Sub-Riemannian structures on 3D Lie groups

Andrei Agrachev
SISSA, Trieste, Italy and MIAN, Moscow, Russia - agrachev@sissa.it

Davide Barilari
SISSA, Trieste, Italy - barilari@sissa.it

July 28, 2010

Abstract

We give a complete classification of left-invariant sub-Riemannian structures on three dimensional Lie groups in terms of the basic differential invariants. As a corollary we explicitly find a sub-Riemannian isometry between the nonisomorphic Lie groups $SL(2)$ and $A^+(\mathbb{R}) \times S^1$, where $A^+(\mathbb{R})$ denotes the group of orientation preserving affine maps on the real line.

1 Introduction

In this paper, by a sub-Riemannian manifold we mean a triple $(M, \Delta, \mathbf{g})$, where M is a connected smooth manifold of dimension n, Δ is a smooth vector distribution of constant rank $k < n$, and $\mathbf{g}$ is a Riemannian metric on Δ, smoothly depending on the point.

In the following we always assume that the distribution Δ satisfies the *bracket generating condition* (also known as *Hörmander condition*), i.e. the Lie algebra generated by vector fields tangent to the distribution spans at every point the tangent space to the manifold.

Under this assumption, M is endowed with a natural structure of metric space, where the distance is the so called *Carnot-Caratheodory distance*

$$d(p,q) = \inf\{\int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\,dt \mid \gamma : [0,T] \to M \text{ is a Lipschitz curve}, \\ \gamma(0) = p, \gamma(T) = q, \quad \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ a.e. in } [0,T]\}.$$

As a consequence of the Hörmander condition this distance is always finite and continuous, and induces on M the original topology (see Chow-Raschevsky Theorem, [4]). Standard references on sub-Riemannian geometry are [5, 9, 11].

A sub-Riemannian structure is said to be *contact* if its distribution is defined as the kernel of a contact differential one form ω, i.e. $n = 2m+1$ and $(\bigwedge^m d\omega) \wedge \omega$ is a nonvanishing n-form on M.

In this paper we focus on the 3-dimensional case. Three dimensional contact sub-Riemannian structures have been deeply studied in the last years (for example see [2, 3]) and they have two basic differential invariants χ and κ (see Section 3 for the precise definition and paper [2] for their role in the asymptotic expansion of the sub-Riemannian exponential map).

The invariants χ and κ are smooth real functions on M. It is easy to understand, at least heuristically, why it is natural to expect exactly *two* functional invariants. Indeed, in local coordinates the sub-Riemannian structure can be defined by its orthonormal frame, i.e. by a couple of smooth vector fields on $\mathbb{R}^3$ or, in other words, by 6 scalar functions on $\mathbb{R}^3$. One function can be normalized by the rotation of the frame within its linear hull and three more functions by smooth change of variables. What remains are two scalar functions.

In this paper we exploit these local invariants to provide a complete classification of left-invariant structures on 3D Lie groups. A sub-Riemannian structure on a Lie group is said to be *left-invariant* if its distribution and the inner product are preserved by left translations on the group. A left-invariant distribution is uniquely determined by a two dimensional subspace of the Lie algebra of the group. The distribution is bracket generating (and contact) if and only if the subspace is not a Lie subalgebra.

Left invariant structures on Lie groups are the basic toy models of sub-Riemannian manifolds and the study of such structures is the starting point to understand the general properties of sub-Riemannian geometry. In particular, thanks to the group structure, in some of these cases it is also possible to compute explicitly the sub-Riemannian distance and geodesics (see in particular [8] for the Heisenberg group and [6] for semisimple Lie groups with Killing form).

A standard result on classification of 3D Lie algebras (see [10]) permit us to restrict our analysis on the Lie algebras of the following Lie groups:

H_2, the Heisenberg group,

$A^+(\mathbb{R}) \oplus \mathbb{R}$, where $A^+(\mathbb{R})$ is the group of orientation preserving affine maps on $\mathbb{R}$,

$SOLV^+, SOLV^-$ are Lie groups whose Lie algebra is solvable and has 2-dim square,

$SE(2)$ and $SH(2)$ are the groups of motions of Euclidean and Hyperbolic plane respectively,

$SL(2)$ and $SU(2)$ are the three dimensional simple Lie groups.

Moreover it is easy to show that in each of these cases but one all left-invariant bracket generating distributions are equivalent by automorphisms of the Lie algebra. The only case where there exists two nonequivalent distributions is the Lie algebra $\mathfrak{sl}(2)$. More precisely a 2-dim subspace of $\mathfrak{sl}(2)$ is called *elliptic* (*hyperbolic*) if the restriction of the Killing form on this subspace is sign-definite (sign-indefinite). Accordingly, we use notation $SL_e(2)$ and $SL_h(2)$ to specify on which subspace the sub-Riemannian structure on $SL(2)$ is defined.

For a left-invariant structure on a Lie group the invariants χ and κ are constant functions and allow us to distinguish non isometric structures. To complete our classification we can restrict our attention to *normalized* sub-Riemannian structures, i.e. structures that satisfies

$$\chi = \kappa = 0 \qquad \text{or} \qquad \chi^2 + \kappa^2 = 1 \tag{1}$$

Indeed χ and κ are homogeneous with respect to dilations of the orthonormal frame, that means rescaling of distances on the manifold. Thus we can always rescale our structure such that (1) is satisfied.

To find missing discrete invariants, i.e. to distinguish between normalized structures with same χ and κ, we then show that it is always possible to select a canonical orthonormal frame for the sub-Riemannian structure such that all structural constants of the Lie algebra of this frame are invariant with respect to local isometries. Then the commutator relations of the Lie algebra generated by the canonical frame determines in a unique way the sub-Riemannian structure.

Collecting together these results we prove the following

Theorem 1. *All left-invariant sub-Riemannian structures on 3D Lie groups are classified by local isometries as in Figure 1, where a structure is identified by the point (κ, χ) and two distinct points represent non locally isometric structures.*

In particular

- *If $\chi = \kappa = 0$ then the structure is locally isometric to the Heisenberg group,*
- *If $\chi^2 + \kappa^2 \neq 0$ then there exist no more than three non isometric normalized sub-Riemannian structures with these invariants; in particular there exists a unique normalized structure on a unimodular Lie group (for every choice of χ, κ).*
- *If $\chi \neq 0$ or $\chi = 0, \kappa \geq 0$, then two structures are locally isometric if and only if their Lie algebras are isomorphic.*

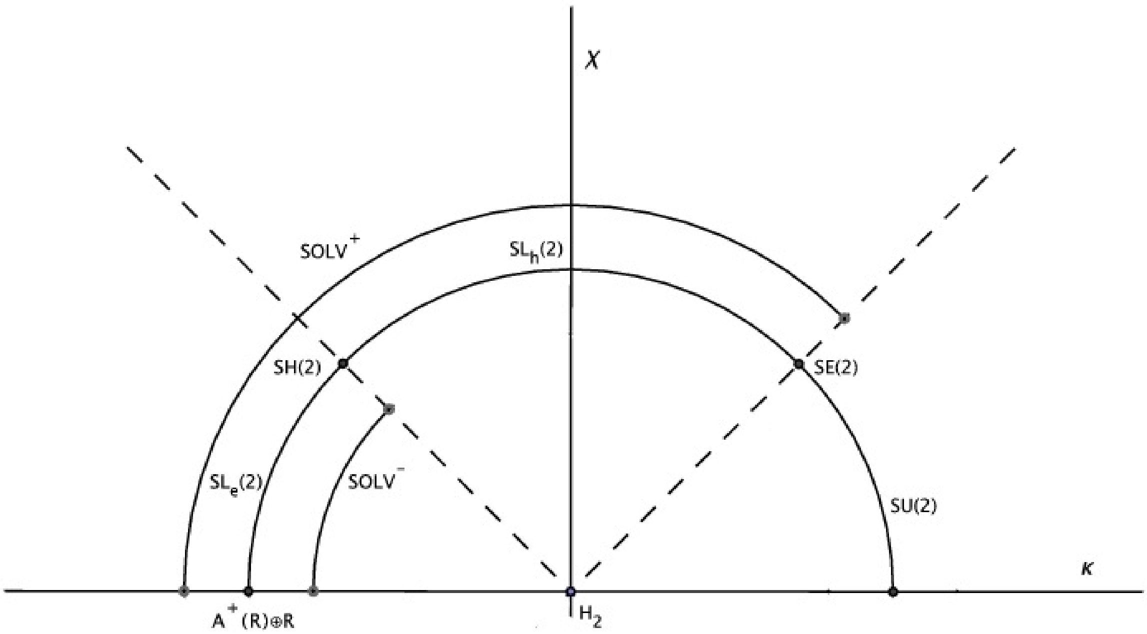

Figure 1: Classification

In Figure 1 we draw points on different circles since we consider equivalence classes of structures up to dilations. In this way it is easier to understand how many normalized structures corresponds to the same local invariants. Notice that unimodular Lie groups are exactly those in the middle circle (except for $A^+(\mathbb{R}) \oplus \mathbb{R}$).

From the proof of Theorem 1 we get also a uniformization-like theorem for "constant curvature" manifolds in the sub-Riemannian setting:

Corollary 2. *Let M be a complete simply connected 3D contact sub-Riemannian manifold. Assume that $\chi = 0$ and κ is costant on M. Then M is isometric to a left-invariant sub-Riemannian structure. More precisely:*

(i) if $\kappa = 0$ it is isometric to the Heisenberg group H_2,

(ii) if $\kappa = 1$ it is isometric to the group $SU(2)$ with Killing metric,

(iii) if $\kappa = -1$ it is isometric to the group $\tilde{SL}(2)$ with elliptic type Killing metric,

where $\tilde{SL}(2)$ is the universal covering of $SL(2)$.

Another byproduct of our classification is the fact that there exist non isomorphic Lie groups with locally isometric sub-Riemannian structures. Indeed from Theorem 1 we get that there exists a unique normalized left-invariant structure defined on $A^+(\mathbb{R}) \oplus \mathbb{R}$ having $\chi = 0, \kappa = -1$. Thus $A^+(\mathbb{R}) \oplus \mathbb{R}$ is locally isometric to the group $SL(2)$ with elliptic type Killing metric.

We write explicitly the global sub-Riemannian isometry between $A^+(\mathbb{R}) \oplus \mathbb{R}$ and the universal covering of $SL(2)$. We then show that this map pass to the quotient as a global isometry between the group $A^+(\mathbb{R}) \times S^1$ and the group $SL(2)$, where the sub-Riemannian structure is defined by the restriction of the Killing form on the elliptic distribution.

The group $A^+(\mathbb{R}) \oplus \mathbb{R}$ can be interpreted as the subgroup of the affine maps on the plane that acts as orientation preserving affinity on one axes and as translations on the other one [1]

$$A^+(\mathbb{R}) \oplus \mathbb{R} := \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \ a > 0, b, c \in \mathbb{R} \right\}$$

The standard left-invariant sub-Riemannian structure on $A^+(\mathbb{R}) \oplus \mathbb{R}$ is defined by the orthonormal frame $\Delta = \text{span}\{e_2, e_1 + e_3\}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

is a basis of the Lie algebra of the group, with $[e_1, e_2] = e_1$.

The subgroup $A^+(\mathbb{R})$ is topologically homeomorphic to the half-plane $\{(a, b) \in \mathbb{R}^2, a > 0\}$ and we can consider standard polar coordinates on this half-plane $\{(\rho, \theta) | \, \rho > 0, -\pi/2 < \theta < \pi/2\}$.

[1] We can recover the action as an affine map identifying $(x, y) \in \mathbb{R}^2$ with $(x\ y\ 1)^T$ and

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ y + c \\ 1 \end{pmatrix}$$

Theorem 3. *The diffeomorphism* $\Psi : A^+(\mathbb{R}) \times S^1 \longrightarrow SL(2)$ *defined by*

$$\Psi(\rho, \theta, \varphi) = \frac{1}{\sqrt{\rho \cos \theta}} \begin{pmatrix} \cos \varphi & \sin \varphi \\ \rho \sin(\theta - \varphi) & \rho \cos(\theta - \varphi) \end{pmatrix} \tag{2}$$

where $(\rho, \theta) \in A^+(\mathbb{R})$ *and* $\varphi \in S^1$*, is a global sub-Riemannian isometry.*

Using this global sub-Riemannian isometry as a change of coordinates one can recover the geometry of the sub-Riemannian structure on the group $A^+(\mathbb{R}) \times S^1$, starting from the analogous properties of $SL(2)$ (e.g. explicit expression of the sub-Riemannian distance, the cut locus). In particular we notice that, since $A^+(\mathbb{R}) \times S^1$ is not unimodular, the canonical sub-laplacian on this group is not expressed as sum of squares. Indeed if X_1, X_2 denotes the left-invariant vector fields associated to the orthonormal frame the sub-laplacian is expressed

$$L_{sr} = X_1^2 + X_2^2 + X_1$$

Moreover in the nonunimodular case the generalized Fourier transfom method cannot apply (see [1]). Hence the heat kernel of the corresponding heat equation cannot be computed directly. On the other hand one can use the map (2) to express the solution in term of the heat kernel on $SL(2)$.

2 Basic definitions

We start recalling the definition of sub-Riemannian manifold.

Definition 4. A *sub-Riemannian manifold* is a triple $(M, \Delta, \mathbf{g})$, where

(*i*) M is a smooth connected n-dimensional manifold ,

(*ii*) Δ is a smooth distribution of constant rank $k < n$, i.e. a smooth map that associates to every $q \in M$ a k-dimensional subspace Δ_q of T_qM,

(*iii*) $\mathbf{g}_q$ is a Riemannian metric on Δ_q, that is smooth with respect to $q \in M$.

The set of smooth sections of the distribution

$$\overline{\Delta} := \{f \in \mathrm{Vec}(M) | \, f(q) \in \Delta_q, \ \forall q \in M\} \subset \mathrm{Vec}(M)$$

is a subspace of the space of the smooth vector fields on M and its elements are said *horizontal* vector fields.

A Lipschitz continuous curve $\gamma : [0, T] \to M$ is *admissible* (or *horizontal*) if its derivative is a.e. horizontal, i.e. if $\dot\gamma(t) \in \Delta_{\gamma(t)}$ for a.e. $t \in [0, T]$. We denote with Ω_{pq} the set of admissible paths joining p to q.

Given an admissible curve γ it is possible to define its *lenght*

$$\ell(\gamma) = \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} \, dt$$

The *Carnot-Caratheodory distance* induced by the sub-Riemannian structure is

$$d(p,q) = \inf\{\ell(\gamma), \gamma \in \Omega_{pq}\}$$

In the following we always assume that the distribution Δ satisfies the *bracket generating condition* (also known as *Hörmander condition*), i.e. the Lie algebra generated by horizontal vector fields spans at every point the tangent space to the manifold

$$\text{span}\{[f_1,\ldots,[f_{j-1},f_j]](q), f_i \in \overline{\Delta}, j \in \mathbb{N}\} = T_qM, \quad \forall q \in M$$

Under this hypothesis it follows from the classical Chow-Raschevsky Theorem [7, 13], that d is a well defined metric on M and it induces on M the original topology.

Definition 5. A *sub-Riemannian isometry* between two sub-Riemannian manifolds $(M, \Delta, \mathbf{g})$ and $(N, \Delta', \mathbf{g}')$ is a diffeomorphism $\phi : M \to N$ that satisfies

(i) $\phi_*(\Delta) = \Delta'$,

(ii) $\mathbf{g}(f_1, f_2) = \mathbf{g}'(\phi_* f_1, \phi_* f_2), \quad \forall f_1, f_2 \in \overline{\Delta}$.

Definition 6. Let M be a $2m+1$ dimensional manifold. A sub-Riemannian structure on M is said to be *contact* if Δ is a contact distribution, i.e. $\Delta = \ker\omega$, where $\omega \in \Lambda^1 M$ satisfies $(\bigwedge^m d\omega) \wedge \omega \neq 0$. Notice that a contact structure is forced to be bracket generating.

The contact structure endows M with a canonical orientation. On the other hand we will not fix an orientation on the distribution Δ.

Now we briefly recall some facts about sub-Riemannian geodesic. In particular we define the sub-Riemannian Hamiltonian.

Let M be a sub-Riemannian manifold and fix $q_0 \in M$. We define the *endpoint map* (at time 1) as

$$F : \mathcal{U} \to M, \quad F(\gamma) = \gamma(1)$$

where $\mathcal{U}$ denotes the set of admissible trajectories starting from q_0 and defined at time $t = 1$. If we fix a point $q_1 \in M$, the problem of finding shortest paths from q_0 to q_1 is equivalent to the following one

$$\min_{F^{-1}(q_1)} J(\gamma), \qquad J(\gamma) := \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt, \tag{3}$$

where J is the action functional. Indeed, it is a standard fact that Cauchy-Schwartz inequality implies that an admissible curve realizes this minimum if and only if it is an arc-lenght parametrized ℓ-minimizer.

Then the Lagrange multipliers rule implies that any solution of (3) is either a critical point of F or a solution of the equation

$$\lambda_1 D_\gamma F = d_\gamma J, \qquad \gamma \in \mathcal{U} \tag{4}$$

for some $\lambda_1 \in T^*_{\gamma(1)}M$. Solution of equation (4) are said *normal geodesics* while critical ponits of F are said *abnormal geodesics.*

Now we can define the *sub-Riemannian Hamiltonian* $h \in C^\infty(T^*M)$ as follows:

$$h(\lambda) = \max_{u \in \Delta_q} \{\langle \lambda, u \rangle - \frac{1}{2}|u|^2\}, \qquad \lambda \in T^*M, \quad q = \pi(\lambda), \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard pairing between vectors and covectors. The Pontryagin Maximum Principle gives a perfect characterization of our geodesics. In fact it can be shown that in contact case there are no abnormal geodesics and a pair (γ, λ_1) satisfies (4) if and only if there exists a curve $\lambda(t) \in T^*_{\gamma(t)}M$ solution of the Hamiltonian system $\dot{\lambda}(t) = \vec{h}(\lambda(t))$ with boundary condition $\lambda(1) = \lambda_1$.

Remark 7. Locally the sub-Riemannian structure can be given assigning a set of k smooth linearly independent vector fields that are orthonormal

$$\Delta_q = \text{span}\{f_1(q), \ldots, f_k(q)\}, \qquad \mathbf{g}_q(f_i(q), f_j(q)) = \delta_{ij}. \tag{6}$$

Notice that if we consider a new orthonormal frame which is a rotation of the previous one, we define the same sub-Riemannian structure.

Following this notation a *local isometry* between two structures defined by orthonormal frames $\Delta_M = \text{span}(f_1, \ldots, f_k), \Delta_N = \text{span}(g_1, \ldots, g_k)$ is given by a diffeomorphism such that

$$\phi : M \to N, \quad \phi_*(f_i) = g_i, \qquad \forall i = 1, \ldots, k.$$

In this setting admissible trajectories are solutions of the equation

$$\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t) f_i(\gamma(t)), \quad \text{for a.e. } t \in [0, T]$$

for some measurable and bounded control functions $u(t) = (u_1(t), \ldots, u_k(t)), u_i \in L^\infty([0, T])$. Lenght and action of this curve turns to be

$$\ell(\gamma) = \int_0^T |u(t)|dt, \quad J(\gamma) = \int_0^T |u(t)|^2 dt$$

where $| \cdot |$ denotes standard Euclidean norm in $\mathbb{R}^k$.

Moreover sub-Riemannian Hamiltonian (5) is written as

$$h(\lambda) = \frac{1}{2} \sum_{i=1}^{k} h_i^2(\lambda), \qquad \text{where} \quad h_i(\lambda) = \langle \lambda, f_i(q) \rangle, \quad q = \pi(\lambda).$$

Notice that $h_i : T^*M \to \mathbb{R}$ are linear on fibers functions associated to vector fields of the frame. The sub-Riemannian Hamiltonian h is a smooth function on T^*M which contains all informations about sub-Riemannian structure. Indeed it does not depend on the orthonormal frame $\{f_1, \ldots, f_k\}$ selected, i.e. is invariant for rotations of the frame, and the annichilator of the distribution at a point $\Delta_q^\perp$ can be recovered as the kernel of the restriction of h to the fiber T_q^*M

$$\ker h|_{T_q^*M} = \{\lambda \in T_q^*M | \; h_i(\lambda) = 0, \; i = 1, \ldots, k\} = \Delta_q^\perp$$

Remark 8. A sub-Riemannian structure on a Lie group G is said to be *left-invariant* if

$$\Delta_{gh} = L_{g*}\Delta_h, \qquad \langle v, w\rangle_h = \langle L_{g*}v, L_{g*}w\rangle_{gh}, \qquad \forall g, h \in G.$$

where L_g denotes the left multiplication map on the group. In particular, to define a left-invariant structure, it is sufficient to fix a subspace of the Lie algebra $\mathfrak{g}$ of the group and an inner product on it.

We also remark that in this case it is possible to have in (6) a global equality, i.e. to select k globally linearly independent orthonormal vector fields.

3 Sub-Riemannian invariants

In this section we study a contact sub-Riemannian structure on a 3D manifold and we give a brief description of its two invariants (see also [2]). We start with the following characterization of contact distributions

Lemma 9. Let M be a 3D manifold, $\omega \in \Lambda^1 M$ and $\Delta = \ker\omega$. The following are equivalent:

(i) Δ is a contact distribution,

(ii) $d\omega\big|_\Delta \neq 0$,

(iii) $\forall f_1, f_2 \in \overline{\Delta}$ linearly independent, then $[f_1, f_2] \notin \overline{\Delta}$.

Moreover, in this case, the contact form can be selected in such a way that $d\omega\big|_\Delta$ coincide with the Euclidean volume form on Δ.

By Lemma 9 it is not restrictive to assume that the sub-Riemannian structure satisfies:

$$\begin{gathered} (M,\omega) \text{ is a 3D contact structure,} \\ \Delta = \operatorname{span}\{f_1, f_2\} = \ker\omega, \\ \mathbf{g}(f_i, f_j) = \delta_{ij}, \quad d\omega(f_1, f_2) = 1. \end{gathered} \tag{7}$$

We stress that in (7) the orthonormal frame f_1, f_2 isn't unique. Indeed every rotated frame (with the angle of rotation that depends smoothly on the point) defines the same structure.

The sub-Riemannian Hamiltonian (5) is written

$$h = \frac{1}{2}(h_1^2 + h_2^2).$$

Definition 10. In the setting (7) we define the *Reeb vector field* associated to the contact structure as the unique vector field f_0 such that

$$\begin{aligned} \omega(f_0) &= 1, \\ d\omega(f_0, \cdot) &= 0. \end{aligned} \tag{8}$$

From the definition it is clear that f_0 depends only on the sub-Riemannian structure (and its orientation) and not on the frame selected.

Condition (8) is equivalent to

$$\begin{aligned} &[f_1, f_0], [f_2, f_0] \in \overline{\Delta}, \\ &[f_2, f_1] = f_0 \pmod{\overline{\Delta}}. \end{aligned}$$

and we deduce the following expression for the Lie algebra of vector fields generated by f_0, f_1, f_2

$$\begin{aligned} [f_1, f_0] &= c_{01}^1 f_1 + c_{01}^2 f_2 \\ [f_2, f_0] &= c_{02}^1 f_1 + c_{02}^2 f_2 \\ [f_2, f_1] &= c_{12}^1 f_1 + c_{12}^2 f_2 + f_0 \end{aligned} \tag{9}$$

where c_{ij}^k are functions on the manifold, called structural constants of the Lie algebra.

If we denote with (ν_0, ν_1, ν_2) the basis of 1-form dual to (f_0, f_1, f_2), we can rewrite (9) as:

$$\begin{aligned} d\nu_0 &= \qquad \nu_1 \wedge \nu_2 \\ d\nu_1 &= c_{01}^1 \nu_0 \wedge \nu_1 + c_{02}^1 \nu_0 \wedge \nu_2 + c_{12}^1 \nu_1 \wedge \nu_2 \\ d\nu_2 &= c_{01}^2 \nu_0 \wedge \nu_1 + c_{02}^2 \nu_0 \wedge \nu_2 + c_{12}^2 \nu_1 \wedge \nu_2 \end{aligned} \tag{10}$$

Let $h_0(\lambda) = \langle \lambda, f_0(q) \rangle$ denotes the linear on fibers Hamiltonian associated with the Reeb field f_0. We now compute the Poisson bracket $\{h, h_0\}$, denoting with $\{h, h_0\}_q$ its restriction to the fiber T_q^*M.

Proposition 11. *The Poisson bracket $\{h, h_0\}_q$ is a quadratic form. Moreover we have*

$$\{h, h_0\} = c_{01}^1 h_1^2 + (c_{01}^2 + c_{02}^1) h_1 h_2 + c_{02}^2 h_2^2, \tag{11}$$

$$c_{01}^1 + c_{02}^2 = 0. \tag{12}$$

In particular, $\Delta_q^\perp \subset \ker \{h, h_0\}_q$ and $\{h, h_0\}_q$ is actually a quadratic form on $T_q^*M/\Delta_q^\perp = \Delta_q^*$.

Proof. Using the equality $\{h_i, h_j\}(\lambda) = \langle \lambda, [f_i, f_j](q) \rangle$ we get

$$\begin{aligned} \{h, h_0\} &= \frac{1}{2}\{h_1^2 + h_2^2, h_0\} = h_1\{h_1, h_0\} + h_2\{h_2, h_0\} \\ &= h_1(c_{01}^1 h_1 + c_{01}^2 h_2) + h_2(c_{02}^1 h_1 + c_{02}^2 h_2) \\ &= c_{01}^1 (h_1)^2 + (c_{01}^2 + c_{02}^1) h_1 h_2 + c_{02}^2 (h_2)^2 \end{aligned}$$

Differentiating first equation in (10) we find:

$$\begin{aligned} 0 = d^2\nu_0 &= d\nu_1 \wedge \nu_2 - \nu_1 \wedge d\nu_2 \\ &= (c_{01}^1 + c_{02}^2) \nu_0 \wedge \nu_1 \wedge \nu_2 \end{aligned}$$

which proves (12). □

Being $\{h, h_0\}_q$ a quadratic form on the Euclidean plane Δ_q (we identify a vector space with its dual with the scalar product), it is a standard fact that it can be interpreted as a symmetric operator on the plane itself. In particular its determinant and its trace are well defined. From (12) we get

$$\text{trace}\{h, h_0\}_q = 0$$

It is natural then to define our *first invariant* as the positive eigenvalue of this operator, namely:

$$\chi(q) = \sqrt{-\det\{h, h_0\}_q} \tag{13}$$

The *second invariant*, which was found in [2] as a term of asymptotic expansion of conjugate locus, is defined in the following way

$$\kappa(q) = f_2(c_{12}^1) - f_1(c_{12}^2) - (c_{12}^1)^2 - (c_{12}^2)^2 + \frac{c_{01}^2 - c_{02}^1}{2}. \tag{14}$$

where we refer to notation (9). A direct calculation shows that κ is preserved by rotations of the frame f_1, f_2 of the distribution, hence it depends only on the sub-Riemannian structure.

χ and κ are functions defined on the manifold; they reflect intrinsic geometric properties of the sub-Riemannian structure and are preserved by the sub-Riemannian isometries. In particular, χ and κ are constant functions for left-invariant structures on Lie groups (since left translations are isometries).

4 Canonical Frames

In this section we want to show that it is always possible to select a canonical orthonormal frame for the sub-Riemannian structure. In this way we are able to find missing discrete invariants and to classify sub-Riemannian structures simply knowing structural constants c_{ij}^k for the canonical frame. We study separately the two cases $\chi \neq 0$ and $\chi = 0$.

We start by rewriting and improving Proposition 11 when $\chi \neq 0$.

Proposition 12. *Let M be a 3D contact sub-Riemannian manifold and $q \in M$. If $\chi(q) \neq 0$, then there exists a local frame such that* (11) *becomes:*

$$\{h, h_0\} = 2\chi h_1 h_2 \tag{15}$$

In particular, in the Lie group case with left-invariant stucture, there exists a unique (up to a sign) canonical frame (f_0, f_1, f_2) such that (9) *become*

$$\begin{aligned} [f_1, f_0] &= c_{01}^2 f_2 \\ [f_2, f_0] &= c_{02}^1 f_1 \\ [f_2, f_1] &= c_{12}^1 f_1 + c_{12}^2 f_2 + f_0 \end{aligned} \tag{16}$$

Moreover we have:

$$\chi = \frac{c_{01}^2 + c_{02}^1}{2}, \qquad \kappa = -(c_{12}^1)^2 - (c_{12}^2)^2 + \frac{c_{01}^2 - c_{02}^1}{2}. \tag{17}$$

Proof. From Proposition 11 we know that the Poisson bracket $\{h, h_0\}_q$ is a non degenerate symmetric operator with zero trace. Hence we have a well defined, up to a sign, orthonormal frame by setting f_1, f_2 as the orthonormal isotropic vectors of this operator (remember that f_0 depends only on the structure and not on the orthonormal frame on the distribution). It is easily seen that in both of these cases we obtain the expression (15). □

Remark 13. Notice that, if we change sign to f_1 or f_2, then c_{12}^2 or c_{12}^1, respectively, change sign in (9), while c_{02}^1 and c_{01}^2 are unaffected. Hence (17) do not depend on the orientation of the sub-Riemannian structure.

If $\chi = 0$ the above procedure cannot apply. Indeed both trace and determinant of the operator are zero, hence we have $\{h, h_0\}_q = 0$. From (11) we get identities

$$c_{01}^1 = c_{02}^2 = 0, \quad c_{01}^2 + c_{02}^1 = 0. \tag{18}$$

so that commutators (9) simplify in (where $c = c_{01}^2$)

$$\begin{aligned} [f_1, f_0] &= \quad c f_2 \\ [f_2, f_0] &= -c f_1 \\ [f_2, f_1] &= c_{12}^1 f_1 + c_{12}^2 f_2 + f_0 \end{aligned} \tag{19}$$

We want to show, with an explicit construction, that also in this case there always exists a rotation of our frame, on a angle that smoothly depend on the point, such that in the new frame κ is the only structural constant which appear in (19).

We begin with a useful lemma

Lemma 14. Let f_1, f_2 be an orthonormal frame on M. If we denote with $\widehat{f}_1, \widehat{f}_2$ the frame obtained from the previous one with a rotation of angle $\theta(q)$ and with $\widehat{c}_{ij}^k$ structural constants of rotated frame, we have:

$$\begin{aligned} \widehat{c}_{12}^1 &= \cos\theta(c_{12}^1 - f_1(\theta)) - \sin\theta(c_{12}^2 - f_2(\theta)) \\ \widehat{c}_{12}^2 &= \sin\theta(c_{12}^1 - f_1(\theta)) + \cos\theta(c_{12}^2 - f_2(\theta)) \end{aligned}$$

Now we can prove the main result of this section.

Proposition 15. *Let M be a 3D simply connected contact sub-Riemannian manifold such that $\chi = 0$. Then there exist a rotation of the original frame $\widehat{f}_1, \widehat{f}_2$ such that:*

$$\begin{aligned} [\widehat{f}_1, f_0] &= \kappa \widehat{f}_2 \\ [\widehat{f}_2, f_0] &= -\kappa \widehat{f}_1 \\ [\widehat{f}_2, \widehat{f}_1] &= f_0 \end{aligned} \tag{20}$$

Proof. Using Lemma 14 we can rewrite the statement in the following way: there exists a function $\theta : M \to \mathbb{R}$ such that

$$f_1(\theta) = c_{12}^1, \quad f_2(\theta) = c_{12}^2. \tag{21}$$

Indeed, this would imply $\widehat{c}^1_{12} = \widehat{c}^2_{12} = 0$ and $\kappa = c$.

Let us introduce simplified notations $c^1_{12} = \alpha_1,\ c^2_{12} = \alpha_2$. Then

$$\kappa = f_2(\alpha_1) - f_1(\alpha_2) - (\alpha_1)^2 - (\alpha_2)^2 + c \tag{22}$$

If (ν_0, ν_1, ν_2) denotes the dual basis to (f_0, f_1, f_2) we have

$$d\theta = f_0(\theta)\nu_0 + f_1(\theta)\nu_1 + f_2(\theta)\nu_2$$

and from (19) we get:

$$\begin{aligned} f_0(\theta) &= ([f_2, f_1] - \alpha_1 f_1 - \alpha_2 f_2)(\theta) \\ &= f_2(\alpha_1) - f_1(\alpha_2) - \alpha_1^2 - \alpha_2^2 \\ &= \kappa - c \end{aligned}$$

Suppose now that (21) are satisfied, we get

$$d\theta = (\kappa - c)\nu_0 + \alpha_1\nu_1 + \alpha_2\nu_2 =: \eta \tag{23}$$

with the r.h.s. independent from θ.

To prove the theorem it is sufficient to show that η a closed 1-form. If we denote $\nu_{ij} := \nu_i \wedge \nu_j$ dual equations of (19) are:

$$\begin{aligned} d\nu_0 &= \nu_{12} \\ d\nu_1 &= -c\nu_{02} + \alpha_1\nu_{12} \\ d\nu_2 &= c\nu_{01} - \alpha_2\nu_{12} \end{aligned}$$

and differentiating we get two nontrivial relations:

$$\begin{aligned} f_1(c) + c\alpha_2 + f_0(\alpha_1) = 0 \\ f_2(c) - c\alpha_1 + f_0(\alpha_2) = 0 \end{aligned} \tag{24}$$

Recollecting all these computations we prove closure of η

$$\begin{aligned} d\eta &= d(\kappa - c) \wedge \nu_0 + (\kappa - c)d\nu_0 + d\alpha_1 \wedge \nu_1 + \alpha_1 d\nu_1 + d\alpha_2 \wedge \nu_2 + \alpha_2 d\nu_2 \\ &= -dc \wedge \nu_0 + (\kappa - c)\nu_{12} + \\ &\quad + f_0(\alpha_1)\nu_{01} - f_2(\alpha_1)\nu_{12} + \alpha_1(\alpha_1\nu_{12} - c\nu_{02}) \\ &\quad + f_0(\alpha_2)\nu_{02} + f_1(\alpha_2)\nu_{12} + \alpha_2(c\nu_{01} - \alpha_2\nu_{12}) \\ &= \quad (f_0(\alpha_1) + \alpha_2 c + f_1(c))\nu_{01} \\ &\quad + (f_0(\alpha_2) - \alpha_1 c + f_2(c))\nu_{02} \\ &\quad + (\kappa - c - f_2(\alpha_1) + f_1(\alpha_2) + \alpha_1^2 + \alpha_2^2)\nu_{12} = 0. \end{aligned}$$

where in the last equality we use (22) and (24). □

5 Proof of Theorem 1

In this section we use results of previous sections to prove Theorem 1.

We will always assume that G is a 3D Lie group with a left-invariant sub-Riemannian structure defined by the orthonormal frame f_1, f_2, i.e.

$$\Delta = \text{span}\{f_1, f_2\} \subset \mathfrak{g}, \qquad \text{span}\{f_1, f_2, [f_1, f_2]\} = \mathfrak{g}.$$

Note that for a 3D left-invariant structure to be contact is equivalent to be bracket generating, moreover the Reeb field f_0 is a left-invariant vector field.

From the fact that, for left-invariant structures, local invariants are constant functions (see Remark 8) we obtain a necessary condition for two structures to be isometric:

Proposition 16. *Let G, H be 3D Lie groups with locally isometric sub-Riemannian structures. Then $\chi_G = \chi_H$ and $\kappa_G = \kappa_H$.*

Notice that this condition is not sufficient. It turns out that there can be up to three mutually non locally isometric normalized structures with the same invariants χ, κ.

Remark 17. It is easy to see that χ and κ are homogeneous of degree 2 with respect to dilations of the frame. In other words, if we consider the two orthonormal frames on the same manifold

$$\Delta = \text{span}\{f_1, f_2\}, \qquad \widetilde{\Delta} = \text{span}\{\lambda f_1, \lambda f_2\}, \qquad \lambda \in \mathbb{R}^+$$

and denote $\widetilde{\chi}$ and $\widetilde{\kappa}$ local invariants for the dilated structure, we find

$$\widetilde{\chi} = \lambda^2 \chi, \quad \widetilde{\kappa} = \lambda^2 \kappa$$

On the other hand, when we dilate the orthonormal frame, we just multiply by a constant all distances in our manifold. Since we are interested in a classification by local isometries, we can always suppose (for a suitable dilation of the orthonormal frame) that local invariants of our structure satisfy

$$\chi = \kappa = 0, \qquad \text{or} \qquad \chi^2 + \kappa^2 = 1$$

and we study equivalence classes with respect to local isometries.

We study separately the two cases $\chi \neq 0$ and $\chi = 0$.

Case $\chi \neq 0$

Let G be a 3D Lie group with a left-invariant sub-Riemannian structure such that $\chi \neq 0$. From Proposition 12 we can suppose that $\Delta = \text{span}\{f_1, f_2\}$ where f_1, f_2 is the canonical frame of the structure and from (16) we obtain dual equations:

$$\begin{aligned} d\nu_0 &= \nu_1 \wedge \nu_2 \\ d\nu_1 &= c_{02}^1 \nu_0 \wedge \nu_2 + c_{12}^1 \nu_1 \wedge \nu_2 \\ d\nu_2 &= c_{01}^2 \nu_0 \wedge \nu_1 + c_{12}^1 \nu_1 \wedge \nu_2 \end{aligned} \tag{25}$$

Using $d^2 = 0$ we obtain structural equations

$$\begin{cases} c_{02}^1 c_{12}^2 = 0 \\ c_{01}^2 c_{12}^1 = 0 \end{cases} \tag{26}$$

We know that the structural constants of the canonical frame are invariant by local isometries (up to change signs of c_{12}^1, c_{12}^2, see Remark 13). Hence, every different choice of coefficients in (16) which satisfy also (26) will belong to a different class of non-isometric structures.

Taking into account that $\chi > 0$ implies that c_{01}^2 and c_{02}^1 cannot be both non positive (see (17)), we have the following cases:

(i) $c_{12}^1 = 0$ and $c_{12}^2 = 0$. In this first case we get

$$\begin{aligned} [f_1, f_0] &= c_{01}^2 f_2 \\ [f_2, f_0] &= c_{02}^1 f_1 \\ [f_2, f_1] &= f_0 \end{aligned}$$

and formulas (17) implies

$$\chi = \frac{c_{01}^2 + c_{02}^1}{2} > 0, \quad \kappa = \frac{c_{01}^2 - c_{02}^1}{2}.$$

In addition, we find relations between invariants

$$\chi + \kappa = c_{01}^2, \quad \chi - \kappa = c_{02}^1.$$

We have the following subcases:

(a) If $c_{02}^1 = 0$ we get the Lie algebra $\mathfrak{se}(2)$ of the group $SE(2)$ of the Euclidean isometries of $\mathbb{R}^2$, and it holds $\chi = \kappa$.

(b) If $c_{01}^2 = 0$ we get the Lie algebra $\mathfrak{sh}(2)$ of the group $SH(2)$ of the Hyperbolic isometries of $\mathbb{R}^2$, and it holds $\chi = -\kappa$.

(c) If $c_{01}^2 > 0$ and $c_{02}^1 < 0$ we get the Lie algebra $\mathfrak{su}(2)$ and $\chi - \kappa < 0$.

(d) If $c_{01}^2 < 0$ and $c_{02}^1 > 0$ we get the Lie algebra $\mathfrak{sl}(2)$ with $\chi + \kappa < 0$.

(e) If $c_{01}^2 > 0$ and $c_{02}^1 > 0$ we get the Lie algebra $\mathfrak{sl}(2)$ with $\chi + \kappa > 0, \chi - \kappa > 0$.

(ii) $c_{02}^1 = 0$ and $c_{12}^1 = 0$. In this case we have

$$\begin{aligned} [f_1, f_0] &= c_{01}^2 f_2 \\ [f_2, f_0] &= 0 \\ [f_2, f_1] &= c_{12}^2 f_2 + f_0 \end{aligned} \tag{27}$$

and necessarily $c_{01}^2 \neq 0$. Moreover we get

$$\chi = \frac{c_{01}^2}{2} > 0, \quad \kappa = -(c_{12}^2)^2 + \frac{c_{01}^2}{2}$$

from which follows:

$$\chi - \kappa \geq 0$$

Lie algebra (27) satisfies $\dim[\mathfrak{g}, \mathfrak{g}] = 2$, hence it can be interpreted as the operator $A = \text{ad } f_1$ which acts on span$\{f_0, f_2\}$. It can be easily seen that it holds

$$\text{trace } A = -c_{12}^2, \quad \det A = c_{01}^2 > 0$$

and we can find the useful relation

$$2\frac{\text{trace}^2 A}{\det A} = 1 - \frac{\kappa}{\chi} \tag{28}$$

(*iii*) $c_{01}^2 = 0$ and $c_{12}^2 = 0$. In this last case we get

$$\begin{aligned} [f_1, f_0] &= 0 \\ [f_2, f_0] &= c_{02}^1 f_1 \\ [f_2, f_1] &= c_{12}^1 f_1 + f_0 \end{aligned} \tag{29}$$

and $c_{02}^1 \neq 0$. Moreover we get

$$\chi = \frac{c_{02}^1}{2} > 0, \quad \kappa = -(c_{12}^1)^2 - \frac{c_{02}^1}{2}$$

from which follows:

$$\chi + \kappa \leq 0$$

As before, Lie algebra (29) has two-dimensional square and it can be interpreted as the operator $A = \text{ad } f_2$ which acts on the plane span$\{f_0, f_1\}$. It can be easily seen that it holds

$$\text{trace } A = c_{12}^1, \quad \det A = -c_{02}^1 < 0$$

and we have an analogous relation

$$2\frac{\text{trace}^2 A}{\det A} = 1 + \frac{\kappa}{\chi} \tag{30}$$

Remark 18. Lie algebras of cases (*ii*) and (*iii*) are *solvable* algebras and we will denote respectively $\mathfrak{solv}^+$ and $\mathfrak{solv}^-$, where the sign depends on the determinant of the operator it represents. In particular, formulas (28) and (30) permits to recover the ratio between invariants (hence to determine a unique normalized structure) only from intrinsic properties of the operator. Notice that if $c_{12}^2 = 0$ we recover the normalized structure (*i*)-(*a*) while if $c_{12}^1 = 0$ we get the case (*i*)-(*b*).

Remark 19. The algebra $\mathfrak{sl}(2)$ is the only case where we can define two nonequivalent distributions which corresponds to the case that Killing form restricted on the distribution is positive definite (case (*d*)) or indefinite (case (*e*)). We will refer to the first one as the *elliptic* structure on $\mathfrak{sl}(2)$, denoted $\mathfrak{sl}_e(2)$, and with *hyperbolic* structure in the other case, denoting $\mathfrak{sl}_h(2)$.

Case $\chi = 0$

A direct consequence of Proposition 15 for left-invariant structures is the following

Corollary 20. *Let G, H be Lie groups with left-invariant sub-Riemannian structures and assume $\chi_G = \chi_H = 0$. Then G and H are locally isometric if and only if $\kappa_G = \kappa_H$.*

Thanks to this result it is very easy to complete our classification. Indeed it is sufficient to find all left-invariant structures such that $\chi = 0$ and to compare their second invariant κ.

A straightforward calculation leads to the following list of the left-invariant structures on simply connected three dimensional Lie groups G with $\chi_G = 0$:

- $G = H_2$ is the Heisenberg nilpotent group; then $\kappa_G = 0$.
- $G = SU(2)$ with the Killing inner product; then $\kappa_G > 0$.
- $G = \tilde{SL}(2)$ with the elliptic distribution and Killing inner product; then $\kappa_G < 0$.
- $G = A^+(\mathbb{R}) \oplus \mathbb{R}$; then $\kappa_G < 0$.

Remark 21. In particular, we have the following:

(*i*) All left-invariant sub-Riemannian structures on H_2 are locally isometric,

(*ii*) There exists on $A^+(\mathbb{R}) \oplus \mathbb{R}$ a unique (modulo dilations) left-invariant sub-Riemannian structure, which is locally isometric to $SL_e(2)$ with the Killing metric.

Proof of Theorem 1 is now completed and we can recollect our result as in Figure 1, where we associate to every normalized structure a point in the (κ, χ) plane: either $\chi = \kappa = 0$, or (κ, χ) belong to the semicircle

$$\{(\kappa, \chi) \in \mathbb{R}^2, \chi^2 + \kappa^2 = 1, \chi > 0\}$$

Notice that different points means that sub-Riemannian structures are not locally isometric.

6 Proof of Theorem 3

In this section we want to explicitly write the sub-Riemannian isometry between $SL(2)$ and $A^+(\mathbb{R}) \oplus S^1$.

Consider the Lie algebra $\mathfrak{sl}(2) = \{A \in M_2(\mathbb{R}),\ \text{trace}(A) = 0\} = \text{span}\{g_1, g_2, g_3\}$ where we consider the basis

$$g_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The sub-Riemannian structure on $SL(2)$ defined by the Killing form on the elliptic distribution is given by the orthonormal frame

$$\Delta_{\mathfrak{sl}} = \text{span}\{g_1, g_2\} \qquad \text{and} \qquad g_0 := -g_3 \tag{31}$$

is the Reeb vector field. Notice that this frame is already canonical since equations (20) are satisfied. Indeed

$$[g_1, g_0] = -g_2 = \kappa g_2$$

Recall that the universal covering of $SL(2)$, which we denote $\tilde{SL}(2)$, is a simply connected Lie group with Lie algebra $\mathfrak{sl}(2)$. Hence (31) define a left-invariant structure also on the universal covering.

On the other hand we consider the following coordinates on the Lie group $A^+(\mathbb{R}) \oplus \mathbb{R}$, that are well-adapted for our further calculations

$$A^+(\mathbb{R}) \oplus \mathbb{R} := \left\{ \begin{pmatrix} -y & 0 & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad y < 0, x, z \in \mathbb{R} \right\} \tag{32}$$

It is easy to see that, in these coordinates, the group law reads

$$(x, y, z)(x', y', z') = (x - yx', -yy', z + z')$$

and its Lie algebra $\mathfrak{a}(\mathbb{R}) \oplus \mathbb{R}$ is generated by the vector fields

$$e_1 = -y\partial_x, \quad e_2 = -y\partial_y, \quad e_3 = \partial_z$$

with the only nontrivial commutator relation $[e_1, e_2] = e_1$.

The left-invariant structure on $A^+(\mathbb{R}) \oplus \mathbb{R}$ is defined by the orthonormal frame

$$\begin{aligned} \Delta_{\mathfrak{a}} &= \text{span}\{f_1, f_2\} \\ f_1 &:= e_2 = -y\partial_y \\ f_2 &:= e_1 + e_3 = -y\partial_x + \partial_z \end{aligned} \tag{33}$$

With straightforward calculations we compute the Reeb vector field $f_0 = -e_3 = -\partial_z$.

This frame is not canonical since it does not satisfy equations (20). Hence we can apply Proposition 15 to find the canonical frame, that will be no more left-invariant.

Following the notation of Proposition 15 we have

Lemma 22. The canonical orthonormal frame on $A^+(\mathbb{R}) \oplus \mathbb{R}$ has the form:

$$\begin{aligned} \widehat{f_1} &= \quad y \sin z\, \partial_x - y \cos z\, \partial_y - \sin z\, \partial_z \\ \widehat{f_2} &= -y \cos z\, \partial_x - y \sin z\, \partial_y + \cos z\, \partial_z \end{aligned} \tag{34}$$

Proof. It is equivalent to show that the rotation defined in the proof of Proposition 15 is $\theta(x, y, z) = z$. The dual basis to our frame $\{f_1, f_2, f_0\}$ is given by

$$\nu_1 = -\frac{1}{y}dy, \qquad \nu_2 = -\frac{1}{y}dx, \qquad \nu_0 = -\frac{1}{y}dx - dz.$$

Moreover we have $[f_1, f_0] = [f_2, f_0] = 0$ and $[f_2, f_1] = f_2 + f_0$ so that, in equation (23) we get $c = 0, \alpha_1 = 0, \alpha_2 = 1$. Hence

$$d\theta = -\nu_0 + \nu_2 = dz.$$

□

Now we have two canonical frames $\{\widehat{f}_1, \widehat{f}_2, f_0\}$ and $\{g_1, g_2, g_0\}$, whose Lie algebras satisfy the same commutator relations:

$$
\begin{aligned}
&[\widehat{f}_1, f_0] = -\widehat{f}_2 \qquad && [g_1, g_0] = -g_2 \\
&[\widehat{f}_2, f_0] = \widehat{f}_1 && [g_2, g_0] = g_1 \\
&[\widehat{f}_2, \widehat{f}_1] = f_0 && [g_2, g_1] = 0
\end{aligned} \tag{35}
$$

Let us consider the two control systems

$$
\begin{aligned}
\dot{q} &= u_1\widehat{f}_1(q) + u_2\widehat{f}_2(q) + u_0 f_0(q), \quad q \in A^+(\mathbb{R}) \oplus \mathbb{R} \\
\dot{x} &= u_1 g_1(x) + u_2 g_2(x) + u_0 g_0(x), \quad x \in \tilde{SL}(2)
\end{aligned}
$$

and denote with $x_u(t), q_u(t),\ t \in [0,T]$ the solutions of the equations relative to the same control $u = (u_1, u_2, u_0)$. Nagano Principle (see [4] and also [12, 14, 15]) ensure that the map

$$
\widetilde{\Psi} : A^+(\mathbb{R}) \oplus \mathbb{R} \to \tilde{SL}(2), \qquad q_u(T) \mapsto x_u(T) \tag{36}
$$

that sends the final point of the first system to the final point of the second one, is well-defined and does not depend on the control u.

Thus we can find the endpoint map of both systems relative to constant controls, i.e. considering maps[2]

$$
\widetilde{F} : \mathbb{R}^3 \to A^+(\mathbb{R}) \oplus \mathbb{R}, \qquad (t_1, t_2, t_0) \mapsto e^{t_0 f_0} \circ e^{t_2 \widehat{f}_2} \circ e^{t_1 \widehat{f}_1}(1_A) \tag{37}
$$

$$
\widetilde{G} : \mathbb{R}^3 \to SL(2), \qquad (t_1, t_2, t_0) \mapsto e^{t_0 g_0} \circ e^{t_2 g_2} \circ e^{t_1 g_1}(1_{SL}) \tag{38}
$$

The composition of these two maps makes the following diagram commutative

$$
\begin{array}{ccc}
A^+(\mathbb{R}) \oplus \mathbb{R} & \xrightarrow{\ \widetilde{\Psi}\ } & \tilde{SL}(2) \\
\Big\downarrow \widetilde{F}^{-1} & \searrow^{\Psi} & \Big\downarrow \pi \\
\mathbb{R}^3 & \xrightarrow{\ \widetilde{G}\ } & SL(2)
\end{array} \tag{39}
$$

where $\pi : \tilde{SL}(2) \to SL(2)$ is the canonical projection and we set $\Psi := \pi \circ \widetilde{\Psi}$.

To simplify computation we introduce the rescaled maps

$$
F(t) := \widetilde{F}(2t), \qquad G(t) := \widetilde{G}(2t), \qquad t = (t_1, t_2, t_0),
$$

and solving differential equations we get from (37) the following expressions:

$$
F(t_1, t_2, t_0) = \left(2e^{-2t_1} \frac{\tanh t_2}{1 + \tanh^2 t_2},\ -e^{-2t_1} \frac{1 - \tanh^2 t_2}{1 + \tanh^2 t_2},\ 2(\arctan(\tanh t_2) - t_0) \right) \tag{40}
$$

The function F is globally invertible on its image and its inverse

[2] we denote with 1_A and 1_{SL} identity element of $A^+(\mathbb{R}) \oplus \mathbb{R}$ and $\tilde{SL}(2)$ respectively.

$$F^{-1}(x,y,z)=\left(-\frac{1}{2}\log\sqrt{x^2+y^2},\ \operatorname{artanh}(\frac{y+\sqrt{x^2+y^2}}{x}),\ \arctan(\frac{y+\sqrt{x^2+y^2}}{x})-\frac{z}{2}\right)$$

is defined for every $y<0$ and for every x (is extended by continuity at $x=0$).

On the other hand the map (38) can be expressed by the product of exponential matrices as follows[3]

$$G(t_1,t_2,t_0)=\begin{pmatrix}e^{t_1} & 0\\ 0 & e^{-t_2}\end{pmatrix}\begin{pmatrix}\cosh t_2 & \sinh t_2\\ \sinh t_2 & \cosh t_2\end{pmatrix}\begin{pmatrix}\cos t_0 & -\sin t_0\\ \sin t_0 & \cos t_0\end{pmatrix} \tag{41}$$

To simplify computations we consider standard polar coordinates (ρ,θ) on the half-plane $\{(x,y),y<0\}$, where $-\pi/2<\theta<\pi/2$ is the angle that the point (x,y) defines with y-axis. In particular it is easy to see that the expression that appear in F^{-1} is naturally related to these coordinates:

$$\xi=\xi(\theta):=\tan\frac{\theta}{2}=\begin{cases}\dfrac{y+\sqrt{x^2+y^2}}{x}, & \text{if}\quad x\neq 0\\ 0, & \text{if}\quad x=0\end{cases}$$

In particular we can rewrite

$$F^{-1}(\rho,\theta,z)=\left(-\frac{1}{2}\log\rho,\ \operatorname{artanh}\xi,\ \arctan\xi-\frac{z}{2}\right)$$

and compute the composition $\Psi=G\circ F^{-1}:A^+(\mathbb{R})\oplus\mathbb{R}\longrightarrow SL(2)$. Once we substitute these expressions in (41), the third factor is a rotation matrix for the rotation on the angle $\arctan\xi-z/2$. Splitting this matrix in two consecutive rotations and using standard trigonometric identities $\cos(\arctan\xi)=\frac{1}{\sqrt{1+\xi^2}}$, $\sin(\arctan\xi)=\frac{\xi}{\sqrt{1+\xi^2}}$, $\cosh(\operatorname{artanh}\xi)=\frac{1}{\sqrt{1-\xi^2}}$, $\sinh(\operatorname{artanh}\xi)=\frac{\xi}{\sqrt{1-\xi^2}}$, for $\xi\in(-1,1)$, we obtain:

$$\Psi(\rho,\theta,z)=$$

$$=\begin{pmatrix}\rho^{-1/2} & 0\\ 0 & \rho^{1/2}\end{pmatrix}\begin{pmatrix}\dfrac{1}{\sqrt{1-\xi^2}} & \dfrac{\xi}{\sqrt{1-\xi^2}}\\ \dfrac{\xi}{\sqrt{1-\xi^2}} & \dfrac{1}{\sqrt{1-\xi^2}}\end{pmatrix}\begin{pmatrix}\dfrac{1}{\sqrt{1+\xi^2}} & -\dfrac{\xi}{\sqrt{1+\xi^2}}\\ \dfrac{\xi}{\sqrt{1+\xi^2}} & \dfrac{1}{\sqrt{1+\xi^2}}\end{pmatrix}\begin{pmatrix}\cos\dfrac{z}{2} & \sin\dfrac{z}{2}\\ -\sin\dfrac{z}{2} & \cos\dfrac{z}{2}\end{pmatrix}$$

[3]since we consider left-invariant system, we must multiply matrices on the right.

Then using identities: $\cos\theta = \dfrac{1-\xi^2}{1+\xi^2}$, $\sin\theta = \dfrac{2\xi}{1+\xi^2}$, we get

$$\Psi(\rho,\theta,z) = \begin{pmatrix} \rho^{-1/2} & 0 \\ 0 & \rho^{1/2} \end{pmatrix} \begin{pmatrix} \dfrac{1+\xi^2}{\sqrt{1-\xi^4}} & 0 \\ \dfrac{2\xi}{\sqrt{1-\xi^4}} & \dfrac{1-\xi^2}{\sqrt{1-\xi^4}} \end{pmatrix} \begin{pmatrix} \cos\dfrac{z}{2} & \sin\dfrac{z}{2} \\ -\sin\dfrac{z}{2} & \cos\dfrac{z}{2} \end{pmatrix}$$

$$= \sqrt{\frac{1+\xi^2}{1-\xi^2}} \begin{pmatrix} \rho^{-1/2} & 0 \\ 0 & \rho^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{2\xi}{1+\xi^2} & \dfrac{1-\xi^2}{1+\xi^2} \end{pmatrix} \begin{pmatrix} \cos\dfrac{z}{2} & \sin\dfrac{z}{2} \\ -\sin\dfrac{z}{2} & \cos\dfrac{z}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{\rho\cos\theta}} \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\dfrac{z}{2} & \sin\dfrac{z}{2} \\ -\sin\dfrac{z}{2} & \cos\dfrac{z}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{\rho\cos\theta}} \begin{pmatrix} \cos\dfrac{z}{2} & \sin\dfrac{z}{2} \\ \rho\sin(\theta-\dfrac{z}{2}) & \rho\cos(\theta-\dfrac{z}{2}) \end{pmatrix}$$

Lemma 23. The set $\Psi^{-1}(I)$ is a normal subgroup of $A^+(\mathbb{R}) \oplus \mathbb{R}$.

Proof. It is easy to show that $\Psi^{-1}(I) = \{F(0,0,2k\pi), k \in \mathbb{Z}\}$. From (40) we see that $F(0,0,2k\pi) = (0,-1,-4k\pi)$ and representation (32) let us to prove that this is a normal subgroup. Indeed it is sufficient to show that $\Psi^{-1}(I)$ is a subgroup of the centre, i.e.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4k\pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -y & 0 & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -y & 0 & x \\ 0 & 1 & z+4k\pi \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -y & 0 & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4k\pi \\ 0 & 0 & 1 \end{pmatrix}$$

□

Remark 24. With a standard topological argument it is possible to prove that actually $\Psi^{-1}(A)$ is a discrete countable set for every $A \in SL(2)$, and Ψ is a representation of $A^+(\mathbb{R}) \oplus \mathbb{R}$ as universal covering of $SL(2)$.

By Lemma 23 the map Ψ is well defined isomorphism between the quotient

$$\frac{A^+(\mathbb{R}) \oplus \mathbb{R}}{\Psi^{-1}(I)} \simeq A^+(\mathbb{R}) \times S^1$$

and the group $SL(2)$, defined by restriction of Ψ on $z \in [-2\pi, 2\pi]$.

If we consider the new variable $\varphi = z/2$, defined on $[-\pi,\pi]$, we can finally write the global isometry as

$$\Psi(\rho,\theta,\varphi) = \frac{1}{\sqrt{\rho\cos\theta}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \rho\sin(\theta-\varphi) & \rho\cos(\theta-\varphi) \end{pmatrix} \tag{42}$$

where $(\rho,\theta) \in A^+(\mathbb{R})$ and $\varphi \in S^1$.

Remark 25. In the coordinate set above defined we have that $1_A = (1, 0, 0)$ and

$$\Psi(1_A) = \Psi(1, 0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{SL}.$$

On the other hand Ψ is not an homomorphism since in $A^+(\mathbb{R}) \oplus \mathbb{R}$ it holds

$$\big(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, \pi\big)\big(\frac{\sqrt{2}}{2}, -\frac{\pi}{4}, -\pi\big) = 1_A,$$

while it can be easily checked from (42) that

$$\Psi\big(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, \pi\big)\Psi\big(\frac{\sqrt{2}}{2}, -\frac{\pi}{4}, -\pi\big) = \begin{pmatrix} 2 & 0 \\ 1/2 & 1/2 \end{pmatrix} \neq 1_{SL}.$$

References

[1] A. Agrachev, U. Boscain, J.-P. Gauthier, and F. Rossi. The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups. *J. Funct. Anal.*, 256(8):2621–2655, 2009.

[2] A. A. Agrachev. Exponential mappings for contact sub-Riemannian structures. *J. Dynam. Control Systems*, 2(3):321–358, 1996.

[3] A. A. Agrachev, G. Charlot, J. P. A. Gauthier, and V. M. Zakalyukin. On sub-Riemannian caustics and wave fronts for contact distributions in the three-space. *J. Dynam. Control Systems*, 6(3):365–395, 2000.

[4] A. A. Agrachev and Y. L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[5] A. Bellaïche. The tangent space in sub-Riemannian geometry. *J. Math. Sci. (New York)*, 83(4):461–476, 1997. Dynamical systems, 3.

[6] U. Boscain and F. Rossi. Invariant Carnot-Caratheodory metrics on S^3, SO(3), SL(2), and lens spaces. *SIAM J. Control Optim.*, 47(4):1851–1878, 2008.

[7] W.-L. Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

[8] V. Gershkovich and A. Vershik. Nonholonomic manifolds and nilpotent analysis. *J. Geom. Phys.*, 5(3):407–452, 1988.

[9] M. Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[10] N. Jacobson. *Lie algebras*. Interscience Tracts in Pure and Applied Mathematics, No. 10. Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.

[11] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[12] T. Nagano. Linear differential systems with singularities and an application to transitive Lie algebras. *J. Math. Soc. Japan*, 18:398–404, 1966.

[13] P. Rashevsky. Any two points of a totally nonholonomic space may be connected by an admissible line. *Uch. Zap. Ped Inst. im. Liebknechta*, 2:83–84, 1938.

[14] H. J. Sussmann. An extension of a theorem of Nagano on transitive Lie algebras. *Proc. Amer. Math. Soc.*, 45:349–356, 1974.

[15] H. J. Sussmann. Lie brackets, real analyticity and geometric control. In *Differential geometric control theory (Houghton, Mich., 1982)*, volume 27 of *Progr. Math.*, pages 1–116. Birkhäuser Boston, Boston, MA, 1983.