

Invariant Lagrange Submanifolds of Dissipative Systems

A. Agrachev *

Abstract

We study smooth solutions of modified Hamilton–Jacobi equations $H(\frac{du}{dq}, q) + \alpha u(q) = 0$, $q \in M$, on a compact manifold M .

Let M be a compact Riemannian manifold of class C^k , $k \geq 2$, with the Riemannian structure $(\xi, \eta) \mapsto \langle I_q^{-1}\xi, \eta \rangle$, $\xi, \eta \in T_qM$, $q \in M$, where $I_q : T_q^*M \rightarrow T_qM$ is a self-adjoint linear map such that the quadratic form $z \mapsto \langle z, I_q z \rangle$, $z \in T_q^*M$, is positive definite.

Let $V \in C^k(M)$ and ω be a closed differential 1-form on M of class C^k such that $\nabla\omega = 0$, where $\nabla\omega$ is the covariant derivative of ω . We consider the Hamiltonian function $H \in C^k(T^*M)$ defined by the formula:

$$H(z) = \frac{1}{2} \langle I_q(z + \omega_q), z + \omega_q \rangle + V(q), \quad z \in T_q^*M.$$

Let \vec{H} be the Hamiltonian vector field on T^*M associated to H and ℓ be the “vertical” Euler vector field of the vector bundle $T^*M \rightarrow M$. In local coordinates, $z = (p, q)$, $p, q \in \mathbb{R}^n$, $T_q^*M = (\mathbb{R}^n, q)$,

$$H(p, q) = \frac{1}{2} (p + \omega(q))^* I_q (p + \omega(q)) + V(q),$$

$$\vec{H}(p, q) = \sum_i \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right), \quad \ell(p, q) = \sum_i p^i \frac{\partial}{\partial p^i}.$$

*SISSA, Trieste and MIAN, Moscow

We study the dissipative system $\dot{z} = \vec{H}(z) - \alpha\ell(z)$, where α is a positive constant. It is not hard to see that any bounded trajectory of this system is contained in the set

$$B_H \stackrel{def}{=} \{z \in T^*M : H(z - \omega_{\pi(z)}) \leq \max_{q \in M} H(0_q)\},$$

where 0_q is the origin of T_q^*M and $\pi : T^*M \rightarrow M$, $\pi(T_q^*M) = q$.

Given $z \in T^*M$, we denote by $\rho(z)$ the maximal eigenvalue of the symmetric operator

$$\xi \mapsto \mathfrak{R}(\xi, I_q z) I_q z + (\nabla_q^2 V)\xi, \quad \xi \in T_q M,$$

where \mathfrak{R} is the Riemannian curvature. Finally, we set

$$r = \max\{\rho(z) : z \in B_H\}.$$

Let Ω^α be the set of all absolutely continuous curves $\gamma : [0, +\infty) \rightarrow M$ such that the integral $\int_0^{+\infty} e^{-\alpha t} \langle I_{\gamma(t)}^{-1} \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$ converges. We introduce the *discounted action* functional

$$\mathfrak{J}_\alpha(\gamma) = \int_0^{+\infty} e^{-\alpha t} \left(\frac{1}{2} \langle I_{\gamma(t)}^{-1} \dot{\gamma}(t), \dot{\gamma}(t) \rangle - V(\gamma(t)) + \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle \right) dt, \quad \gamma \in \Omega_\alpha.$$

Theorem 1 *Let $u(q) = -\inf\{\mathfrak{J}_\alpha(\gamma) : \gamma \in \Omega_\alpha, \gamma(0) = q\}$, $q \in M$. If $r \leq 0$ or $0 < r < \frac{\alpha^2}{4}$ and $k < \frac{2}{1 - \sqrt{1 - \frac{4r}{\alpha^2}}}$, then:*

- $u \in C^k(M)$ and the map $(H, \alpha) \mapsto u$ is continuous in the C^2 -topology.
- The function u satisfies the modified Hamilton–Jacobi equation $H(du) + \alpha u = 0$ and $\{d_q u : q \in M\} \subset T^*M$ is an invariant submanifold of the system $\dot{z} = \vec{H}(z) - \alpha\ell(z)$.
- There exists a containing 0 neighborhood \mathcal{O} of u in $C^2(M)$ such that $\forall v_0 \in \mathcal{O}$ the classical solution v_t of the Cauchy problem $\frac{\partial u_t}{\partial t} + H(du_t) + \alpha u_t = 0$, $u_0 = v_0$, is defined for all $t \geq 0$ and $\|dv_t - du\|_{C^1} \rightarrow 0$ as $t \rightarrow +\infty$ with the exponential convergence rate.

Remark. Theorem 1 is applied to the Hamiltonians in $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$H(p, q) = \frac{1}{2}|p + a|^2 + V(q),$$

where $a \in \mathbb{R}^n$ is a constant vector and V is a smooth periodic potential. Then r is the maximum of the eigenvalues of the matrices $\frac{d^2V}{dq^2}$, $q \in \mathbb{R}^n$. If $r < \frac{\alpha^2}{4}$, then the equation $\frac{1}{2}|\frac{du}{dq} + a|^2 + V(q) + \alpha u = 0$ has a periodic C^k -solution u , where k is maximal integer that is strictly smaller than $\frac{2}{1 - \sqrt{1 - \frac{4r}{\alpha^2}}}$. Moreover, $\{(\frac{du}{dq}, q) : q \in \mathbb{R}^n\}$ is an invariant submanifold of the system

$$\dot{q} = p + a, \quad \dot{p} = -\frac{dV}{dq} - \alpha p.$$

The proof of Theorem 1 can be derived from [1] and [2]. Indeed, Theorem 1 is an improvement of results of paper [1]. The improvement concerns more general Hamiltonians (nonzero forms ω are available), better smoothness of u , and stability properties. One can check that ω does not affect the canonical connection and the curvature operators; hence more general Hamiltonians do not require essential changes in the proof.

The better smoothness and stability follow from [2, Th.4.1]. Indeed, Prop. 1 in [1] implies that $\{d_q u : q \in M\}$ is a normally hyperbolic invariant submanifold (see [2] for the definition) of the flow generated by the vector field $\vec{H}(z) - \alpha \ell(z)$. Moreover, this normally hyperbolic invariant submanifold has zero unstable subbundle and can be actually called the “normally stable” invariant submanifold. Theorem 4.1 in [2] contains the estimate for the smoothness class of the normally hyperbolic invariant submanifold in terms of the Lyapunov exponents while the analysis of the proof of Prop. 1 in [1] gives explicit estimates for the Lyapunov exponents in terms of r and α .

References

- [1] A. Agrachev, *Well-posed infinite horizon variational problems on a compact manifold*, arXiv:0906.4433, 23pp.
- [2] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lecture Notes in Math., **583**. Springer Verlag, 1977, 149pp.