Any Sub-Riemannian Metric has Points of Smoothness

A. A. Agrachev*

Abstract

We prove the result stated in the title; it is equivalent to the existence of a regular point of the sub-Riemannian exponential mapping. We also prove that the metric is analytic on an open everywhere dense subset in the case of a complete real-analytic sub-Riemannian manifold.

1. Preliminaries

Let M be a smooth (i. e. C^∞) Riemannian manifold and $\Delta \subset TM$ a smooth vector distribution on M (a vector subbundle of TM). We denote by $\bar\Delta$ the space of smooth sections of Δ that is a subspace of the space $\mathrm{Vec}M$ of smooth vector fields on M. The Lie bracket of vector fields X, Y is denoted by $[X, Y]$. We assume that Δ is bracket generating; in other words, $\forall q \in M$,

$$span\{[X_1, [\cdots, [X_{m-1}, X_m] \cdots](q) : X_i \in \bar\Delta,\ i = 1, \dots m,\ m \in \mathbb{N}\} = T_qM.$$

Given $q_0, q_1 \in M$, we define the space of starting from q_0 *admissible paths*:

$$\Omega_{q_0} = \{\gamma \in H^1([0,1], M) : \gamma(0) = q_0,\ \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ for almost all } t\}$$

and the sub-Riemannian distance:

$$\delta(q_0, q_1) = \inf\{\ell(\gamma) : \gamma \in \Omega_{q_0},\ \gamma(1) = q_1\},$$

where $\ell(\gamma) = \int_0^1 |\dot\gamma(t)|\, dt$ is the length of γ and $\Delta_q = \Delta \cap T_qM$. Classical Rashevskij–Chow theorem implies that δ is a well-defined continuous function on $M \times M$.

*SISSA, Trieste & Steklov Math. Inst., Moscow

An admissible path γ is a *length-minimizer* if $\ell(\gamma) = \delta(\gamma(0), \gamma(1))$. Given $s \in [0, 1]$, we define a re-scaled path $\gamma^s : t \mapsto \gamma(st),\ t \in [0, 1]$. The re-scaled paths of any length-minimizer are also length-minimizers. According to the standard Filippov existence theorem, any $q \in M$ belongs to the interior of the set of points connected with q by a length-minimizer. If M is a complete Riemannian manifold, then length-minimizers connect q with all points of M.

From now on, the point $q_0 \in M$ is supposed to be fixed. Note that Ω_{q_0} is a smooth Hilbert submanifold of $H^1([0, 1], M)$. A smooth *endpoint mapping* $f : \Omega_{q_0} \to M$ is defined by the formula: $f(\gamma) = \gamma(1)$. Let $J : \Omega_{q_0} \to \mathbb{R}$ be the action functional, $J(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2\, dt$. The Cauchy–Schwartz inequality implies that an admissible curve realizes $\min\limits_{\gamma \in f^{-1}(q)} J(\gamma)$ if and only if this curve is a connecting q_0 with q and parameterized proportionally to the length length-minimizer. Then, according to the Lagrange multipliers rule, any starting at q_0 and parameterized proportionally to the length length-minimizer is either a critical point of f or a solution of the equation

$$\lambda D_\gamma f = d_\gamma J \tag{1}$$

for some $\lambda \in T^*_{\gamma(1)}M$, where $\lambda D_\gamma f \in T^*_\gamma \Omega_{q_0}$ is the composition of linear mappings $D_\gamma f : T_\gamma \Omega_{q_0} \to T_{\gamma(1)}M$ and $\lambda : T_{\gamma(1)}M \to \mathbb{R}$.

Now let $a \in C^1(M)$ and a curve γ realizes $\min\limits_{\gamma \in \Omega_{q_0}(M,\Delta)} (J(\gamma) - a(\gamma(1)))$; then $D_\gamma J - d_{\gamma(1)} a D_\gamma f = 0$. Hence γ satisfies (1) with $\lambda = d_{\gamma(1)}a$.

Solutions of (1) are called *normal (sub-Riemannian) geodesics* while critical points of f are *abnormal geodesics*. If a geodesic satisfies (1) for at least two different λ, then it is simultaneously normal and abnormal; all other geodesics are either strictly normal or strictly abnormal.

The *sub-Riemannian Hamiltonian* is a function $h : T^*M \to \mathbb{R}$ defined by the formula: $h(\lambda) = \max\limits_{v \in \Delta_q} \left(\langle \lambda, v \rangle - \frac{1}{2}|v|^2 \right),\ \lambda \in T^*_q M,\ q \in M$. We denote by $\vec{h}$ the associated to h Hamiltonian vector field on T^*M. A pair (γ, λ) satisfies (1) if and only if there exists a solution ψ of the Hamiltonian system $\dot\psi = \vec{h}(\psi)$ such that $\psi(1) = \lambda$ and $\psi(t) \in T^*_{\gamma(t)}M,\ \forall t \in [0, 1]$; this fact is a very special case of the Pontryagin maximum principle.

Note that $h|_{T^*_q M}$ is a nonnegative quadratic form on $T^*_q M$ whose kernel equals $\Delta_q^\perp = \{\lambda \in T^*_q M : \lambda \perp \Delta_q\}$. Given $\xi \in T^*_{q_0} M$ we denote by γ_ξ normal geodesic that is the projection to M of the solution of the Cauchy problem:

$\dot\psi = \vec h(\psi),\ \psi(0) = \xi$. This notation is well-coordinated with re-scalings: $\gamma^s_\xi = \gamma_{s\xi},\ \forall s \in [0,1]$. Finally, we define the *exponential map* $\mathcal{E} : \xi \mapsto \gamma_\xi(1)$; this is a smooth map of a neighborhood of $\Delta^\perp_{q_0}$ in $T^*_{q_0}M$ to M and $\mathcal{E}(\Delta^\perp_{q_0}) = q_0$.

2. Statements

A point $q \in M$ is called a *smooth point* (for the triple (M, Δ, q_0)) if $\exists\, \xi \in T^*_{q_0}M$ such that $q = \mathcal{E}(\xi)$, ξ is a regular point of $\mathcal{E}$ and γ_ξ is a unique length-minimizer connecting q_0 with q. We denote by Σ the set of all smooth points and assume (for all statements of this section) that M is a complete Riemannian manifold.

Theorem 1. *Σ is an open everywhere dense subset of M.*

The term “smooth point” is justified by the following fact, which should be well-known to the experts even if it is not easy to find an appropriate reference. A very close statement is contained in [4].

Theorem *. (i) *If $q \in \Sigma$, then the sub-Riemannian distance δ is smooth in a neighborhood of (q_0, q). If, additionally, M and Δ are real-analytic, then δ is analytic in a neighborhood of (q_0, q).*

(ii) *If δ is C^2 in a neighborhood of (q_0, q), then $q \in \Sigma$ and δ is actually smooth at (q_0, q).*

Corollary 1. *Sub-Riemannian distance δ is smooth on an open everywhere dense subset $\mathcal{S} \subset M \times M$. Moreover, $\mathcal{S} \cap \{q_0\} \times M$ is everywhere dense in $\{q_0\} \times M$, $\forall q_0 \in M$. If M and Δ are real-analytic, then δ is analytic on $\mathcal{S}$.*

2. Proofs

We have: $\mathrm{im} D_\xi \mathcal{E} \subset \mathrm{im} D_{\gamma_\xi} f,\ \forall \xi \in T^*_{q_0}M$, since $\mathcal{E}(\xi) \equiv f(\gamma_\xi)$. Given a normal geodesic γ we say that the point $\gamma(1)$ is conjugate to q_0 along γ if $\mathrm{im} D_\xi \mathcal{E} \neq \mathrm{im} D_{\gamma_\xi} f$ for some ξ such that $\gamma = \gamma_\xi$. The following three properties of conjugate points are specifications of more general facts whose proofs can be found in [3, Ch. 21].

i) If $\mathcal{E}(s\xi) = \gamma_\xi(s)$ is not conjugate to q_0, $\forall s \in [0,1]$, then γ_ξ is strictly shorter than any other connecting q_0 with $\gamma_\xi(1)$ admissible path from a C^0-neighborhood of γ_ξ.

ii) If ξ is a regular point of $\mathcal{E}$ and γ_ξ is strictly shorter than any other connecting q_0 with $\gamma_\xi(1)$ admissible path from a C^∞-neighborhood of γ_ξ, then $\gamma_\xi(s)$ is not conjugate to q_0 along $\gamma_{s\xi}$, $\forall s \in [0,1]$.

iii) The set $\{s \in [0,1] : \gamma_\xi(s)$ is conjugate to q_0 along $\gamma_{s\xi}\}$ is a closed subset of $[0,1]$ which does not contain 0. Moreover, this is a finite subset of $[0,1]$ if M and Δ are real-analytic.

We say that that $q \in M$ is a *RT-point* (after Rifford and Trelat) if $q = \mathcal{E}(\xi)$ for some $\xi \in T^*M$ such that γ_ξ is a unique length-minimizer connecting q_0 with q. Obviously, any smooth point is a RT-point but not vice versa! In particular, a normal geodesic γ_ξ from the definition of the RT-point can be also abnormal. If M is complete, then the set of RT-points is everywhere dense in M. This fact is proved in [5]; the proof is simple and we present it here.

Given an open subset $O \subset M$ we denote by RT_O the set of all RT-points of O. We have to show that RT_O is not empty. Let $a : O \to \mathbb{R}$ be a smooth function such that $a^{-1}([c,\infty))$ is compact for any $c \in \mathbb{R}$. Then the function $q' \mapsto \delta(q_0, q') - a(q'),\ q' \in O$, attains minimum at some point $q \in O$. Hence any connecting q_0 with q length-minimizer γ satisfies equation (1) with $\lambda = d_q a$. Then γ is the projection to M of the solution to the Cauchy problem $\dot\psi = \vec{h}(\psi),\ \psi(1) = d_q a$; in other words, $\gamma = \gamma_{\psi(0)}$.

Now we prove that Σ is everywhere dense in M. Suppose that there exists an open subset $O \subset M$ such that any point of RT_O is connected with q_0 by an abnormal length-minimizer. Given $q \in RT_O$ we set $\mathrm{rk}(q) = \mathrm{rank}\, D_\gamma f$, where γ is the length-minimizer connecting q_0 with q. Finally, let $k_O = \max\limits_{q \in RT_O} \mathrm{rk}(q)$. According to our assumption, $k_O < \dim M$.

Now take $\hat q \in RT_O$ such that $\mathrm{rk}(\hat q) = k_O$. Then $\mathrm{rk}(q) = k_O$ for any sufficiently close to $\hat q$ point $q \in RT_O$. Indeed, take a convergent to $\hat q$ sequence $q_n \in RT_O,\ n = 1, 2, \dots$. Let γ_n be the length-minimizer connecting q_0 with q_n and $\hat\gamma$ be the unique length-minimizer connecting q with $\hat q$. The uniqueness property and compactness of the space of length-minimizers (see [1]) imply that $\gamma_n \to \hat\gamma$ in the H^1-topology and $D_{\gamma_n} f \to D_{\hat\gamma} f$ as $n \to \infty$. Hence $\mathrm{rk}(\hat q) \le \mathrm{rk}(q_n)$ for all sufficiently big n.

Now, if necessary, we can substitute O by a smaller open subset and assume, without lack of generality, that $\mathrm{rk}(q) = k_O,\ \forall q \in O$. Given $q \in RT_O$ and the connecting q_0 with q length-minimizer γ we set

$$\Pi_q = \{\xi \in T^*_{q_0} M : \gamma_\xi = \gamma\}.$$

It is easy to see that Π_q is an affine subspace of $T^*_{q_0}M$; moreover, $\xi \in \Pi_q$ if and only if $\lambda = e^{\vec{h}}(\xi)$ satisfies (1), where $e^{t\vec{h}} : T^*M \to T^*M,\ t \in \mathbb{R}$, is the generated by $\vec{h}$ Hamiltonian flow. The already used compactness–uniqueness argument implies that the affine subspace $\Pi_q \subset T^*_{q_0}M$ continuously depends on $q \in RT_O$.

Consider again $\hat{q} = \hat{\gamma}(1) \in RT_O$ and take a containing $\hat{\xi}$ and transversal to $\Pi_{\hat{q}}$ $(\dim M - k_O)$-dimensional ball $\mathfrak{B}$ in $T^*_{q_0}M$. There exists a neighborhood $\hat{O}$ of $\hat{q}$ such that $\Pi_q \cap \mathfrak{B} \neq \emptyset,\ \forall q \in \hat{O} \cap RT_O$. Hence any sufficiently close to $\hat{q}$ element of RT_O belongs to the compact zero measure subset $\mathcal{E}(\mathfrak{B})$. We obtain a contradiction with the fact that RT_O is everywhere dense in M.

This contradiction proves that the set

$$RT'_O \stackrel{def}{=} \{q \in RT_O : \mathrm{rk}(q) = \dim M\}$$

of RT-points connected with q_0 by a strictly normal length-minimizer is everywhere dense in M. Now we are going to find a smooth point arbitrary close to the given point $q \in RT'_O$. According to the definitions, a point $q' \in RT'_O$ is smooth if and only if it is not conjugate to q_0 along a length-minimizer γ_ξ, where $q' = \gamma_\xi(1)$. If M and Δ are real-analytic, then $\gamma(s)$ is a smooth point for any sufficiently close to 1 number $s < 1$ (see iii)). In the general C^∞ situation we need the following

Lemma 1 (L. Rifford)**.** *Let $q \in RT'_O$; then the function*

$$q' \mapsto \delta(q_0, q') \tag{2}$$

is Lipschitz in a neighborhood of q.

Proof. In order to prove this local statement, we may fix some local coordinates in Ω_{q_0} and M and make all the computations under assumption that Ω_{q_0} is a Hilbert space and $M = \mathbb{R}^n$. Let γ_ξ be the connecting q_0 with q length-minimizer; then γ_ξ is a regular point of the endpoint map $f : \gamma \mapsto \gamma(1),\ \gamma \in \Omega_{q_0}$. Let $B_\gamma(\varepsilon)$ and $B_q(\varepsilon)$ be the centered at γ and q radius ε balls in Ω_{q_0} and $\mathbb{R}^n$. The implicit function theorem implies the existence of constants $c, \alpha > 0$ and a neighborhood $\mathcal{O}_{\gamma_\xi}$ of γ_ξ in Ω_{q_0} such that $B_{\gamma(1)}(\varepsilon) \subset f\left(B_\gamma(c\varepsilon)\right)$ for any $\gamma \in \mathcal{O}_{\gamma_\xi}$ and any $\varepsilon \in (0, \alpha]$. Then $\delta(q_0, q') \leq \ell(\gamma) + \bar{c}|\gamma(1) - q'|$, for some constant $\bar{c} > 0$ and any $\gamma \in \mathcal{O}_{\gamma_\xi},\ q' \in B_{\gamma(1)}(\alpha)$.

Now take a neighborhood U_q of q in M such that $U_q \subset B_q(\alpha),\ RT_{U_q} = RT'_{U_q}$ and the connecting q_0 with points from RT'_{U_q} length-minimizers belong to $\mathcal{O}_{\gamma_\xi}$. Then

$$\delta(q_0, q_1) - \delta(q_0, q_2) \leq \bar{c}|q_1 - q_2|, \quad \forall q_1, q_2 \in RT_{U_q}.$$

Hence function (2) is Lipschitz on RT_{U_q} and, by the continuity, on $U_q = \overline{RT}_{U_q}$. $\square$

A Lipschitz function is differentiable almost everywhere. It is easy to see that any point of differentiability of function (2) is an RT-point. Indeed, set $\phi(q) = \frac{1}{2}\delta^2(q_0, q),\ q \in M.$ The functional $J(\gamma) - \phi(\gamma(1)),\ \gamma \in \Omega_{q_0}$, attains minimum exactly on the length-minimizers. Hence a connecting q_0 with a differentiability point of ϕ length-minimizer γ must satisfy the equation $d_{\gamma(1)}\phi D_\gamma f = d_\gamma J$ and is thus unique and normal. All RT-points are values of the exponential map; according to the Sard Lemma, almost all of them must be regular values and thus smooth points!

In the next lemma, we allow to perturb the sub-Riemannian structure, i. e. the Riemannian structure on the given smooth manifold and the vector distribution Δ. The space of sub-Riemannian structures (shortly SR-structures) is endowed with the standard C^∞ topology.

Lemma 2. *Assume that M is complete and q is a smooth point. Then any sufficiently close to q point is smooth. Moreover, all sufficiently close to q points remain to be smooth after a small perturbation of q_0 and the SR-structure; the connecting q_0 with q length-minimizer smoothly depends on the triple $(q,\ q_0, \text{SR-structure})$.*

Proof. Let γ_ξ be the connecting q_0 with q length-minimizer. The fact that ξ is a regular point of $\mathcal{E}$ allows to find normal geodesics connecting any close to q_0 point with any close to q point for any sufficiently close to (M, Δ) SR-structure in such a way that the geodesic smoothly depends on all the data. It remains to show that the found geodesic is a unique length-minimizer connecting corresponding points!

It follows from property ii) of the conjugate points that $\gamma_\xi(s)$ is not conjugate to q_0 along $\gamma_{s\xi}$, $\forall s \in [0, 1]$. This fact implies the existence of a containing ξ Lagrange submanifold $\mathcal{L} \subset T^*M$ such that $\pi \circ e^{t\vec{h}}|_{\mathcal{L}}$ is a diffeomorphism of $\mathcal{L}$ on a neighborhood of $\gamma_\xi(t)$, $\forall t \in [0, 1]$, where $\pi : T^*M \to M$ is the standard projection (see [2] or [3, Ch.21]). Then γ_ξ is strictly shorter than any connecting q_0 with q admissible path γ such that

$$\gamma(t) \in \pi \circ e^{t\vec{h}}(\mathcal{L}), \quad \forall t \in [0, 1]. \qquad (3)$$

Moreover, compactness of the space of length-minimizers implies that $\exists\, \varepsilon > 0$ such that $\ell(\gamma) - \ell(\gamma_\xi) \geq \varepsilon$ for any connecting q_0 with q admissible γ which does not satisfy (3). It remains to mention that construction of $\mathcal{L}$ survives small perturbations of the SR-structure and of the initial data for normal geodesics. Hence found geodesics are indeed unique length minimizers connecting their endpoints. $\square$

Lemma 2 implies that Σ is open as soon as M is complete. This finishes proof of Theorem 1. Moreover, statement (i) of Theorem * is also a direct corollary of Lemma 2. Here is the proof of statement (ii): Let $\phi(q') = \frac{1}{2}\delta^2(q_0, q'),\ q' \in M$, then the functional $\gamma \mapsto J(\gamma) - \phi(\gamma(1)),\ \gamma \in \Omega_{q_0}(M, \Delta)$ attains minimum on the length-minimizers. Hence q is an RT-point and the connecting q_0 with q length-minimizer is γ_ξ, where $\xi = e^{-t\vec{h}}(d_q\phi)$. Now the mapping $q' \mapsto e^{-t\vec{h}}(d_{q'}\phi)$ defines a local inverse of $\mathcal{E}$ on a neighborhood of $q = \mathcal{E}(\xi)$. Hence ξ is a regular point of $\mathcal{E}$ and $q \in \Sigma$.

Acknowledgments. I am grateful to professors Morimoto, Rifford and Zelenko for very useful discussions.

Remark. The first version of this paper with a little bit weaker result to appear in the "Russian Math. Dokl.". The improvement made in the present updated version concerns the density of the set of smooth points in the general C^∞ case. The density was originally proved only for real-analytic sub-Riemannian structures while in the general smooth case we guaranteed the existence of a nonempty open subset of smooth points. I am indebted to Ludovic Rifford for the nice observation (see Lemma 1) which allows to get rid of the analyticity assumption.

References

[1] A. A. Agrachev, *Compactness for sub-Riemannian length-minimizers and subanalyticity.* Rend. Semin. Mat. Torino, 1998, v.56, 1–12

[2] A. A. Agrachev, G. Stefani, P. Zezza, *Strong minima in optimal control.* Proc. Steklov Math. Inst., 1998, v.220, 4–22

[3] A. A. Agrachev, Yu. L. Sachkov, *Control theory from the geometric viewpoint.* Springer-Verlag, Berlin, 2004, xiv+412pp.

[4] S. Jacquet, *Regularity of the sub-Riemannian distance and cut locus.* In: Nonlinear control in the year 2000, Springer Verlag, 2001, 521–533

[5] L. Rifford, E. Trelat, *Morse–Sard type results in sub-Riemannian geometry*. Math. Ann., 2005, v.332, 145–159