Hamiltonian Systems of Negative Curvature are Hyperbolic

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Abstract

The *curvature* and the *reduced curvature* are basic differential invariants of the pair: ⟨Hamiltonian system, Lagrange distribution⟩ on the symplectic manifold. We show that negativity of the curvature implies that any bounded semi-trajectory of the Hamiltonian system tends to a hyperbolic equilibrium, while negativity of the reduced curvature implies the hyperbolicity of any compact invariant set of the Hamiltonian flow restricted to a prescribed energy level. Last statement generalizes a well-known property of the geodesic flows of Riemannian manifolds with negative sectional curvatures.

1 Regularity and Monotonicity

Smooth objects are supposed to be C^∞ in this note; the results remain valid for the class C^k with a finite and not large k but we prefer not to specify the minimal possible k.

Let M be a $2n$-dimensional symplectic manifold endowed with a symplectic form σ. A *Lagrange distribution* $\Delta \subset TM$ is a smooth vector subbundle of TM such that each fiber $\Delta_x = \Delta \cap T_xM,\ x \in M$, is a Lagrange subspace of the symplectic space T_xM; in other words, $\dim \Delta_x = n$ and $\sigma_x(\xi, \eta) = 0\ \forall \xi, \eta \in \Delta_x$.

Basic examples are cotangent bundles endowed with the standard symplectic structure and the "vertical" distribution:

$$M = T^*N,\ \Delta_x = T_x(T^*_qN), \quad \forall x \in T^*_qN,\ q \in N. \tag{1}$$

Let $h \in C^\infty(M)$; then $\vec{h} \in \mathrm{Vec}M$ is the associated to h Hamiltonian vector field: $dh = \sigma(\cdot, \vec{h})$. We assume that $\vec{h}$ is a complete vector field,

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i.e. solutions of the Hamiltonian system $\dot{x} = \vec{h}(x)$ are defined on the whole time axis. We may assume that without a lack of generality since we are going to study dynamics of the Hamiltonian system on compact subsets of M and may reduce the general case to the complete one by the usual cut-off procedure.

The generated by $\vec{h}$ Hamiltonian flow is denoted by $e^{t\vec{h}},\ t \in \mathbb{R}$. Other notations: $\bar{\Delta} \subset \mathrm{Vec}M$ is the space of sections of the Lagrange distribution Δ; $[v_1, v_2] \in \mathrm{Vec}M$ is the Lie bracket (the commutator) of the fields $v_1, v_2 \in \mathrm{Vec}M,\ [v_1, v_2] = v_1 \circ v_2 - v_2 \circ v_1$.

Definition 1 *We say that $\vec{h}$ is regular at $x \in M$ with respect to the Lagrange distribution Δ if $\{[\vec{h}, v](x) : v \in \bar{\Delta}\} = T_xM$.*

An effective version of Definition 1 is as follows: Let $v_i \in \bar{\Delta},\ i = 1, \ldots, n$ be such that the vectors $v_1(x), \ldots, v_n(x)$ form a basis of Δ_x; then $\vec{h}$ is regular at $x$with respect to Δ if and only if the vectors

$$v_1(x), \ldots, v_n(x), [\vec{h}, v_1](x), \ldots, [\vec{h}, v_n](x)$$

form a basis of T_xM.

We define a bilinear mapping $g^h : \bar{\Delta} \times \bar{\Delta} \to C^\infty(M)$ by the formula:

$$g^h(v_1, v_2) = \sigma([\vec{h}, v_1], v_2).$$

Lemma 1 *$g^h(v_2, v_1) = g^h(v_1, v_2),\ \forall v_1, v_2 \in \bar{\Delta}$ and $g^h(v_1, v_2)(x)$ depends only on $v_1(x), v_2(x)$.*

Proof. Hamiltonian flows preserve σ and σ vanishes on $\bar{\Delta}$. Using these facts, we obtain:

$$0 = \sigma(v_1, v_2) = \left(e^{t\vec{h}*}\sigma\right)(v_1, v_2) = \sigma(e^{t\vec{h}}_* v_1, e^{t\vec{h}}_* v_2).$$

Differentiation of the identity $0 = \sigma(e^{t\vec{h}}_* v_1, e^{t\vec{h}}_* v_2)$ with respect to t at $t = 0$ gives: $0 = \sigma([\vec{h}, v_1], v_2) + \sigma(v_1, [\vec{h}, v_2])$. Now the anti-symmetry of σ implies the symmetry of g^h. Moreover, g^h is $C^\infty(M)$-linear with respect to each argument, hence $g^h(v_1, v_2)(x)$ depends only on $v_1(x), v_2(x)$. □

Let $x \in M,\ \xi_i \in \Delta_x,\ \xi_i = v_i(x),\ v_i \in \Delta,\ i = 1, 2$. We set $g^h_x(\xi_1, \xi_2) = g^h(v_1, v_2)(x)$. According to Lemma 1, g^h_x is a well-defined symmetric bilinear form on Δ_x. It is easy to see that the regularity of h at x is equivalent to the nondegeneracy of g^h_x.

If $M = T^*N$ and Δ is the vertical distribution (see (1)), then $g^h_x = D^2_x(h|_{T^*_qN})$, where $x \in T^*_qN$. The last equation can be easily checked in local coordinates. Indeed, local coordinates defined on a neighborhood $O \subset N$ provide the identification of $T^*N|_O$ with $\mathbb{R}^n \times \mathbb{R}^n = \{(p,q) : p,q \in \mathbb{R}^n\}$ such that T^*_qN is identified with $\mathbb{R}^n \times \{q\}$, the form σ is identified with $\sum\limits_{i=1}^n dp_i \wedge dq_i$ and the field $\vec{h}$ with $\sum\limits_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\right)$. The fields $\frac{\partial}{\partial p_i}$ form a basis of the vertical distribution and $g^h\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = -\left\langle dq_j, \left[\sum\limits_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\right), \frac{\partial}{\partial p_i}\right]\right\rangle = \frac{\partial^2 h}{\partial p_i \partial p_j}$.

Definition 2 *We say that a regular Hamiltonian field $\vec{h}$ is monotone at $x \in M$ with respect to Δ if g^h_x is a sign-definite form.*

2 The Curvature

Let X_1, X_2 be a pair of transversal n-dimensional subspaces of T_xM, then $T_xM = X_1 \oplus X_2$. We denote by $\pi_x(X_1, X_2)$ the projector of T_xM on X_2 parallel to X_1. In other words, $\pi_x(X_1, X_2)$ is a linear operator characterized by the relations $\pi_x(X_1,X_2)|_{X_1} = 0,\ \pi_x(X_1,X_2)|_{X_2} = \mathbf{1}$.

Now consider the family of subspaces $J_x(t) = e_*^{-t\vec{h}}\Delta_{e^{t\vec{h}}(x)} \subset T_xM$, where $\vec{h}$ is a regular Hamiltonian field; in particular, $J_x(0) = \Delta_x$. It is easy to check that the regularity of $\vec{h}$ implies the transversality of $J_x(t)$ and $J_x(\tau)$ for $t \neq \tau$, if t and τ are close enough to 0. Hence $\pi_x(J_x(t), J_x(\tau))$ is well-defined and smooth with respect to (t,τ) in a neighborhood of $(0,0)$ with the removed diagonal $t = \tau$. The mapping $(t,\tau) \mapsto \pi_x(J_x(t), J_x(\tau))$ has a singularity at the diagonal, but this singularity can be controlled. In particular, the following statement is valid:

Lemma 2 *(see [1]). For any regular field $\vec{h}$,*

$$\frac{\partial^2}{\partial t \partial \tau}\left(\pi_x(J_x(t), J_x(\tau))|_{\Delta_x}\right)\Big|_{\tau=0} = t^{-2}\mathbf{1} + R^h_x + O(t) \quad \text{as } t \to 0,$$

where $R^h_x \in \mathrm{gl}(\Delta_x)$ is a self-adjoint operator with respect to the scalar product g^h_x, i.e. $g^h_x(R^h_x\xi_1, \xi_2) = g^h_x(\xi_1, R^h_x\xi_2),\ \forall \xi_1, \xi_2 \in \Delta_x$.

We set $r^h_x(\xi) = g^h_x(R^h_x\xi, \xi)$.

Definition 3 *Operator R^h_x and quadratic form r^h_x are called the curvature operator and the curvature form of $\vec{h}$ at x with respect to Δ. We say that*

$\vec{h}$ has a negative (positive) curvature at x if $r_x^h(\xi)g_x^h(\xi,\xi) < 0$ (> 0), $\forall \xi \in \Delta_x \setminus \{0\}$.

It follows from the definition that only monotone fields may have negative or positive curvature. If $\vec{h}$ is monotone at x, then R_x^h has only real eigenvalues and negativity (positivity) of the curvature is equivalent to the negativity (positivity) of all eigenvalues of R_x^h.

Let us give a coordinate presentation of R_x^h. Fix local coordinates $(p,q),\ p,q \in \mathbb{R}^n$ in a neighborhood of x in M in such a way that $\Delta_x \cong \{(p,0) : p \in \mathbb{R}^n\}$. Let $(p(t;p_0), q(t;p_0))$ be the trajectory of the field $\vec{h}$ with the initial conditions $p(0;q_0) = p_0,\ q(0;p_0) = 0$. We set $S_t = \frac{\partial q(t;p_0)}{\partial p_0}|_{p_0=0}$; regularity of $\vec{h}$ is equivalent to the nondegeneracy of the $n \times n$-matrix $\dot{S}_0 = \frac{dS_t}{dt}|_{t=0}$. The curvature operator is presented by the matrix Schwartzian derivative:

$$R_x^h = 1/2\dot{S}_0^{-1}\dddot{S}_0 - 3/4(\dot{S}_0^{-1}\ddot{S}_0)^2.$$

Examples:

1. *Natural mechanical system*, $M = \mathbb{R}^n \times \mathbb{R}^n,\ \sigma = \sum_{i=1}^{n} dp_i \wedge dq_i,\ \Delta_{(p,q)} = (\mathbb{R}^n, 0),\ h(p,q) = 1/2\|p|^2 + U(q)$; then $R_{(p,q)}^h = \frac{d^2U}{dq^2}$.

2. *Riemannian geodesic flow*, $M = T^*N$ and $h|_{T_q^*N}$ is a positive quadratic form $\forall q \in N$; then h is actually a Riemannian structure on N which identifies the tangent and cotangent bundles and we have: $R_x^h\xi = \mathcal{R}(x',\xi')x'$, where $\mathcal{R}$ is the Riemanian curvature tensor and $x',\xi' \in T_qM$ are obtained from $x,\xi \in T_q^*M$ by the "raising of the indices".

3. *Mechanical system on a Riemannian manifold*, $M = T^*N$ and h is the sum of the Riemannian Hamiltonian from Example 2 and the function $U \circ \pi$, where $\pi : T^*N \to N$ is standard projection and U is a smooth function on N. Then $R_x^h\xi = \mathcal{R}(x,\xi)x + \nabla_\xi(\nabla U)$, where ∇_ξ is the Riemannian covariant derivative.

Now we introduce a *reduced curvature form* $\hat{r}_x^h$ defined on $\Delta_x \cap \ker d_xh$ and related to the restriction of the Hamiltonian system on the prescribed energy level. To do that, we need some notations. Symplectic form σ_x on T_xM induces a nondegenerate pairing of Δ_x and T_xM/Δ_x. Hence there exists a unique linear mapping $G_x : \Delta_x \to T_xM/\Delta_x$ such that $g_x(\xi_1,\xi_2) = \sigma_x(G_x\xi_1,\xi_2),\ \forall \xi_1,\xi_2 \in \Delta_x$. The mapping G_x is invertible since the form g_x is nondegenerate. Let $\Pi_x : T_xM \to T_xM/\Delta_x$ be the canonical projection. We set $v(x) = G_x^{-1}\Pi_x\vec{h}(x)$; then v is a smooth section of Δ, i.e. $v \in \bar{\Delta}$.

Assume that $\vec{h}$ is a monotone field and $\vec{h}(x) \notin \Delta_x$; the reduced curvature form is defined by the formula:

$$\hat{r}^h_x(\xi) = r^h_x(\xi) + \frac{3\sigma_x([\vec{h},[\vec{h},v]](x),\xi)^2}{4g_x(v(x),v(x))}, \quad \xi \in \Delta_x \cap \ker d_x h.$$

In Ex. 1, we obtain: $\hat{r}^h_{(p,q)}(\xi) = r^h_{(p,q)}(\xi) + \frac{3}{|p|^2}\langle \frac{dU}{dq}, \xi\rangle^2$. In Ex. 2, $\hat{r}^h_x(\xi) = r^h_x(\xi)$. Finally, in Ex. 3 (which includes both Ex. 1 and Ex. 2) we have: $\hat{r}^h_x(\xi) = r^h_x(\xi) + \frac{3g_x(d_qU,\xi)^2}{2(h(x)-U(q))}$, where $q = \pi(x)$.

We say that $\vec{h}$ has a *negative (positive) reduced curvature at* x if $\hat{r}^h_x(\xi) g^h_x(\xi,\xi) < 0$ (> 0), $\forall \xi \in \Delta_x \cap \ker d_x h \setminus \{0\}$.

3 Main Results

Theorem 1 *Let $\vec{h}$ be a monotone field and $x_0 \in M$. Assume that the semi-trajectory $\{e^{t\vec{h}}(x_0) : t \geq 0\}$ has a compact closure and $\vec{h}$ has a negative curvature at each point of its closure. Then there exists $x_\infty = \lim_{t\to+\infty} e^{t\vec{h}}(x_0)$, where $\vec{h}(x_\infty) = 0$ and $D_{x_\infty}\vec{h}$ is hyperbolic (i.e. $D_{x_\infty}\vec{h}$ has no eigenvalues on the imaginary axis).*

Remark. Monotonicity of $\vec{h}$ is equivalent to the monotonicity of $-\vec{h}$ and $R^{-h}_x = R^h_x$; hence Theorem 1 can be applied to the negative time semi-trajectories of the field $\vec{h}$ as well.

Example. Consider a natural mechanical system (Ex. 1 in Sec. 2) where $U(q)$ is a strongly concave function, then any bounded semi-trajectory of $\vec{h}$ satisfies conditions of Theorem 1.

Theorem 2 *Let $\vec{h}$ be a monotone field, S be a compact invariant subset of the flow $e^{t\vec{h}}$ contained in a fix level set of h, $S \subset h^{-1}(c)$, and $\vec{h}(x) \notin \Delta_x$ $\forall x \in S$. If $\vec{h}$ has a negative reduced curvature at each point of S, then S is a hyperbolic set of the flow $e^{t\vec{h}}\Big|_{h^{-1}(c)}$ (see [2, Sec. 17.4] for the definition of a hyperbolic set).*

Example. Mechanical system on a Riemannian manifold (Ex. 3 in Sec. 2). Let κ_q be the maximal sectional curvature of the Riemannian manifold N at $q \in N$. Then any compact invariant set S of the flow $e^{t\vec{h}}\Big|_{h^{-1}(c)}$ such that the projection of S to N is contained in the domain

$$\left\{q \in N : \kappa_q < 0,\ 2\max\{\|\nabla^2_q U\|/|\kappa_q|,\ |\nabla_q U|^2\} < c - U(q)\right\}$$

is hyperbolic. In particular, if N is a compact Riemannian manifold of a negative sectional curvature, then $e^{t\vec{h}}\Big|_{h^{-1}(c)}$ is an Anosov flow for any big enough c. Last statement generalizes a classical result on geodesic flows.

Both theorems are based on the structural equations derived in [1]. These equations are similar to the standard linear differential equation for Jacobi vector fields in Riemannian Geometry with the curvature operators R^h_x playing the same role as the Riemannian curvature. In particular, the proof of Theorem 2 simply simulates the proof of the correspondent classical result on geodesic flows. Theorem 1 describes a new phenomenon, which is not performed by geodesic flows. Indeed, if the curvature is negative, then the operators R^h_x are nondegenerate, while in the Riemannian case (Ex. 2 in Sec. 2) we have $R^h_x e(x) = 0$, where e is the Euler field (i.e. the field generating homothety of the fibers $T^*_q N$).

References

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