} \mathbb{P}^n.$$

We recall the existence for every $k \in \mathbb{Z}$ of the following exact sequence, which is a direct consequence of Alexander-Pontryagin Duality:

$$0 \to \ker(c_*) \to H_k(\mathbb{P}^n \backslash X) \xrightarrow{c_*} H_k(\mathbb{P}^n) \simeq H^{n-k}(\mathbb{P}^n) \xrightarrow{i^*} H^{n-k}(X) \to \mathrm{coker}(i^*) \to 0$$

In particular we have the following equality for the k-th $\mathbb{Z}_2$-Betti number of $\mathbb{P}^n$:

$$b_k(\mathbb{P}^n) = \mathrm{rk}(c^*)_k + \mathrm{rk}(j_*)_{n-k} \tag{7}$$

Consider now $p : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} \supseteq K$ such that

$$\mathrm{i}^+(\bar{p}\eta) = \mu \quad \forall \eta \in \Omega.$$

Then in this case $\Omega^1 = \cdots = \Omega^\mu = \Omega$ and $\Omega^{\mu+1} = \cdots = \Omega^{n+1} = \emptyset$. For *any* scalar product g on $\mathbb{R}^{n+1}$ we have $D_\mu = \Omega^\mu = \Omega$ and we denote by $w_{k,\mu}$ the k-th Stiefel-Whitney class of the $\mathbb{R}^\mu$-bundle $\bar{p}^*\mathcal{L}^+_j \to \Omega$ (notice that this class does not depend on g). We define $\gamma_{k,\mu} \in H^{k+1}(C\Omega,\Omega) \simeq H^k(\Omega)$ by

$$\gamma_{k,\mu} \doteq \partial^* w_{k,\mu}$$

(notice that this notation agrees with the one previously used for $\gamma_{1,j}$).
Letting (E_r, d_r) be the spectral sequence of Theorem A convergent to $H_{n-*}(X)$, where as usual $X = p^{-1}(K) \subseteq \mathbb{P}^n$, we have that (E_r, d_r) degenerates at $(k+2)$-th

step and $E_2 = \cdots = E_{k+1}$. Moreover E_{k+1} has entries only in the 0-th and the $(k+1)$-th column:

$$E_{k+1}^{a,b} = \begin{cases} \mathbb{Z}_2 & \text{if } a = 0 \text{ and } \mu \le b \le n \text{ or} \\ & a = k+1 \text{ and } 0 \le b < \mu \\ 0 & \text{otherwise} \end{cases}$$

Thus the only possible nonzero differential is d_{k+1}, for which we prove the following.

Theorem 31. *Suppose* $\mathrm{i}^+ \equiv \mu$*. Then* $E_2 = \cdots = E_{k+1}$ *and the only possible nonzero differential is* $d_{k+1} : E_{k+1}^{0,b} \to E_{k+1}^{k+1,b-k}$ *for* $\mu \le b \le n$ *and it is given by:*

$$d_{k+1}(x) = x \smile \gamma_{k,\mu}$$

Remark 5. Notice that $\gamma_{k,\mu}$ and x are nothing but numbers modulo 2, thus since $E_{k+1}^{0,b} = \mathbb{Z}_2 = E_{k+1}^{k+1,b-k}$ the element $d_{k+1}(x)$ is nothing but the product $x\gamma_{k,\mu}$.

Proof. By Theorem C we have that $d_{k+1} : E_{k+1}^{0,b} \to E_{k+1}^{k+1,b-k}$ is identically zero if and only if $\mathrm{rk}(j_*)_{n-b} = 1$ and formula (7) implies

$$(d_{k+1})_{0,b} \equiv 0 \quad \text{iff} \quad \mathrm{rk}(c^*)_b = 0$$

where c^* is the map induced by $c : \mathbb{P}^n \backslash X \hookrightarrow \mathbb{P}^n$. Consider now the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[d, "\beta_{r|B}"'] \arrow[dr, "\beta_r \circ \iota"] & \Omega \times \mathbb{P}^n \arrow[d, "\beta_r"] \\ \mathbb{P}^n \backslash X \arrow[r, "c"] & \mathbb{P}^n \end{tikzcd}$$

Since $\beta_{r|B}$ is a homotopy equivalence, then

$$\mathrm{rk}(c^*)_b = \mathrm{rk}(\iota^* \beta_r^*)_b.$$

Let $\mathbb{P}^{\mu-1} \hookrightarrow P(\bar{p}^*\mathcal{L}_\mu) \to \Omega$ be the projectivization of the bundle $\mathbb{R}^\mu \hookrightarrow \bar{p}^*\mathcal{L}_\mu \to \Omega$. It is easily seen that the inclusion

$$P(\bar{p}^*\mathcal{L}_\mu) \hookrightarrow B$$

is a homotopy equivalence. From this, letting $l : P(\bar{p}^*\mathcal{L}_\mu) \to \mathbb{P}^n$ be the restriction of $\beta_r \circ \iota$ to $P(\bar{p}^*\mathcal{L}_\mu)$, it follows that:

$$\mathrm{rk}(c^*)_b = \mathrm{rk}(l^*)_b.$$

Let $y \in H^1(\mathbb{P}^n)$ be the generator; since l is a linear embedding on each fiber, then by Leray-Hirsch, it follows that

$$H^*(P(\bar{p}^*\mathcal{L}_\mu)) = H^*(\Omega) \otimes \{1, l^*y, \ldots, (l^*y)^{\mu-1}\}.$$

Thus for $\mu \le b \le n$ we have:

$$\begin{aligned} l^*y^b =& (l^*y)^b = (l^*y)^\mu \smile (l^*y)^{b-\mu} \\ =& \beta_l^* w_{k,\mu} \smile (l^{*y})^{\mu-k} \smile (l^*y)^{b-\mu} \\ =& \beta_l^* w_{k,\mu} \smile (l^*y)^{b-k}. \end{aligned}$$

Thus $(d_{k+1})_{0,b}$ is zero if and only if $w_{k,\mu}=0$ and by looking at the definition of $\gamma_{k,\mu}$ we see that

$$d_{k+1}(x)=x\smile\gamma_{k,\mu}.$$

□

Example 2 (The case of one quadric). This is the most elementary example we can consider, namely the homology of a single quadric in $\mathbb{P}^n$. Let $q\in\mathcal{Q}$ be a quadratic form on $\mathbb{R}^{n+1}$ with signature (a,b) with $a\leq b$ (otherwise we can replace q with $-q$) and $a+b=\mathrm{rk}(q)\leq n+1$. Consider

$$X_{a,b}=\{q=0\}\subset\mathbb{P}^n.$$

For example, in the case q is nondegenerate (i.e. $a+b=n+1$) then $X_{a,b}$ is smooth and $S^{a-1}\times S^{b-1}$ is a double cover of it.
Define the two vectors $h^-(X_{a,b}),h^+(X_{a,b})\in\mathbb{N}^n$ by:

$$h^-(X_{a,b})=(\underbrace{1,\ldots,1}_{n+1-b},0,\ldots,0),\quad h^+(X_{a,b})=(0,\ldots,0,\underbrace{1,\ldots,1}_{a}).$$

Then a straightforward application of Theorem A gives the following identity for the array whose components are the $\mathbb{Z}_2$-Betti numbers of $X_{a,b}$:

$$(b_0(X_{a,b}),\ldots,b_n(X_{a,b}))=h^-(X_{a,b})+h^+(X_{a,b}).$$

Moreover if we let $j:X_{a,b}\to\mathbb{P}^n$ be the inclusion, then Theorem C gives the following:

$$(\mathrm{rk}(j_*)_0,\ldots,\mathrm{rk}(j_*)_n)=h^-(X_{a,b}).$$

Example 3 (The case of two quadrics). In the case $p=(q_1,q_2)$ and i^+ not constant, then the spectral sequence of Theorem A degenerates at the second step and $E_2=E_\infty$. In the case of constant positive index we can use Theorem 31 to find $H_*(p^{-1}(K))$ (notice that $K\neq\{0\}$ again implies $E_2=E_\infty$.)

Example 4 (see [14]). For $a=1,2,4,8$ consider the isomorphism $\mathbb{R}^a\simeq A$ where A denotes respectively $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. Consider the quadratic map

$$h_a:\mathbb{R}^a\oplus\mathbb{R}^a\to\mathbb{R}^a\oplus\mathbb{R}$$

defined, using the previous identification $\mathbb{R}^a\simeq A$, by

$$(z,w)\mapsto(2z\overline{w},|w|^2-|z|^2).$$

Then it is not difficult to prove that h_a maps S^{2a-1} into S^a by a Hopf fibration. Hence it follows that

$$\emptyset=K_a\doteq h_a^{-1}(0)\subset\mathbb{P}^{2a-1}.$$

In each case we have $\mathrm{i}^+(\omega h_a)=a$ for every $\omega\in\Omega=S^a$. Using Theorem 31, since $K_a=\emptyset$ then d_{a+1} must be an isomorphism, hence

$$0\neq w_{a,a}=w_a(\bar{h}_a^*\mathcal{L}_a)\in H^a(S^a).$$

For example in the case $a=2$ we have the standard Hopf fibration $h_2|_{S^3}:S^3\to S^2$ and the table for $E_2=E_3$ is:

$$\begin{array}{|c|c|c|c}
\mathbb{Z}_2 & 0 & 0 & 0\\
\mathbb{Z}_2 & 0 & 0 & 0\\
0 & 0 & 0 & \mathbb{Z}_2\\
0 & 0 & 0 & \mathbb{Z}_2\\
\hline
\end{array}$$

The bundle $\mathbb{R}^2 \hookrightarrow \bar{h}_a^*\mathcal{L}_2 \to S^2$ has total Stiefel-Whitney class

$$w(\bar{h}_a^*\mathcal{L}_2) = 1 + w_{2,2}, \quad w_{2,2} \neq 0$$

and the differential d_3 is an isomorphism.
Notice that for $a = 1, 2, 4, 8$ we have $\ker(\omega h_a) = 0$ for every $\omega \in \Omega$. It is an interesting fact that the contrary also is true.

Fact 1. if $p : \mathbb{R}^m \to \mathbb{R}^l$ is such that $\ker(\omega p) = \{0\}$ for every $\omega \in S^l$ and $p_{|S^{m-1}} : S^{m-1} \to S^{l-1}$ then, up to orthonormal change of coordinates $p = h_a$ for some $a \in \{1, 2, 4, 8\}$.

Proof. First observe that $\mathrm{i}^+ \equiv c$ for a constant c and that $m = 2c$. Then, since p maps the sphere S^{2c-1} to the sphere S^{l-1}, we have

$$\emptyset = p^{-1}(\{0\}) \subset \mathbb{P}^{2c-1}.$$

Thus Theorem 31 implies that the differential d_l must be an isomorphism and this forces $l = c + 1$. Moreover the condition $\ker(\omega p) = \{0\}$ for every $\omega \in S^{c-1}$ says also $p_{|S^{2c-1}} : S^{2c-1} \to S^c$ is a submersion. It is a well-known result (see [15]) that the preimage of a point trough a quadratic map between spheres is a sphere, and thus $p_{|S^{2c-1}}$ is the projection of a sphere-bundle between spheres, hence it must be a Hopf fibration. $\square$

The situation in the case $\{\omega \in S^{l-1} : \ker(\omega p) \neq 0\} = \emptyset$ with only the assumption $X = \emptyset$ (which is weaker than $p(S^{m-1}) \subset S^{l-1}$) is more delicate.

Example 5. For $i = 1, \ldots, l$ let $p_i : \mathbb{R}^{n_i} \to \mathbb{R}^{k+1}$ be a quadratic map and set $N = \sum_i n_i$. Define the map

$$\oplus_i p_i : \mathbb{R}^N \to \mathbb{R}^{k+1}$$

by the formula

$$(x_1, \ldots, x_l) \mapsto \sum_{i=1}^{l} p_i(x_i) \quad x_i \in \mathbb{R}^{n_i}.$$

Then for every $\omega \in S^k$ we have

$$\mathrm{i}^+(\omega(\oplus_i p_i)) = \sum_{i=1}^{l} \mathrm{i}^+(\omega p_i).$$

In particular if each p_i has constant positive index function with constant value μ_i, then $\oplus_i p_i$ has also constant positive index function with constant value $\sum_i \mu_i$.
Generalizing the previous example, we consider now for $a = 1, 2, 4, 8$ the map $h_a : \mathbb{R}^{2a} \to \mathbb{R}^{a+1}$ defined above and we take for $n \in \mathbb{N}$ the map

$$n \cdot h_a \doteq \oplus_{i=1}^n h_a : \mathbb{R}^{2an} \to \mathbb{R}^{a+1}.$$

In coordinate the map $n \cdot h_a$ is written by:

$$(w, z) \mapsto (2\langle z, w\rangle, \|w\|^2 - \|z\|^2), \quad w, z \in A^n.$$

Then for this map we have

$$\mathrm{i}^+ \equiv na, \quad \text{and} \quad (n \cdot \bar{h}_a)^*\mathcal{L}_{na} = n(\bar{h}_a^*\mathcal{L}_a) = \underbrace{\bar{h}_a^*\mathcal{L}_a \oplus \cdots \oplus \bar{h}_a^*\mathcal{L}_a}_{n}$$

The solution of $\{n \cdot h_a = 0\}$ on the sphere S^{2a-1} is diffeomorphic to the Stiefel manifold of 2-frames in A^n, and it is a double cover of

$$n \cdot K_a \doteq \{n \cdot h_a = 0\} \subset \mathbb{P}^{2na-1}.$$

We can proceed now to the calculation of the $\mathbb{Z}_2$-cohomology of $n \cdot K_a$, using Theorem 31: we only need to compute d_{a+1}, i.e. $w_a(n\bar{h}_a^*\mathcal{L}_a)$. Since $w_a(\bar{h}_a^*\mathcal{L}_a) = w_{a,a} \neq 0$, and $w_k(\bar{h}_a^*\mathcal{L}_a) = 0$ for $k \neq 0, k \neq a$, then we have

$$w_a(n\bar{h}_a^*\mathcal{L}_a) = n \bmod 2 \in \mathbb{Z}_2 = H^a(S^a).$$

There are some cases in which the problem of describing the index function can be reduced to a simpler problem; this is the case of a quadratic map defined by a bilinear one. We start noticing the following.

Fact 2. Let L be a $n \times n$ real matrix and Q_L be the symmetric $2n \times 2n$ matrix defined by:

$$Q_L = \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix}$$

Then, setting q_L for the quadratic form defined by $x \mapsto \langle x, Q_L x\rangle$ we have:

$$\mathrm{i}^+(q_L) = \mathrm{rk}(L).$$

Proof. Let $x = (z, w) \in \mathbb{R}^{2n} \simeq \mathbb{R}^n \oplus \mathbb{R}^n$; then $Q_L \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} Lw \\ {}^tLz \end{smallmatrix}\right)$. Hence $\ker Q_L = \ker {}^tL \oplus \ker L$ and

$$\dim(\ker Q_L) = 2\dim(\ker L).$$

Consider now the characteristic polynomial f of Q_L:

$$f(t) = \det(Q_L - tI) = \det(t^2 I - {}^tLL) = (-1)^n \det({}^tLL - t^2 I) = (-1)^n g(t^2)$$

where g is the characteristic polynomial of ${}^tLL$. Let now $\lambda \in \mathbb{R}$ be such that $g(\lambda) = 0$; since ${}^tLL \geq 0$, then $\lambda \geq 0$ and $f(\pm\sqrt{\lambda}) = 0$. Since Q_L is diagonalizable, then for each one of its eigenvalues algebraic and geometric multiplicity coincide, hence

$$\mathrm{i}^+(q_L) = \mathrm{i}^-(q_L) = \frac{1}{2}\mathrm{rk}(Q_L).$$

It follows that

$$\mathrm{i}^+(q_L) = \frac{1}{2}(2n - \dim(\ker Q_L)) = \mathrm{rk}(L).$$

□

In particular if $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{k+1}$ is a bilinear antisymmetric map whose components are defined by

$$(x, y) \mapsto \langle (x, y), \left(\begin{smallmatrix} 0 & B_i \\ {}^tB_i & 0 \end{smallmatrix}\right)(x, y)\rangle$$

for certain real squared matrices B_i, $i = 1, \ldots, k+1$, then we can consider the quadratic map

$$p_b : \mathbb{R}^{2n} \to \mathbb{R}^{k+1}$$

defined by $(x, y) \mapsto b(x, y)$. In this case we define for $\omega \in S^k$ the matrix ωB by

$$\omega B = \omega_1 B_1 + \cdots + \omega_{k+1} B_{k+1}.$$

By the previous fact we have

$$\mathrm{i}^+(\omega p_b) = \mathrm{rk}(\omega B).$$

Example 6. Let $\mathbb{R}^8$ be identified with the space of pairs of 2×2 real matrices. We apply the previous consideration to describe the topology of

$$\Gamma=\{(X,Y)\in\mathbb{R}^8\,:\,[X,Y]=0\}.$$

Since the equation for Γ are homogeneous, it is a cone, and we can study the homology of its projectivization

$$\mathbb{P}(\Gamma)\subset\mathbb{P}^7.$$

If we define $V=\{(X,Y)\in\mathbb{R}^8\,:\,\mathrm{tr}(X)=\mathrm{tr}(Y)=0\}$ and $\Gamma_V=\Gamma\cap V$, then it is readily seen that

$$\Gamma=\Gamma_V\oplus\mathbb{R}^2.$$

We proceed first to the computation of $H_*(\mathbb{P}(\Gamma_V))$ using the above theorems.
In coordinates $(X,Y)=(\left(\begin{smallmatrix} x & y\\ z & -x\end{smallmatrix}\right),\left(\begin{smallmatrix} w & t\\ s & -w\end{smallmatrix}\right))$ we have

$$\{[X,Y]=0\}\cap V=\{tz-ys=xt-yw=sx-wz=0\}.$$

Consider the following matrices

$$B_1=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad B_2=\begin{pmatrix}0&0&1\\-1&0&0\\0&0&0\end{pmatrix},\quad B_3=\begin{pmatrix}0&1&0\\0&0&0\\-1&0&0\end{pmatrix}$$

and the bilinear map $b:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ whose components are $(x,y)\mapsto\langle x,B_iy\rangle$. Then $p_b:V\to\mathbb{R}^3$ equals the quadratic map defined by $(X,Y)\mapsto[X,Y]$ (we are using the above notations for the quadratic map p_b defined by a bilinear map b). It follows that

$$\Gamma_V=V\cap\Gamma=\{p_b=0\}.$$

Using ωB for the matrix $\omega_1B_1+\omega_2B_2+\omega_3B_3$, then by the previous fact we have

$$\mathrm{i}^+(\omega p_b)=\mathrm{rk}(\omega B)\ \ \forall\omega\in S^2.$$

Let ωQ_b the symmetric matrix associated to ωp_b by the rule $(\omega p_b)(x)=\langle x,\omega Q_b x\rangle$. Then

$$\omega Q_b=\begin{pmatrix}0&\omega B\\ {}^t\omega B&0\end{pmatrix}$$

The matrix ωB, for $\omega=(\omega_1,\omega_2,\omega_3)\in S^2$ has the following form:

$$\begin{pmatrix}0&\omega_3&\omega_2\\-\omega_2&-\omega_1&0\\-\omega_3&0&\omega_1\end{pmatrix}$$

and we immediatly see that $\mathrm{rk}(\omega B)=2$ for $\omega\neq 0$; this gives

$$\mathrm{i}^+(\omega p_b)=2\quad\forall\omega\in S^2.$$

Since $\mathrm{i}^+\equiv 2$, we can apply Theorem 31; letting (E_r,d_r) be the spectral sequence of Theorem A converging to $H_{n-*}(\mathbb{P}(\Gamma_V))$, we have the following picture for $E_2=E_3$:

$$\begin{array}{|c|c|c|c}
\mathbb{Z}_2 & 0 & 0 & 0\\
\mathbb{Z}_2 & 0 & 0 & 0\\
\mathbb{Z}_2 & 0 & 0 & 0\\
\mathbb{Z}_2 & 0 & 0 & 0\\
0 & 0 & 0 & \mathbb{Z}_2\\
0 & 0 & 0 & \mathbb{Z}_2\\
\hline
\end{array}$$

Consider the section $\sigma : S^2 \to S^2 \times \mathbb{R}^6$ defined for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ by:

$$\sigma(\omega) = (\omega_2, 0, \omega_1, -\omega_1\omega_3, \omega_2\omega_3, \omega_1^2 + \omega_2^2).$$

Since for every $\omega \in S^2$

$$(\omega Q_b)\sigma(\omega) = \sigma(\omega)$$

then it follows that σ is a section of the bundle $\bar{p}_b^*\mathcal{L}_2$. The index sum of the zeroes of σ (which occur only at $(0,0,1), (0,0,-1) \in S^2$) is even, thus the euler class e of $\bar{p}_b^*\mathcal{L}_2$ is even. This implies

$$w_2(\bar{p}_b^*\mathcal{L}_2) = e \bmod 2 = 0.$$

Thus by Theorem 31 we have $d_3 \equiv 0$ and $E_2 = E_3 = E_\infty$. It follows that the only nonzero homology groups of $\mathbb{P}(\Gamma_V)$ are:

$$H_0(\mathbb{P}(\Gamma_V)) = H_3(\mathbb{P}(\Gamma_V)) = \mathbb{Z}_2 \quad \text{and} \quad H_1(\mathbb{P}(\Gamma_V)) = H_2(\mathbb{P}(\Gamma_V)) = (\mathbb{Z}_2)^2.$$

Actually since the equations for Γ_V are given by the vanishing of the minors of the matrix $\left(\begin{smallmatrix} x & z & y \\ w & s & t \end{smallmatrix}\right)$, then Γ_V is the Segre variety $\Sigma_{2,1} \simeq \mathbb{P}^1 \times \mathbb{P}^2$.
Notice also that in the case $\mathrm{i}^+ \equiv \mu$ if we take $l_v^+ = \{t^2 v\}_{t\in\mathbb{R}}$, then we can easily calculate the homology of $X_{l_v^+} = \{x \in \mathbb{P}^n : p(x) \in l_v^+\}$) (the preimage of a half line): using Theorem 2 we immediatly see that $E_2 = E_\infty$ which implies $H_*(X_{l_v^+}) \simeq H_*(\mathbb{P}^{n-\mu})$.

Example 7. Consider the map $p : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2).$$

Then $C = \{p = 0\} \subset \mathbb{P}^3$ is the rational normal curve, the so called twisted cubic. In this case $\Omega = S^2$ and the set $\{\omega \in \Omega : \ker(\omega p) \neq 0\}$ consists of two disjoint ovals in S^2, bounding two disks B_1, B_2. Then S^2 is the disjoint union of the sets $\mathrm{Int}(B_1), \partial B_1, R, \partial B_2, \mathrm{Int}(B_2)$, on which the function i^+ is constant with value respectively $2, 1, 2, 2, 2$. Then

$$\Omega^1 = S^2, \quad \Omega^2 = S^2 \backslash \partial B_1, \quad \Omega^3 = \emptyset$$

and the second term of the spectral sequence (E_r, d_r) of Theorem A converging to $H_{3-*}(C)$ is the following:

$$\begin{array}{|c|c|c|c}
\mathbb{Z}_2 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 \\
0 & \mathbb{Z}_2 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}_2 \\
\hline
\end{array}$$

The differential $d_2 : E_2^{1,1} \to E_2^{3,0}$ is an isomorphism; hence $E_3 = E_\infty$ has the following picture:

$$\begin{array}{|c|c|c|c}
\mathbb{Z}_2 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
\end{array}$$

From the previous, using Theorem C, we see that $j_* : H_1(C) \to H_1(\mathbb{P}^3)$ is an isomorphism (we can check this fact also by noticing that, since C is a curve of degree 3, then the intersection number of C with a generic hyperplane $H \subset \mathbb{P}^3$ is odd).

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