Rigid Carnot algebras: Classification

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Abstract

A Carnot algebra is a graded nilpotent Lie algebra $L = L_1 \oplus \cdots \oplus L_r$ generated by L_1. The bi-dimension of the Carnot algebra L is the pair $(\dim L_1, \dim L)$. A Carnot algebra is called rigid if it is isomorphic to any of its small perturbations in the space of Carnot algebras of the prescribed bi-dimension. In this paper we give a complete classification of rigid Carnot algebras. Besides free nilpotent Lie algebras there are two infinite series and 29 exceptional rigid algebras of 16 exceptional bi-dimensions.

1 Introduction

One main motivation to study Carnot algebras is their role as local nilpotent approximations of regular vector distributions.

Let M be a (C^∞-) smooth n-dimensional manifold and $\mathcal{F} \subset \mathrm{Vec}M$ be a set of smooth vector fields on M. Given $q \in M$ and an integer $l > 0$ we set

$$\Delta_q^l = \mathsf{span}\,\{[f_1, [\cdots, [f_{i-1}, f_i]\cdots](q) : f_j \in \mathcal{F},\ 1 \le j \le i,\ i \le l\} \subseteq T_qM.$$

Of course, $\Delta_q^l \subseteq \Delta_q^m$ for $l < m$. The set $\mathcal{F}$ is called bracket generating (or completely nonholonomic) at q if there exists r such that $\Delta_q^r = T_qM$. The minimal among these r is called the degree of nonholonomy of $\mathcal{F}$ at q. The set $\mathcal{F}$ is called bracket generating if it is bracket generating at every point.

Definition 1 *We say that $\mathcal{F} \subset$ VecM is regular at $q_0 \in M$ if $\dim \Delta_q^i$ is constant in a neighborhood of q_0, $\forall i > 0$.*

Let $\mathcal{F}$ be regular at q_0 and $\dim \Delta_{q_0}^1 = d$. Take $f_1, \ldots, f_d \in \mathcal{F}$ such that vectors $f_1(q_0), \ldots, f_d(q_0)$ form a basis of $\Delta_{q_0}^1$. Then $f_1(q), \ldots, f_d(q)$ form a basis of Δ_q^1 for any q from a neighborhood of q_0. Hence, for any $f \in \mathcal{F}$ there exist smooth functions $a_1, \ldots, a_d$ such that $f(q) = \sum_{i=1}^{d} a_i(q) f_i(q)$ for any q from the same neighborhood. It follows that

$$\Delta_q^l = \mathsf{span}\,\{[f_{i_1}, [\ldots, f_{i_l}] \cdots](q) : 1 \le i_j \le d\} + \Delta_q^{l-1}, \quad l = 1, 2, \ldots.$$

The regularity implies that one can select vector fields from the collection $\{[f_{i_1}, [\ldots, f_{i_l}] \cdots](q) : 1 \le i_j \le d\}$ in such a way that the values of the selected fields at q form a basis of $\Delta_q^l / \Delta_q^{l-1}$ for all q close enough to q_0. With these bases in hands we easily obtain the following well-known fact:

Lemma 1 *Assume that $\mathcal{F} \subset Vec\, M$ is regular at q_0, $v_i, v_j \in Vec\, M$, $v_i(q) \in \Delta_q^i$, $v_j(q) \in \Delta_q^j$ $\forall q$, and $v_i(q_0) = 0$. Then $[v_i, v_j](q_0) \in \Delta_{q_0}^{i+j-1}$.*

It follows immediately from this lemma that the Lie brackets of the vector fields with values in Δ_q^i, $i = 1, 2, \ldots$, induce the structure of a graded Lie algebra on the space $\sum_{i>0} \Delta_{q_0}^i / \Delta_{q_0}^{i-1}$. We denote this graded Lie algebra by $\mathrm{Lie}_{q_0} \mathcal{F}$. Obviously, $\mathrm{Lie}_{q_0} \mathcal{F}$ is generated by $\Delta_{q_0}^1$. In particular, $\mathrm{Lie}_{q_0} \mathcal{F}$ is a Carnot algebra.

Moreover, any Carnot algebra L can be realized as $\mathrm{Lie}_{q_0} \mathcal{F}$ for some $\mathcal{F}$. Indeed, let M be a Lie group with Lie algebra L and q_0 be the unit element of this group. Then L_1 is a regular bracket generating set of left-invariant vector fields on M and $L = \mathrm{Lie}_{q_0} L_1$.

We now turn to the generic case. Let $\mathcal{L}_d$ be the free Lie algebra with d generators (all algebras in this paper are over $\mathbb{R}$); in other words, $\mathcal{L}_d$ is the Lie algebra of commutator polynomials of d variables. We have $\mathcal{L}_d = \bigoplus_{i=1}^{\infty} \mathcal{L}_d^i$, where $\mathcal{L}_d^i$ is the space of degree i homogeneous commutator polynomials. Then $\mathcal{L}_d^{(r)} \stackrel{def}{=} \bigoplus_{j=1}^{\infty} \mathcal{L}_d^j / \bigoplus_{j=r+1}^{\infty} \mathcal{L}_d^j$ is the free nilpotent Lie algebra of "length" r. We set $\ell_d(i) = \dim \mathcal{L}_d^i$, $\ell_d^{(r)} = \sum_{i=1}^{r} \ell_d(i) = \dim \mathcal{L}_d^{(r)}$. The classical recursion expression of $\ell_d(i)$ is: $i\ell_d(i) = d^i - \sum_{j|i} j\ell_d(j)$.

Any Carnot algebra of bi-dimension (d, n) is a factor-algebra of $\mathcal{L}_d^{(n)}$ with respect to some graded ideal of codimension n. These algebras can be realized as follows. Any surjective linear mapping $A : \mathcal{L}_d^{(n)} \to \mathbb{R}^n$ induces a filtration of $\mathbb{R}^n$ by the subspaces $E_A^k = \sum_{i=1}^{k} A\mathcal{L}_d^i$, $k = 1, \ldots, n$. We set $\bar{A}_k : \mathcal{L}_d^k \to E_A^k / E_A^{k-1}$, the composition of $A|_{\mathcal{L}_d^k}$ with the canonical factorization, and $\bar{A} = \bigoplus_{k=1}^{n} \bar{A}_k$, the induced mapping of the graded linear spaces.

Let $\mathfrak{A}(d,n) \subset \mathrm{Hom}(\mathcal{L}_d^{(n)}, \mathbb{R}^n)$ be the set of all surjective linear mappings $A : \mathcal{L}_d^{(n)} \to \mathbb{R}^n$ such that $\ker \bar{A}$ is an ideal of $\mathcal{L}_d^{(n)}$. If $A \in \mathfrak{A}(d,n)$, then $\mathcal{L}_d^{(n)}/\ker \bar{A}$ is a Carnot algebra and any Carnot algebra can be realized in this way. Of course, different ideals may provide isomorphic Carnot algebras.

Definition 2 *A Carnot algebra L of bi-dimension (d,n) is called rigid if the set of $A \in \mathfrak{A}(d,n)$ such that $L \cong \mathcal{L}_d^{(n)}/\ker \bar{A}$ is an open subset of $\mathfrak{A}(d,n)$.*

Here symbol $\cong$ denotes the isomorphism relation for Carnot algebras.

So a Carnot algebra is rigid if it does not admit deformations: any admissible small perturbation of A gives an isomorphic Carnot algebra. As a first step towards the classification of rigid cases we describe a more general class of "generic" A which characterizes Carnot algebras $\mathrm{Lie}_{q_0}\{f_1, \dots, f_d\}$ for generic germs of d-tuples of vector fields.

Proposition 1 *Let $\mathfrak{A}_0(d,n)$ be the set of all surjective linear mappings $A : \mathcal{L}_d^{(n)} \to \mathbb{R}^n$ such that $\ker \bar{A}_i = \begin{cases} 0, & \ell_d^{(i)} < n \\ \mathcal{L}_d^i, & \ell_d^{(i)} \geq n \end{cases}$. Then $\mathfrak{A}_0(d,n) \subset \mathfrak{A}(d,n)$ and $\mathfrak{A}_0(d,n)$ is an open everywhere dense subset of Hom$(\mathcal{L}_d^{(n)}, \mathbb{R}^n)$.*

Proof. Let $r = \min\{i : \ell_d^{(i)} \geq n\}$. Then $A \in \mathfrak{A}_0(d,n)$ if and only if $A\Big|_{\bigoplus_{i=1}^{r-1} \mathcal{L}_d^i}$ is an injective map and $A\Big|_{\bigoplus_{i=1}^{r} \mathcal{L}_d^i}$ is a surjective map. Surely, these properties are valid for an open dense subset of $\mathrm{Hom}(\mathcal{L}_d^{(n)}, \mathbb{R}^n)$. Moreover, if $A \in \mathfrak{A}_0(d,n)$ then $\ker \bar{A} = \left(\ker \bar{A}_r\right) \bigoplus A\left(\bigoplus_{i=r+1}^{n} \mathcal{L}_d^i\right)$. In other words, $\ker \bar{A}$ is the direct sum of a linear subspace of $\mathcal{L}_d^r$ and $A\left(\bigoplus_{i=r+1}^{n} \mathcal{L}_d^i\right)$. Obviously, any such a subspace is an ideal of $\mathcal{L}_d^{(n)}$. $\square$

Corollary 1 *Any rigid Carnot algebra of bi-dimension (d,n) is isomorphic to $\mathcal{L}_d^{(r)}/E$, where $r = \min\{i : \ell_d^{(i)} \geq n\}$ and E is a $\left(\ell_d^{(r)} - n\right)$-dimensional subspace of $\mathcal{L}_d^r$.*

We set $\bar{m} = \ell_d^{(r)} - n$. Rigid Carnot algebras of bi-dimension (d,n) are thus characterized by $\bar{m}$-dimensional subspaces of $\mathcal{L}_d^r$. Let $\mathrm{Gr}_{\bar{m}}(\mathcal{L}_d^r)$ be the Grassmannian of $\bar{m}$-dimensional subspaces of $\mathcal{L}_d^r$. Of course, not any $E \in$

$\mathrm{Gr}_{\bar m}(\mathcal{L}_d^r)$ gives a rigid Carnot algebra. Moreover, not any bi-dimension admits a rigid Carnot algebra.

Definition 3 *A bi-dimension (d,n) is called rigid if there exists a rigid Carnot algebra of bi-dimension (d,n).*

Two Carnot algebras $\mathcal{L}_d^{(r)}/E_i,\ E_i \in \mathrm{Gr}_{\bar m}(\mathcal{L}_d^r),\ i=1,2$, are isomorphic if and only if there exists an automorphism of $\mathcal{L}_d^{(r)}$ which transforms E_1 into E_2. The automorphisms of the free nilpotent Lie algebra $\mathcal{L}_d^{(r)}$ are in one-to-one correspondence with linear transformations of $\mathbb{R}^d = \mathcal{L}_d^1$. More precisely, the rule

$$V^{(i)}[x_1,[\dots,x_i]\dots] \overset{def}{=} [Vx_1,[\dots,Vx_i]\dots], \quad x_1,\dots,x_i \in \mathcal{L}_d^1,$$

provides a canonical extension of $V \in \mathrm{GL}(\mathbb{R}^d)$ to the automorphism $V^{(1)} \oplus \cdots \oplus V^{(r)}$ of $\mathcal{L}_d^{(r)}$. In particular, we obtain a canonical action $V \mapsto V^{(r)}$ of $\mathrm{GL}(\mathbb{R}^d)$ on $\mathcal{L}_d^r$; Carnot Lie algebras $\mathcal{L}_d^{(r)}/E_i,\ i=1,2$ are isomorphic if and only if there exists $V \in \mathrm{GL}(\mathbb{R}^d)$ such that $V^{(r)}E_1 = E_2$.

Let $\bar\Phi(V) : \mathrm{Gr}_{\bar m}(\mathcal{L}_d^r) \to \mathrm{Gr}_{\bar m}(\mathcal{L}_d^r),\ V \in \mathrm{GL}(\mathbb{R}^d)$, be the induced action of $\mathrm{GL}(\mathbb{R}^d)$ on the Grassmannian so that $\bar\Phi(V)(E) = V^{(r)}E,\ E \in \mathrm{Gr}_{\bar m}(\mathcal{L}_d^r)$. The Carnot algebra $\mathcal{L}_d^{(r)}/E$ is rigid if and only if E belongs to a full-dimensional orbit of the action $\bar\Phi$. In particular, the bi-dimension (d,n) is rigid if and only if there exists a full-dimensional orbit of $\bar\Phi$. Moreover, such orbits are actually in one-to-one correspondence with the isomorphism classes of rigid Carnot algebras. The action $\bar\Phi$ is algebraic; this implies the following

Corollary 2 *Let (d,n) be a rigid bi-dimension. Then the set of $E \in Gr_{\bar m}(\mathcal{L}_d^r)$ such that $\mathcal{L}_d^{(r)}/E$ is rigid is a Zarisski open (in particular, open dense) subset of $Gr_{\bar m}(\mathcal{L}_d^r)$ and there is only a finite number of mutually nonisomorphic rigid Carnot algebras of the bi-dimension (d,n).*

In the next theorem we list all rigid bi-dimensions. It is convenient to give special names to some infinite series of bi-dimensions. For $d = 2,3,4,\dots$, the bi-dimensions $(d,\ell_d^{(i)}),\ i=1,2,3,\dots$, are called *free*; the bi-dimension $(d,d+1)$ is called *Darboux bi-dimension*, and the bi-dimension $(d,(d-1)(d+2)/2)$ is called *dual Darboux bi-dimension.*

Theorem 1 *All free, Darboux, and dual Darboux bi-dimensions are rigid; any of these bi-dimensions admits a unique up to an isomorphism rigid Carnot algebra. Besides that, there are 16 exceptional rigid bi-dimensions:*

$(2,4)_1,\ (2,6)_2,\ (2,7)_2,\ (4,6)_2,\ (4,7)_2,\ (4,8)_2,$

$(5,7)_1,\ (5,8)_2,\ (5,9)_3,\ (5,11)_3,\ (5,12)_2,\ (5,13)_1,$
$(6,8)_2,\ (6,19)_2,\ (7,9)_1,\ (7,26)_1,$

where index j in the expression $(d,n)_j$ indicates the number of isomorphism classes of rigid Carnot algebras for the given bi-dimension (d,n).

All other bi-dimensions are not rigid.

In the rest of the paper we will prove this theorem: in Section 2 we will give a necessary condition for a bi-dimension to be rigid. We obtain that only free bi-dimensions are rigid if the degree of nonholonomy r is bigger than 4. Few following sections are devoted to the analysis of bi–dimensions corresponding to $r = 2, 3, 4$: Section 3 for $r = 2$, Section 4 for $r = 3$ and Section 5 for $r = 4$. We present a canonical basis and the multiplication table for any isomorphism class of rigid Carnot algebras. These multiplication tables are then used in Section 6 to give the normal forms for all possible rigid Lie algebras of vector fields.

2 Rigidity: Necessary Condition

We have the following:

Proposition 2 *Let (d,n) be a rigid bi–dimension; then*

$$d^2 > (\ell_d(r) - \bar{m})\bar{m}. \tag{1}$$

Proof. It was shown in the previous section that to rigid Carnot algebras there correspond full-dimensional orbits of the action of $\mathrm{GL}(\mathbb{R}^d)$ on $\mathrm{Gr}_{\bar{m}}(\mathcal{L}_d^r)$. Let us compair the dimensions. We have $\dim \mathrm{GL}(\mathbb{R}^d) = d^2$, $\mathcal{L}_d^r = \mathbb{R}^{\ell_d(r)}$, $\dim \mathrm{Gr}_{\bar{m}}(\mathcal{L}_d^r) = \bar{m}(\ell_d(r) - \bar{m})$. Taking into account that scalar multiples of the identity matrix from $\mathrm{GL}(\mathbb{R}^d)$ act trivially on the Grassmannian, we obtain that a necessary condition for the existence of a full-dimensional orbit is:

$$d^2 - 1 \geq (\ell_d(r) - \bar{m})\bar{m}$$

and the proposition is proved. $\square$

First of all we observe that condition (1) is trivially satisfied when $\bar{m} = 0$. Moreover, the condition is satisfied for some $\bar{m}$ if and only if it is satisfied for $m = \ell_d(r) - \bar{m}$.

For $r = 1$ we have $\ell_d(1) = d$, hence, by definition of $\bar{m}$, it must be $\bar{m} = 0$. These cases correspond to the free bi–dimension (d,d).
For $r = 2$, since $\ell_d(2) = \frac{1}{2}d(d-1)$, condition (1) is verified for all $\bar{m} = 0, 1, \ldots, \ell_d(2) - 1$ if $d \leq 4$, for $\bar{m} = 0, 1, \ldots, \ell_d(2) - 1$, with $\bar{m} \neq 5$, if $d = 5$

and for all $\bar{m} = 0, 1, 2, \ell_d(2) - 2, \ell_d(2) - 1$ if $d \geq 6$. Notice that bi–dimensions corresponding to $\bar{m} = 0, 1, \ell_d(2) - 1$ are respectively free, dual Darboux and Darboux bi–dimensions.
For $r = 3$, $\ell_d(3) = \frac{1}{3}(d^3 - d)$ and condition (1) holds for $\bar{m} = 0, 1 = \ell_d(3) - 1$ if $d = 2$ and for $\bar{m} = 0, 1, \ell_d(3) - 1$ if $d = 3$. The bi–dimensions corresponding to $\bar{m} = 0$ are free.
Finally, for $r = 4$, $\ell_d(4) = \frac{1}{4}(d^4 - d^2)$ and condition (1) holds for all $\bar{m} = 0, 1, \ldots, \ell_d(4) - 1$ if $d = 2$.
For $r > 4$ condition (1) is never satisfied for $\bar{m} > 0$.

In synthesis, beside the free bi-dimensions, we have the following cases to analyse:

$r = 2$	$d = 3$	$\bar{m} = 1, 2$
	$d = 4$	$\bar{m} = 1, 2, 3, 4, 5$
	$d = 5$	$\bar{m} = 1, 2, 3, 4, 6, 7, 8, 9$
	$d \geq 6$	$\bar{m} = 1, 2, \ell_d(2) - 2, \ell_d(2) - 1$
$r = 3$	$d = 2$	$\bar{m} = 1$
	$d = 3$	$\bar{m} = 1, 7$
$r = 4$	$d = 2$	$\bar{m} = 1, 2$

Let $m = \ell_d(r) - \bar{m}$ and $\mathcal{L}_d^{r^\star}$ be the adjoint space to $\mathcal{L}_d^r$. The involution $E \mapsto E^\perp$ sends $\bar{m}$-dimensional subspaces of $\mathcal{L}_d^r$ into m-dimensional subspaces of $\mathcal{L}_d^{r^\star}$. Denote by Φ the corresponding action of $\mathrm{GL}(\mathbb{R}^d)$ on $\mathrm{Gr}_m(\mathcal{L}_d^{r^\star})$; it acts according to the rule: $\Phi(V)(E^\perp) = (\bar{\Phi}(V)E)^\perp$.

In forthcoming sections we deal with the action Φ on $\mathrm{Gr}_m(\mathcal{L}_d^{r^\star})$ rather than with the action $\bar{\Phi}$ on $\mathrm{Gr}_{\bar{m}}(\mathcal{L}_d^r)$; this makes shorter the way from the classification of subspaces to the tables of products of the Lie algebras. Moreover, we mainly work in a fixed Hall basis of $\mathcal{L}_d^r$ and do not make difference between $\mathcal{L}_d^r$ and $\mathcal{L}_d^{r^\star}$.

3 Cases with $r = 2$

The following proposition allows us to reduce the analysis of possibly rigid bi-dimensions for $r = 2$.

Proposition 3 *If $r = 2$ then the bi–dimension $(d, \ell_d^{r-1} + m)$ is rigid if and only if the dual bi–dimension $(d, \ell_d^r - m)$ is rigid. Moreover the number of isomporphism classes of rigid algebras for the bi–dimension $(d, \ell_d^{r-1} + m)$ and the dual bi–dimension $(d, \ell_d^r - m)$ is the same.*

Proof. Let Φ be the action on $\mathrm{Gr}_m(\mathcal{L}_d^2)$ as in Section 1. Fix $E \in \mathrm{Gr}_m(\mathcal{L}_d^2)$ and consider the following maps

$$\begin{array}{rccc} \Psi: & \mathrm{GL}(\mathbb{R}^d) & \to & \mathrm{Gr}_m(\mathcal{L}_d^2) \\ & V & \mapsto & \Phi(V)(E) \\ \tilde{\Psi}: & \mathrm{GL}(\mathbb{R}^d) & \to & \mathrm{Gr}_{l_d(2)-m}(\mathcal{L}_d^2) \\ & V & \mapsto & \Phi(V)(E^{\perp}). \end{array}$$

Let $V \in \mathrm{GL}(\mathbb{R}^d)$, and denote by $V^{-T} = (V^T)^{-1}$. We show that for all $V \in \mathrm{GL}(\mathbb{R}^d)$, $\tilde{\Psi}(V^{-T}) = (\Psi(V))^{\perp}$ so that a one to one correspondence between the image of Ψ and that of $\tilde{\Psi}$ is established and the proposition is proved.

Let $V \in \mathrm{GL}(\mathbb{R}^d),\ V = (v_{il})_{i,l=1}^d$ and $g_i(q) = \sum_{l=1}^d v_{il}(q) f_l(q),\ i = 1, \ldots, d$. Then

$$[g_i, g_j] = \Phi(V)[f_i, f_j] = \sum_{l<k} \mathsf{det} \left(\begin{bmatrix} v_{il} & v_{ik} \\ v_{jl} & v_{jk} \end{bmatrix} \right) [f_l, f_k].$$

Hence, $\Phi(V^T) = (\Phi(V))^T$ and $\Phi(V_1 V_2) = \Phi(V_1)\Phi(V_2)$ for all $V, V_1, V_2 \in \mathrm{GL}(\mathbb{R}^d)$. It follows that $\Phi(V^{-T}) = (\Phi(V))^{-T}$ and for all $w_1 \in E, w_2 \in E^{\perp}$,

$$< \Phi(V^{-T}) w_2, \Phi(V) w_1 >= w_2^T (\Phi(V))^{-1} \Phi(V) w_1 =< w_2, w_1 >= 0,$$

which proves that $\tilde{\Psi}(V^{-T}) = (\Psi(V))^{\perp}$. $\square$

Assume that we know a multiplication table for some m then we obtain the dual multiplication table as follows.

Let f^{π_i}, $i = 1, \ldots, \ell_d(2)$, be Lie brackets of order 2 which are linearly independent with respect to the Jacobi identity. Assume that the multiplication table gives: $f^{\pi_i} = \sum_{j=1}^m \lambda_{ij} f^{\pi_j}$, for $i = m+1, \ldots, \ell_d(2)$, i.e. $\Lambda f = 0$ where $\Lambda = [\hat{\Lambda} \,|\, -I_{\ell_d(2)-m}]$,

$$\hat{\Lambda} = \begin{bmatrix} \lambda_{(m+1)1} & \cdots & \lambda_{(m+1)m} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell_d(2)1} & \cdots & \lambda_{\ell_d(2)m} \end{bmatrix},$$

and $f = [f^{\pi_1}, \ldots, f^{\pi_{\ell_d(2)}}]$. Then $\Lambda^{\perp} = [I_m \,|\, \hat{\Lambda}^T] f$ represents the orthogonal space to that generated by Λ and the dual multiplication table is given by $\Lambda^{\perp} f$, i.e. $f^{\pi_j} = -\sum_{i=m+1}^{\ell_d(2)} \lambda_{ij} f^{\pi_i}$, for $j = 1, \ldots, m$. As example, in this paper we will give the dual multiplication table and the corresponding normal form for $m = 1$. The other dual cases may be obtained as seen so far.

The space $\mathcal{L}_d^2$ is identified with the wedge square $\bigwedge^2 \mathbb{R}^d$. Hence any $E \in \mathrm{Gr}_m(\mathcal{L}_d^2)$ is identified with an m-dimensional vector space of $d \times d$ antisymmetric matrices. In order to fix notations we next describe this identification (and the corresponding action of $\mathrm{GL}(\mathbb{R}^d)$) in more detail.

Fix $f_1, \ldots, f_d$ generators for $\mathcal{L}_d^1$ and let $f^{\pi_1}, \ldots, f^{\pi_m}$ be such that they form a basis of $E \in \mathrm{Gr}_m(\mathcal{L}_d^2)$. Then we can write $[f_l, f_k] = \sum_{h=1}^m \omega_{lk}^h f^{\pi_h}$. Notice that, since $[f_l, f_k] = -[f_k, f_l]$, for all $h = 1, \ldots, m$, $\omega^h = \{\omega_{lk}^h\}_{lk}$ is a $d \times d$ antisymmetric matrix. For a different choice of the set $\{f^{\pi_h}, h = 1, \ldots, m\}$, that is $f^{\pi_h} = \sum_{i=1}^m x_{hi} \tilde{f}^{\pi_i}$, we have

$$[f_l, f_k] = \sum_{i=1}^m \tilde{\omega}_{lk}^i \tilde{f}^{\pi_i} \tag{2}$$

where $\tilde{\omega}^i = \sum_{h=1}^m x_{hi} \omega^h$.

Consider the space generated by ω^h, $h = 1, \ldots, m$ and write any element of the space under consideration as $\omega(x) = \sum_i x_i \omega^i$, $x = (x_1, \ldots, x_m)$. Let $V \in \mathrm{GL}(\mathbb{R}^d)$. Then

$$\Phi(V)[f_i, f_j] = \sum_{lk} v_{il} v_{jk} [f_l, f_k] = \sum_h \sum_{lk} v_{il} v_{jk} \omega_{lk}^h f^{\pi_h} = \sum_h (V \omega^h V^T)_{ij} f^{\pi_h}.$$

Hence $\Phi(V)\omega(x) = V\omega(x)V^T$.

Next we analyse all the possibly rigid bi–dimension for $r = 2$ up to duality:

$(d, d+1)$ for any d, corresponding to $m = 1$, section 3.1;

$(d, d+2)$ for $d \geq 4$, corresponding to $m = 2$, section 3.2;

$(d, d+3)$ for $d = 4$ and $d = 5$ corresponding to $m = 3$, section 3.3;

$(d, d+4)$ for $d = 5$ corresponding to $m = 4$, section 3.4.

3.1 The case $m = 1$

In this case we have a 1 dimensional space of antisymmetric $d \times d$ matrices. A generic $d \times d$ antisymmetric matrix ω can be written as $\omega = VDV^T$ where V is nonsingular and D is block diagonal matrix with blocks D_i as described next.

- If ω is a $d \times d$ matrix with d even then $D_i, i = 1, \ldots d/2$ are all 2×2 of type $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- if ω is a $d \times d$ matrix with d odd then $D_i, i = 1, \ldots (d-1)/2$ are all 2×2 of type $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and D_i for $i = (d-1)/2 + 1$ is the zero 1-dimensional block.

Then we have:

Proposition 4 *The Darboux bi–dimension* $(d, d+1)$ *is rigid with unique isomorphism class. The representing family* $\mathcal{F}$ *is completely described by the following multiplication table for d even (d odd):*

$$[f_{2i-1}, f_j] = \begin{cases} [f_1, f_2] & \text{if } j = 2i \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, \ldots, \frac{d}{2} \; (\frac{d-1}{2} \text{ for } d \text{ odd }) \quad (3)$$

By duality, also the dual Darboux bi–dimension $(d, (d-1)(d+1)/2)$ *is rigid with unique isomorphism class. The multiplication table is:*

$$[f_1, f_2] = \begin{cases} \sum_{i=2}^{\frac{d}{2}} [f_{2i-1}, f_{2i}] & \text{if } d \text{ is even} \\ \sum_{i=2}^{\frac{d-1}{2}} [f_{2i-1}, f_{2i}] & \text{if } d \text{ is odd} \end{cases} . \quad (4)$$

The normal forms are given in Section 6, equations (34) and (35), for the Darboux and dual Darboux bi–dimension, respectively.

3.2 The case $m = 2$

If $m = 2$, then any $E \in \mathrm{Gr}_m(\mathcal{L}_d^2)$ is identified with a 2–dimensional subspace of the vector space $\bigwedge^2 \mathbb{R}^d$ of $d \times d$ antisymmetric matrices . We distinguish among d even or odd.

Assume first that d is even and that $\mathsf{Pf}\,(\omega)$ is the Pfaffian of the $d \times d$ antisymmetric matrix ω. Recall that Pf is a degree $\frac{d}{2}$ homogeneous polynomial such that $(\,\mathsf{Pf}\,(\omega))^2 = \,\mathsf{det}\,(\omega)$.

Let $\omega^1, \omega^2 \in \bigwedge^2 \mathbb{R}^d$ form a basis of the subspace E under consideration so that any element of the subspace can be written as $\omega(x_1, x_2) = x_1\omega^1 + x_2\omega^2$. Consider the polynomial $p(x_1, x_2) = \,\mathsf{Pf}\,(\omega(x_1, x_2))$. A change of the basis of E induces a linear change of variables of the polynomial $p(x_1, x_2)$ and the transformation $\omega \mapsto V^T \omega V$, $V \in \mathrm{GL}(\mathbb{R}^d)$ preserves $p(x_1, x_2)$ up to a scalar multiplier since $\mathsf{Pf}\,(V^T\omega(x_1, x_2)V) = \,\mathsf{det}\, V \,\mathsf{Pf}\,(\omega(x_1, x_2))$. The following holds:

Proposition 5 *If d is even then the codimension of any orbit of the action* Φ *in* $Gr_2(\mathcal{L}_d^2)$ *is no less than* $\frac{d}{2} - 3$.

Proof. The space of degree $\frac{d}{2}$ homogeneous polynomials of 2 variables has dimension $\frac{d}{2} + 1$ and the group $\mathrm{GL}(2)$ of linear changes of variables in the plane is 4-dimensional. The polynomials $p(x_1, x_2) = \,\mathsf{Pf}\,(\omega(x_1, x_2))$ are invariant by the action Φ up to linear changes of variables. We have $\frac{d}{2} + 1 - 4 = \frac{d}{2} - 3$. What remains to show is that any polynomial of degree $\frac{d}{2}$ is realized as $\mathsf{Pf}\,(\omega(x_1, x_2))$. This is easy. Consider for example

$$\omega^1 = \alpha_0 \begin{bmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix} \quad \omega^2 = \begin{bmatrix} \alpha_1 J & 0 & \cdots & 0 \\ 0 & \alpha_2 J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{\frac{d}{2}} J \end{bmatrix},$$

where by J we denote the 2×2 antisymmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
We have that $\mathsf{Pf}\,(\omega(x_1, x_2)) = (\alpha_0 x_1 + \alpha_1 x_2)(\alpha_0 x_1 + \alpha_2 x_2) \cdots (\alpha_0 x_1 + \alpha_{\frac{d}{2}} x_2)$, hence any polynomial of degree $\frac{d}{2}$ in the variables x_1, x_2 can be obtained by suitably choosing $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{\frac{d}{2}}$. □

Corollary 3 *Let d be even and $(d, d+2)$ be rigid; then $d < 8$.*

By Corollary 3 we only have to analyse the cases with $d < 8$, that is $d = 4$ and $d = 6$. For these cases we have the following:

Proposition 6 *For $d = 4$ and $d = 6$ the bi–dimension $(d, d+2)$ is rigid with two isomorphism classes distinguished by the sign of the discriminant of the polynomial $\mathsf{Pf}\,(\omega(x_1, x_2))$.*

Proof. Observe that the roots of

$$\mathsf{Pf}\,(\omega(x_1, x_2)) = 0 \tag{5}$$

can be:

$d = 4$**:** either real or complex conjugate.

$d = 6$**:** either three real or one real and two complex conjugate.

Next we provide the multiplication table for a representing family $\mathcal{F}$ for each of the above cases. This will show that the bi–dimensions $(d, d+2)$ are rigid and the isomorphism class is uniquely reconstructed from the number of real roots of equation (5).

$d = 4$, real case

Consider a generic 2 dimensional subspace of $\bigwedge^2 \mathbb{R}^d$. Then equation (5) has simple roots and to form a basis of the subspace under consideration we can choose two corank 2, 4×4 antisymmetric matrices, ω^1 and ω^2, with transversal kernel. Let $e_1, e_2, e_3, e_4 \in \mathbb{R}^4$, be linearly independent and such that $e_1, e_2 \in \mathsf{ker}\,\omega^2$ with $e_1^T\omega^1 e_2 = 1$ and $e_3, e_4 \in \mathsf{ker}\,\omega^1$ with $e_3^T\omega^2 e_4 = 1$. By writing the equation (2) in these coordinates we obtain the following multiplication table:

$$\begin{aligned}
&[f_1, f_2] = f^{\pi_1}\\
&[f_3, f_4] = f^{\pi_2}\\
&[f_i, f_j] = 0 \ \text{ otherwise.}
\end{aligned} \tag{6}$$

The normal form for $\mathcal{F}$ is reported in Section 6 equation (36).

$d = 4$, complex case

Let $(x_1, 1)$, $(\bar{x}_1, 1)$ a pair of conjugate complex solutions to equation (5). Then $\omega^1 = x_1\tilde{\omega}^1 + \tilde{\omega}^2$ and $\omega^2 = \bar{x}_1\tilde{\omega}^1 + \tilde{\omega}^2$ are two corank 2, 4×4 antisymmetric matrices with complex coefficients such that

$$\begin{aligned}
\omega^1 + \omega^2 &= 2\Re(\omega^1)\\
\omega^1 - \omega^2 &= 2\imath\Im(\omega^1).
\end{aligned}$$

Observe that it is sufficient to find a normal form for $\Re(\omega^1)$ and $\Im(\omega^1)$, indeed

$$\begin{aligned}
&\omega^1 f^{\pi_1} + \omega^2 f^{\pi_2} = (\Re(\omega^1) + \imath\Im(\omega^1))f^{\pi_1} + (\Re(\omega^1) - \imath\Im(\omega^1))f^{\pi_2} =\\
&(\Re(\omega^1) + \imath\Im(\omega^1))\left(\tfrac{1+\imath}{4}\tilde{f}^{\pi_1} + \tfrac{1-\imath}{4}\tilde{f}^{\pi_2}\right) + (\Re(\omega^1) - \imath\Im(\omega^1))\left(\tfrac{1-\imath}{4}\tilde{f}^{\pi_1} + \tfrac{1+\imath}{4}\tilde{f}^{\pi_2}\right)\\
&= (\Re(\omega^1) + \imath\Im(\omega^1))\left(\tfrac{1}{4}(\tilde{f}^{\pi_1} + \tilde{f}^{\pi_2}) + \tfrac{1}{4}\imath(\tilde{f}^{\pi_1} - \tilde{f}^{\pi_2})\right)\\
&\qquad\qquad\qquad + (\Re(\omega^1) - \imath\Im(\omega^1))\left(\tfrac{1}{4}(\tilde{f}^{\pi_1} + \tilde{f}^{\pi_2}) - \tfrac{1}{4}\imath(\tilde{f}^{\pi_1} - \tilde{f}^{\pi_2})\right)\\
&= \tfrac{1}{2}\left(\Re(\omega^1)(\tilde{f}^{\pi_1} + \tilde{f}^{\pi_2}) - \Im(\omega^1)(\tilde{f}^{\pi_1} - \tilde{f}^{\pi_2})\right)\\
&= \tfrac{1}{2}\left(\Re(\omega^1) - \Im(\omega^1)\right)\tilde{f}^{\pi_1} + \tfrac{1}{2}\left(\Re(\omega^1) + \Im(\omega^1)\right)\tilde{f}^{\pi_2}.
\end{aligned}$$

Let $p = p_1 + \imath p_2,\ q = p_3 + \imath p_4 \ \in\ \mathsf{ker}\,\omega^1$, with $p_1\Re(\omega^1)p_3 = 1$, $p_2\Im(\omega^1)p_4 = 1$. Then, in the coordinates p_1, p_2, p_3, p_4, we can write:

$$\Re(\omega^1) = \begin{bmatrix} 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\\ -1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0 \end{bmatrix} \quad \Im(\omega^1) = \begin{bmatrix} 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0 \end{bmatrix}.$$

Finally, writing the equation $\omega = \omega^1\tilde{f}^{\pi_1} + \omega^2\tilde{f}^{\pi_2}$ in the new coordinates, we obtain the following multiplication table:

$$\begin{aligned}
&[f_1, f_2] = [f_3, f_4] = 0,\\
&[f_1, f_3] = -[f_2, f_4] = \tilde{f}^{\pi_1},\\
&[f_1, f_4] = [f_2, f_3] = \tilde{f}^{\pi_2}.
\end{aligned} \tag{7}$$

The normal form for $\mathcal{F}$ is reported in Section 6 equation (37).

$d = 6$, real case

Consider a generic 2 dimensional subspace of $\bigwedge^2 \mathbb{R}^d$. Then equation (5) has simple roots: $(x_1, 1)$, $(x_2, 1)$ and $(x_3, 1)$. Let $\lambda_1, \lambda_2, \lambda_3 \neq 0$ be such that $\lambda_1(x_1, 1) + \lambda_2(x_2, 1) = \lambda_3(x_3, 1)$. Hence $\omega^i = \lambda_i x_i \tilde{\omega}^1 + \lambda_i \tilde{\omega}^2$, for $i = 1, 2, 3$, are 6×6 antisymmetric matrices such that $\omega^3 = \omega^1 + \omega^2$. Moreover, by generic assumptions, we also have that the kernels of the above matrices are transversal. Then let p_i, $i = 1, \ldots, 6 \in \mathbb{R}^6$ be linearly independent, with $p_1, p_2 \in \ \mathsf{ker}\, \omega^2$, $p_3, p_4 \in \ \mathsf{ker}\, \omega^1$ and $p_5, p_6 \in \ \mathsf{ker}\, \omega^3$ such that $p_1 \omega^1 p_2 = 1$ and $p_3 \omega^2 p_4 = 1$. In these coordinates we can write

$$\omega^1 = \begin{bmatrix} J & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -J \end{bmatrix}, \quad \omega^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}, \quad \omega^3 = \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, equation (2) gives the following multiplication table for $\mathcal{F}$:

$$\begin{aligned} &[f_5, f_6] = -[f_1, f_2] + [f_3, f_4] \\ &[f_i, f_j] = 0 \quad \text{otherwise.} \end{aligned} \tag{8}$$

The normal form is given in Section 6, equation (39).

$d = 6$, complex case

Let $(x_1, 1)$, $(x_2, 1)$ and $(x_3, 1)$ be the three solutions to equation (5) with $x_3 \in \mathbb{R}$ and $x_2 = \bar{x}_1$, where by $\bar{x}$ we denote the conjugate of x. There exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, with $\lambda_2 = \bar{\lambda}_1$ and $\lambda_3 \in \mathbb{R}$, such that $\omega^i = \lambda_i x_i \tilde{\omega}^1 + \lambda_i \tilde{\omega}^2$, for $i = 1, 2, 3$, is antisymmetric with

$$\begin{aligned} &\omega^1 + \omega^2 = 2\Re(\omega^1) \\ &\omega^1 - \omega^2 = 2\imath\Im(\omega^1) \\ &\omega^3 = \omega^1 + \omega^2 = 2\Re(\omega^1). \end{aligned}$$

Let $p_1, p_2 \in \ \mathsf{ker}\ (\omega^3)$ with $p_1 \Im(\omega^1) p_2 = 1$ and $p_3, \ldots, p_6$ orthogonal to the 2–dimensional space generated by $\{\Im(\omega^1) p_i,\ i = 1, 2\}$. In these coordinates we write:

$$\Re(\omega^1) = \begin{bmatrix} 0 & 0 \\ 0 & \Re(\omega^1)_{22} \end{bmatrix} \quad \Im(\omega^1) = \begin{bmatrix} J & 0 \\ 0 & \Im(\omega^1)_{22} \end{bmatrix},$$

where $\Re(\omega^1)_{22}$ and $\Im(\omega^1)_{22}$ are 4×4 antisymmetric matrices with $\Re(\omega^1)_{22} \pm \imath\Im(\omega^1)_{22}$ of corank 2. Therefore we are reduced to consider the complex case for $d = 4$ and, with the same arguments, we can write

$$\Re(\omega^1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}, \Im(\omega^1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix},$$

and

$$\omega = \frac{1}{2}(\Re(\omega^1) - \Im(\omega^1))\tilde{f}^{\pi_1} + \frac{1}{2}(\Re(\omega^1) + \Im(\omega^1))\tilde{f}^{\pi_2}.$$

Finally we obtain the following multiplication table:

$$\begin{aligned} &[f_1, f_2] = \tilde{f}^{\pi_1} - \tilde{f}^{\pi_2}, \\ &[f_3, f_5] = -[f_4, f_6] = \tilde{f}^{\pi_1}, \\ &[f_3, f_6] = [f_4, f_5] = \tilde{f}^{\pi_2}, \\ &[f_i, f_j] = 0 \ \text{ otherwise.} \end{aligned} \tag{9}$$

The normal form for $\mathcal{F}$ is reported in Section 6 equation (40). □

To complete the analysis for $m = 2$ it remains to study the cases where d is odd.

Proposition 7 *Let $d = 2k+1$. Then the codimension of any orbit of the action Φ on $Gr_2(\mathcal{L}_d^2)$ is no less than $k - 3$.*

Proof. Consider the action $(V; (\omega^1, \omega^2)) \mapsto (V^T\omega^1 V, V^T\omega^2 V)$ of the group $\mathrm{GL}(\mathbb{R}^d)$ on the space of pairs of $d \times d$ antisymmetric matrices. It is enough to show that the codimension of orbits of this action is no less than k. Indeed, the space of bases (ω_1, ω_2) of a fixed 2–dimensional subspace is 4–dimensional, but the difference of the codimensions of the given orbit in the space of pairs of matrices and in the Grassmannian cannot be greater than 3 since the action of scalar matrices $V = cI$ on (ω_1, ω_2) does not change the subspace.

Let $\omega^1 \in \bigwedge^2 \mathbb{R}^{2k+1}$ with $\omega^1 = \left[\begin{array}{c|c} \tilde{\omega}^1 & 0 \\ \hline 0 & 0 \end{array}\right]$, where $\tilde{\omega}^1 \in \bigwedge^2 \mathbb{R}^{2k}$,

$$\tilde{\omega}^1 = \begin{bmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix}.$$

The subspace of $\Omega(2k+1) \subset \mathrm{GL}(\mathbb{R}^{2k+1})$ which preserves ω^1 is given by the matrices $V = \left[\begin{array}{c|c} \tilde{V} & 0 \\ \hline v & 1 \end{array}\right]$, where $\tilde{V} \in Sp(k)$, the group of symplectic transformations, and $v \in \mathbb{R}^{2k}$. The codimension of the orbit of the pair (ω_1, ω_2) is equal to the codimension of the orbit of a matrix $\omega^2 \in \bigwedge^2 \mathbb{R}^{2k+1}$ under the action of $\Omega(2k+1)$. We have $\omega^2 = \left[\begin{array}{c|c} \tilde{\omega}^2 & -\nu^T \\ \hline \nu & 0 \end{array}\right]$, $\nu \in \mathbb{R}^{2k}$. The codimension of the orbit of ω^2 under the action of $\Omega(2k+1)$ is no less than the codimension of $\tilde{\omega}^2$ under the action of $Sp(k)$. Indeed, let $Stab(\tilde{\omega}^2) \subset Sp(k)$ be the stabilizer of $\tilde{\omega}^2$. Then $\left[\begin{array}{c|c} Stab(\tilde{\omega}^2) & 0 \\ \hline 0 & 1 \end{array}\right]$ is contained in the stabilizer of ω^2 under the action of $\Omega(2k+1)$ and the codimension of $Sp(k)$ in $\Omega(2k+1)$ equals the codimension of $\bigwedge^2 \mathbb{R}^{2k}$ in $\bigwedge^2 \mathbb{R}^{2k+1}$.

On the other hand, the codimension of the orbit of $\tilde{\omega}^2$ under the action of $Sp(k)$ equals the codimension of the orbit of the pair $(\tilde{\omega}^1, \tilde{\omega}^2)$ under the action of $\mathrm{GL}(\mathbb{R}^{2k})$. The codimension of the last orbit is no less than k since the action of $\mathrm{GL}(\mathbb{R}^{2k})$ leaves invariant the Pfaffian $p(x_1, x_2) = \mathsf{Pf}\,(x_1\tilde{\omega}^1 + x_2\omega^2)$, up to a scalar multiplier. (cf. the proof of Proposition 5). $\square$

As corollary of the above proposition we have that $(d, d+2)$ is not rigid for all $d \geq 8$. Then the only bi–dimensions to analyse are $(5,7)$ and $(7,9)$. Next we show the following:

Proposition 8 *Let $d = 5$ or $d = 7$. Then the bi–dimensions $(d, d+2)$ are rigid with only one isomorphism class.*

Proof.

$d = 5$

Consider a pair of 5×5 antisymmetric matrices, ω^1 and ω^2. Recall that $d \times d$ antisymmetric matrices with d odd, have always corank at least 1 therefore we take $p_1 \in \ker \omega^1$ and $p_2 \in \ker \omega^2$. By generic assumptions we have that p_1 and p_2 are linearly independent. In these coordinates we write

$$\omega^1 = \begin{bmatrix} \omega^1_{11} & v_1 \\ -v_1^T & 0 \end{bmatrix}, \quad \omega^2 = \begin{bmatrix} \omega^2_{11} & v_2 \\ -v_2^T & 0 \end{bmatrix}$$

where $v_1, v_2 \in \mathbb{R}^4$ and, for $i = 1, 2$, ω^i_{11} is the first 4×4 principal submatrix of ω^i and has corank 2. Hence, using the same arguments as for the real case with

$d = 4$, we can assume the ω^1 and ω^2 have the following form:

$$\omega^1 = \left[\begin{array}{cc|c} J & 0 & \multirow{2}{*}{v_1} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{-v_1} & 0 \end{array}\right],$$

$$\omega^2 = \left[\begin{array}{cc|c} 0 & 0 & \multirow{2}{*}{v_2} \\ 0 & J & \\ \hline \multicolumn{2}{c|}{-v_2^T} & 0 \end{array}\right].$$

Let now $P = \left[\begin{array}{cc|c} P_1 & 0 & \multirow{2}{*}{α} \\ 0 & P_2 & \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]$, where P_1 and P_2 are 2×2 matrices with determinant equal to 1 and $\alpha \in \mathbb{R}^4$. We have that

$$(P)^T \tilde{\omega}^1 (P) = \left[\begin{array}{cc|c} J & 0 & \multirow{2}{*}{v_1'} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{-(v_1')^T} & 0 \end{array}\right] \text{ and}$$

$$(P)^T \tilde{\omega}^2 (P) = \left[\begin{array}{cc|c} 0 & 0 & \multirow{2}{*}{v_2'} \\ 0 & J & \\ \hline \multicolumn{2}{c|}{-(v_2')^T} & 0 \end{array}\right],$$

where

$$v_1' = \begin{bmatrix} P_1^T & 0 \\ 0 & P_2^T \end{bmatrix} \left(\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \alpha + v_1 \right) \text{ and}$$

$$v_2' = \begin{bmatrix} P_1^T & 0 \\ 0 & P_2^T \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} \alpha + v_2 \right).$$

For a suitable choice of α, P_1 and P_2, we can write $v_1' = [0, 0, 1, 0]^T$, $v_2' = [1, 0, 0, 0]^T$ and

$$\omega^1 = \left[\begin{array}{cccc|c} \multicolumn{2}{c}{\multirow{2}{*}{J}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 \\ & & & & 0 \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 \\ & & & & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \end{array}\right],$$

$$\omega^2 = \left[\begin{array}{cccc|c} \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 \\ & & & & 0 \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{J}} & 0 \\ & & & & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Equation (2) in these coordinates gives the following multiplication table for a representative family $\mathcal{F}$:

$$\begin{array}{l} [f_3, f_5] = [f_1, f_2] = f^{\pi_1} \\ [f_1, f_5] = [f_3, f_4] = f^{\pi_2} \\ [f_i, f_j] = 0 \ \text{ otherwise.} \end{array} \tag{10}$$

The normal form for $\mathcal{F}$ is reported in Section (6), equation (38).

$d = 7$

With the same arguments as for the previous case, we can reduce to the case of a pair of $d \times d$ antisymmetric matrices of the form:

$$\omega^1 = \left[\begin{array}{ccc|c} J & 0 & 0 & \\ 0 & 0 & 0 & v_1 \\ 0 & 0 & -J & \\ \hline & -v_1^T & & 0 \end{array}\right]$$

$$\omega^2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & J & 0 & v_2 \\ 0 & 0 & J & \\ \hline & -v_2^T & & 0 \end{array}\right]$$

By letting $P = \left[\begin{array}{ccc|c} P_1 & 0 & 0 & \\ 0 & P_2 & 0 & \alpha \\ 0 & 0 & P_3 & \\ \hline & 0 & & 1 \end{array}\right]$, where P_1, P_2 and P_3 are 2×2 matrices with determinant equal to 1 and $\alpha \in \mathbb{R}^6$, we obtain, for suitable choices of

P_1, P_2, P_3 and α, $v_1' = [0,\, 0,\, 1,\, 0,\, 1,\, 0]^T$ and $v_2' = [1,\, 0,\, 0,\, 0,\, 0,\, 1]^T$,

$$\omega^1 = \left[\begin{array}{cccccc|c} \multicolumn{2}{c}{J} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{0} & \begin{matrix}0\\0\end{matrix} \\ \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{0} & \begin{matrix}1\\0\end{matrix} \\ \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{-J} & \begin{matrix}1\\0\end{matrix} \\ \hline 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{array}\right] \text{ and}$$

$$\omega^2 = \left[\begin{array}{cccccc|c} \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{0} & \begin{matrix}1\\0\end{matrix} \\ \multicolumn{2}{c}{0} & \multicolumn{2}{c}{J} & \multicolumn{2}{c|}{0} & \begin{matrix}0\\0\end{matrix} \\ \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{J} & \begin{matrix}0\\1\end{matrix} \\ \hline -1 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

Finallay we have the following multiplication table:

$$\begin{array}{l} [f_3, f_7] = [f_5, f_7] = [f_1, f_2] = f^{\pi_1} \\ [f_1, f_7] = [f_6, f_7] = [f_3, f_4] = f^{\pi_2} \\ [f_5, f_6] = -f^{\pi_1} + f^{\pi_2} \\ [f_i, f_j] = 0 \text{ otherwise.} \end{array} \tag{11}$$

See Section (6), equation (41) for the normal form. □

3.3 The case $m = 3$

In this case we deal with a three dimensional space of $d \times d$ antisymmetric matrices.

Proposition 9 *The bi–dimension $(4, 7)$ is rigid with two isomorphism classes distinguished by the signature of the quadratic form* $\mathsf{Pf}\,(x_1\omega^1 + x_2\omega^2 + x_3\omega^3)$.

Proof. The equation $\mathsf{Pf}\,(x_1\omega^1 + x_2\omega^2 + x_3\omega^3) = 0$ can be rewritten as:

$$[x_1, x_2, x_3]A[x_1, x_2, x_3]^T = 0, \tag{12}$$

with A a 3×3 symmetric matrix. Depending on the signature of A we either have real roots (corresponding to non definite A) or complex roots (corresponding to sign definite A). Next we provide the multiplication table for a representing family $\mathcal{F}$ for each of the above cases. This will show that the bi-dimensions $(d, d+3)$ are rigid and the isomorphism class is uniquely reconstructed from the number of real roots of equation (12).

$d = 4$, real case

Consider a generic 3 dimensional subspace of $\bigwedge^2 \mathbb{R}^d$. Then the matrix A of equation (12) is non degenerate one. If A is not sign definite, then we can assume that

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the real solutions to equation (12) are

$$\begin{aligned} [x_{11}, x_{21}, x_{31}] &= [1, 0, 0], \\ [x_{12}, x_{22}, x_{32}] &= [0, 1, 0], \text{ and} \\ [x_{13}, x_{23}, x_{33}] &= [a, b, c], \text{ with } ab + c^2 = 0. \end{aligned}$$

Then $\omega^1 = \tilde{\omega}^1$, $\omega^2 = \tilde{\omega}^2$ and $\omega^3 = a\tilde{\omega}^1 + b\tilde{\omega}^2 + c\tilde{\omega}^3$, with $ab + c^2 = 0$, have corank 2. Moreover, under generic assumptions, we have that the kernels of ω^1, ω^2 and ω^3 are transversal. In the coordinates $p_1, p_2 \in \mathsf{ker}\,(\omega^2)$, $p_3, p_4 \in \mathsf{ker}\,(\omega^1)$, with $p_1\omega^1 p_2 = 1$ and $p_3\omega^2 p_4 = 1$, we write:

$$\omega^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \omega^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \omega^3 = \quad & a\tilde{\omega}^1 + b\tilde{\omega}^2 + c\tilde{\omega}^3 = \\ & \begin{bmatrix} 0 & a + cw_{12} & cw_{13} & cw_{14} \\ -a - cw_{12} & 0 & cw_{23} & cw_{24} \\ -cw_{13} & -cw_{23} & 0 & b + cw_{34} \\ -cw_{14} & -cw_{24} & -b - cw_{34} & 0 \end{bmatrix}, \end{aligned}$$

where w_{ij} are the components of $\tilde{\omega}^3$. Since ω^3 has corank 2, the following condition holds true, for all a, b, c such that $ab + c^2 = 0$:

$$0 = (a + cw_{12})(b + cw_{34}) - c^2(w_{13}w_{24} - w_{23}w_{14}). \tag{13}$$

By equation (13) it follows that $w_{12} = w_{34} = 0$ and

$$w_{13}w_{24} - w_{23}w_{14} = -1. \tag{14}$$

In particular, setting $a = 1, b = -1, c = 1$, ω^3 has the form:

$$\omega^3 = \begin{bmatrix} J & (\tilde{\omega}^3)_{12} \\ -(\tilde{\omega}^3)_{12}^T & -J \end{bmatrix},$$

where by $(\tilde{\omega}^3)_{ij}$ we denote the (ij)–th, 2×2 block of the block matrix decomposition of $\tilde{\omega}^3$. Let now $P_1 = (\tilde{\omega}^3)_{12}^{-T}$. Then, by equation (14), $\mathsf{det}\,(P_1) = -1$ and, setting $P = \begin{bmatrix} P_1 & 0 \\ 0 & I_2 \end{bmatrix}$, we get

$$\begin{aligned} &(P)^T \omega^1 P = \begin{bmatrix} -J & 0 \\ 0 & 0 \end{bmatrix}, \quad (P)^T \omega^2 P = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} \text{ and} \\ &(P)^T \omega^3 P = \begin{bmatrix} -J & I_2 \\ -I_2 & -J \end{bmatrix}. \end{aligned} \tag{15}$$

Finally,

$$\omega = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} f^{\pi_1} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} f^{\pi_2}$$

$$+ \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} f^{\pi_3}$$

which gives the following multiplication table.

$$\begin{aligned} &[f_1, f_2] = -f^{\pi_1} - f^{\pi_3} \\ &[f_3, f_4] = f^{\pi_2} - f^{\pi_3} \\ &[f_1, f_3] = [f_2, f_4] = f^{\pi_3} \\ &[f_1, f_4] = [f_2, f_3] = 0 \end{aligned} \tag{16}$$

The normal form is given in Section 6 equation (42).

$d = 4$, complex case

A positive definite matrix A can be put in the following form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the complex solutions $[x_{11}, x_{21}, x_{31}] = [1, i, 0]$, $[x_{12}, x_{22}, x_{32}] = [1, -i, 0]$ and $[x_{13}, x_{23}, x_{33}] = [a, b, c]$, with $a^2 + b^2 + c^2 = 0$. With this choice, $\omega^1 = \tilde{\omega}^1 + \imath\tilde{\omega}^2$, $\omega^2 = \tilde{\omega}^1 - \imath\tilde{\omega}^2$ and $\omega^3 = a\tilde{\omega}^1 + b\tilde{\omega}^2 + c\tilde{\omega}^3$, with $a^2 + b^2 + c^2 = 0$, have corank 2. Moreover, following the same arguments as for the complex case with $m = 2$ and $d = 4$, we can write

$$\Re(\omega^1) = \tilde{\omega}^1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix},$$
$$\Im(\omega^1) = \tilde{\omega}^2 = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}.$$

In the same coordinates we write

$$\begin{aligned} \omega^3 = \; & a\tilde{\omega}^1 + b\tilde{\omega}^2 + c\tilde{\omega}^3 = \\ & \begin{bmatrix} 0 & cw_{12} & a-b+cw_{13} & a+b+cw_{14} \\ -cw_{12} & 0 & a+b+cw_{23} & -a+b+cw_{24} \\ -a+b-cw_{13} & -a-b-cw_{23} & 0 & cw_{34} \\ -a-b-cw_{14} & a+b-cw_{24} & -cw_{34} & 0 \end{bmatrix}, \end{aligned}$$

where w_{ij} are the components of $\tilde{\omega}^3$. Since ω^3 has corank 2, the following condition holds true for all a, b, c such that $a^2 + b^2 + c^2 = 0$:

$$2(a^2+b^2)+bc(w_{23}+w_{14}+w_{24}-w_{13})+ac(w_{23}+w_{14}-w_{24}+w_{13})+c^2\sqrt{\mathsf{det}\ (\tilde{\omega}^3)}. \tag{17}$$

By equation (17) it follows that $w_{13} = w_{24}$, $w_{23} = -w_{14}$ and $\sqrt{\mathsf{det}\ (\tilde{\omega}_3)} = 1$, hence $w_{12}w_{34} - w_{13}^2 - w_{14}^2 = 1$. Let now $P_1 = -(\tilde{\omega}^3)_{11}^{-1}(\tilde{\omega}^3)_{12}$, where by $(\tilde{\omega}^3)_{ij}$ we denote the (ij)–th, 2×2–block of the block matrix decomposition of $\tilde{\omega}^3$, and $P = \begin{bmatrix} \frac{1}{\sqrt{w_{12}}} I_2 & \sqrt{w_{12}} P_1 \\ 0 & \sqrt{w_{12}} I_2 \end{bmatrix}$. Then we get:

$$(P)^T \tilde{\omega}^3 P = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix},$$

while $\tilde{\omega}_1$ and $\tilde{\omega}_2$ remain invaried. In particular, by setting $a = 0, b = \imath, c = 1$, we have

$$(P)^T \omega_3 P = \imath (P)^T \tilde{\omega}_2 P + (P)^T \tilde{\omega}_3 P$$

and

$$\omega = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} f^{\pi_1} + \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} f^{\pi_2}$$

$$+ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} f^{\pi_3}.$$

The corresponding multiplication table is:

$$\begin{aligned} &[f_1, f_2] = [f_3, f_4] = f^{\pi_3} \\ &[f_1, f_3] = -[f_2, f_4] = f^{\pi_1} - f^{\pi_2} \\ &[f_1, f_4] = [f_2, f_3] = f^{\pi_1} + f^{\pi_2}. \end{aligned} \tag{18}$$

The normal form is reported in Section 6, equation (43). □

Let now $d = 5$ and recall that an antisymmetric matrix can be seen as a skew form of degree 2, we consider the wedge products $v^{ij} = \omega^i \wedge \omega^j$ for $i < j$ which are 4–forms in $\mathbb{R}^5$. We then have that v^{ij}, $i \leq j$, $i, j = 1, 2, 3$ are 6 vectors in $\mathbb{R}^5$. Let $\alpha_{ij} \in \mathbb{R}, i \leq j, i = 1, 2, 3$ such that $\sum \alpha_{ij} v^{ij} = 0$. Taking $\tilde{\omega}^i = \sum_{h=1}^{3} x_{hi} \omega^h$ gives $\tilde{v}_{ij} = \sum_{hk} x_{hi} x_{kj} v_{hk}$ and $\alpha_{hk} = \sum_{hk} x_{hi} x_{kj} \tilde{\alpha}_{ij}$. That is the symmetric matrix A of coefficients α_{ij},

$$A = \begin{bmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} & \frac{\alpha_{13}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} & \frac{\alpha_{23}}{2} \\ \frac{\alpha_{13}}{2} & \frac{\alpha_{23}}{2} & \alpha_{33} \end{bmatrix}$$

is mapped to $x^{-T} A x^{-1}$ where

$$x = \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

Matrix A is defined up to a nonzero scalar multiplier, hence the transformation $A \mapsto -A$ is also allowed. We have the following

Proposition 10 *The bi–dimension* $(5, 8)$ *is rigid with two isomorphism classes distinguished by the signature of the symmetric matrix* A.

Proof. Observe first that $\omega \wedge \omega \in \mathsf{ker}\,\omega$, if $\mathsf{corank}\,(\omega) = 1$, and $\omega \wedge \omega = 0$, if $\mathsf{corank}\,(\omega) > 1$. Then $V^T \omega V (V^{-1}\omega \wedge \omega) = V^T \omega(\omega \wedge \omega) = 0$. Now, since

$$(x_1\omega^1 + x_2\omega^2 + x_3\omega^3) \wedge (x_1\omega^1 + x_2\omega^2 + x_3\omega^3) = \sum_{ij} x_i x_j \omega^i \wedge \omega^j = \sum_{ij} x_i x_j v_{ij},$$

we have that, under the action of $V \in \mathrm{GL}(\mathbb{R}^5)$, each vector v_{ij} is mapped into $V^{-1}v_{ij}$. Hence the coefficients α_{ij} of A remain invaried under the action of $\mathrm{GL}(\mathbb{R}^5)$. This fact shows that the signature of the symmetric matrix A is an invariant for the bi–dimension $(5, 8)$.

Under generic assumptions the matrix A is non degenerate and the possibly arising signatures of A are $+++$ and $++-$. Next we provide the multiplication table for a representing family $\mathcal{F}$ for each of the two cases. Thus we will show that $(5, 8)$ is a rigid bi–dimension with two isomorphism class.

We can assume that A has either the form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

or the form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

depending on the signature of A. Recalling the geometric meaning of the coefficients of A, we have $v_{11} + v_{22} = v_{33}$ or $v_{11} + v_{22} = -v_{33}$, in the first and in the second case respectively. Choose first a coordinate systems as for the case $m = 2$ and $d = 5$. Then we can write

$$\omega^1 = \left[\begin{array}{cc|c} J & 0 & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & \begin{matrix} 1 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} -1 & 0 \end{matrix} & 0 \end{array}\right] \quad \text{and}$$

$$\omega^2 = \left[\begin{array}{cc|c} 0 & 0 & \begin{matrix} 1 \\ 0 \end{matrix} \\ 0 & J & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} -1 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & 0 \end{array}\right].$$

In these coordinates $v_{11} = [0,0,0,1,0]$ and $v_{22} = [0,1,0,0,0]$. Therefore, depending if we are in the first or in the second case, $v_{33} = \pm[0,1,0,1,0]$. Being $\omega^3 v_{33} = 0$, we have that ω^3 has the form:

$$\omega^3 = \begin{bmatrix} 0 & w_{12} & w_{13} & -w_{12} & w_{15} \\ -w_{12} & 0 & w_{23} & 0 & w_{25} \\ -w_{13} & -w_{23} & 0 & w_{23} & w_{35} \\ w_{12} & 0 & -w_{23} & 0 & -w_{25} \\ -w_{15} & -w_{25} & -w_{35} & w_{25} & 0 \end{bmatrix},$$

with $w_{12}w_{35} + w_{15}w_{23} - w_{13}w_{25} = \pm 1$. Computing now the v_{ij}'s, for $i < j$ gives: $v_{12} = [0,0,0,0,-1]$, $v_{13} = \pm[0,-w_{12},-w_{25},-w_{35} - w_{12}, w_{23}]$ and $v_{12} = \pm[w_{25}, -w_{25} - w_{23}, 0, -w_{23}, w_{12}]$.

By taking $P = \frac{1}{\sqrt{w_{25}}}[v_{11}, v_{22}, v_{12}, v_{13}, v_{23}]$ we obtain:

$$P^T\omega^1 P = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \end{array}\right],$$

$$P^T\omega^2 P = \left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

and, either

$$P^T\omega^3 P = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

if the first case holds, or

$$P^T\omega^3 P = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right],$$

if the second case holds. Finally, by

$$\omega = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \end{array}\right] f^{\pi_1} + \left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right] f^{\pi_2}$$

$$+ \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mp 1 \\ \hline 0 & 0 & 0 & \pm 1 & 0 \end{array}\right] f^{\pi_3}$$

we have the following multiplication table:

$$\begin{aligned} &[f_2, f_5] = [f_3, f_4] = f^{\pi_1} \\ &[f_1, f_4] = [f_3, f_5] = -f^{\pi_2} \\ &[f_1, f_3] = -[f_2, f_3] = \mp[f_4, f_5] = f^{\pi_3} \\ &[f_1, f_2] = [f_1, f_5] = [f_2, f_4] = 0. \end{aligned} \tag{19}$$

The normal forms are shown in Section 6, equation (44). □

3.4 The case $m = 4$

Recall that, for $m = 4$, the only case to analyse is that with $d = 5$. A simple calculation shows that the submanifold of rank 2 antisymmetric 5×5 matrices has codimension 3 in the projectivized space $\bigwedge^2 \mathbb{R}^5$ of all antisymmetric 5×5 matrices. Let $x = [x_1, x_2, x_3, x_4]$ and

$$\omega(x) = x_1\omega^1 + x_2\omega^2 + x_3\omega^3 + x_4\omega^4$$

be a generic 4-dimensional vector subspace (or 3-dimensional projective subspace) of $\bigwedge^2 \mathbb{R}^5$. Then $\omega(x)$ meets the submanifold of rank 2 matrices in a finite number of points. We'll show that the bi–dimension (5,9) is rigid with isomorphism classes distinguished by the number of these points which we are going to locate effectively.

First of all, we may assume without lack of generality that $w = \sum_{i=1}^{4}(x_i\omega^i)_{12}$ does not vanish in rank 2 points. Provided that $w \neq 0$, we can assume the following block matrix decomposition (Schur theorem):

$$\omega(x) = \begin{bmatrix} \omega_{11}(x) & 0 \\ 0 & \omega_{22}(x) \end{bmatrix},$$

where $\omega_{11}(x) = wJ$ is 2×2 nonsingular antisymmetric matrix and

$$\omega_{22}(x) = \frac{1}{w} \begin{bmatrix} 0 & d_5 & d_4 \\ -d_5 & 0 & d_3 \\ -d_4 & -d_3 & 0 \end{bmatrix}$$

is 3×3 antisymmetric with $d_i = d_i(\omega(x))$ being the Pfaffian of the i–th principal minor of order 4 of $\omega(x)$. Then we have that $\omega(x)$ has rank 2 if and only if d_3, d_4 and d_5 are zero, that is if x is the root of 3 homogeneous polynomials of degree 2, with the additional condition that $w \neq 0$. Of course, d_1 and d_2 also vanish at such a root (otherwise $\omega(x)$ would have rank 4).

Proposition 11 *The bi–dimension* $(5, 9)$ *is rigid with three classes of isomorphism distinguished by the number of real solutions of the system* $d_i = 0,\ i = 1, \ldots, 5$.

Proof. As in the proof Proposition 10 we have that under the action of $V \in \mathrm{GL}(\mathbb{R}^5)$, each vector v_{ij} is mapped into $\tilde{v}_{ij} = V^{-1} v_{ij}$. Then

$$\begin{aligned} \tilde{d}_k = \ & d_k(V^T \omega(x) V) = \textstyle\sum_{ij} x_i x_j (\tilde{v}_{ij})_k = \sum_{ij} x_i x_j (\sum_h (V^{-1})_{kh} (v_{ij})_h) = \\ & \textstyle\sum_h (V^{-1})_{kh} (\sum_{ij} x_i x_j (v_{ij})_h) = \sum_h (V^{-1})_{kh} d_k, \end{aligned}$$

where by $(v)_k$ (resp. $(V)_{kh}$) we denote the k–th (resp. (kh)–th) component of the vector v (resp. matrix V); thus we obtain that $\tilde{d}_k$ belongs to the linear space generated by $d_1, \ldots, d_5$. This shows that such linear space is invariant by the action of Φ.

Assume now that ω^1, ω^2 and ω^3 are in the normal form obtained for $m = 3$ and $d = 5$, then $\omega(x) =$

$$\begin{bmatrix} 0 & x_4 w_{12} & x_3 + x_4 w_{13} & -x_2 + x_4 w_{14} & x_4 w_{15} \\ -x_4 w_{12} & 0 & -x_3 + x_4 w_{23} & x_4 w_{24} & x_1 + x_4 w_{25} \\ -x_3 - x_4 w_{13} & x_3 - x_4 w_{23} & 0 & x_1 + x_4 w_{34} & -x_2 + x_4 w_{35} \\ x_2 - x_4 w_{14} & -x_4 w_{24} & -x_1 - x_4 w_{34} & 0 & \pm x_3 + x_4 w_{45} \\ -x_4 w_{15} & -x_1 - x_4 w_{25} & x_2 - x_4 w_{35} & \mp x_3 - x_4 w_{45} & 0 \end{bmatrix},$$

where the w_{ij}'s are the coefficients of ω^4. The computation of d_i, for $i = 3, 4, 5$, gives:

$$\begin{cases} d_3 = d_3(\omega^4) x_4^2 + (-w_{14} x_1 + w_{25} x_2 \pm w_{12} x_3) x_4 + x_1 x_2 \\ d_4 = d_4(\omega^4) x_4^2 + (-w_{13} x_1 - w_{12} x_2 - (w_{25} + w_{15}) x_3) x_4 - x_1 x_3 \\ d_5 = d_5(\omega^4) x_4^2 + (w_{12} x_1 - w_{23} x_2 - (w_{24} + w_{14}) x_3) x_4 + x_3 x_2. \end{cases}$$

Generically there are 8 solutions to the system $d_i = 0,\ i = 3, 4, 5$. Notice that, 3 out of the 8 solutions correspond to solutions with $x_4 = 0$. Since such kind of solution violates the condition $w \neq 0$, it must be discarded. There remains 5 solutions. We may have:

1) five real solutions;

2) three real and two complex conjugate solutions;

3) one real and two pairs of complex conjugate solutions.

Next we provide the multiplication table for a representing family $\mathcal{F}$ for each of the above cases. This will show that $(5, 9)$ is a rigid bi–dimension with three isomorphism classes.

Case 1)

Assume that the 5 solutions $x^i = [x_{1i}, x_{2i}, x_{3i}, x_{4i}],\ i = 1, \ldots, 5$, are all real and, since $\omega(x)$ is a generic subspace, 4 by 4 linearly independent. Let $\lambda_1, \ldots, \lambda_5 \neq 0$ be such that

$$\sum_{i=1}^{4} \lambda_i x^i = \lambda_5 x^5.$$

Let $\tilde{\omega}^i = \omega(\lambda_i x^i)$. By the choice of the λ_i's we have that

$$\sum_{i=1}^{4} \tilde{\omega}_i = \tilde{\omega}_5.$$

Assume now that, for all $i = 1, \ldots, 4$, $\tilde{\omega}^i$ has rank 2 and denote by V_i its kernel, which is a 3–dimensional space, and by v_{ij} the 1–dimensional space $V_i \cap V_j$ (recall that generically V_i are transversal). With this notation we have that

$$\begin{aligned} V_1 &= \{v_{12}, v_{13}, v_{14}\}, \\ V_2 &= \{v_{12}, v_{23}, v_{24}\}, \\ V_3 &= \{v_{13}, v_{23}, v_{34}\} \text{ and} \\ V_4 &= \{v_{14}, v_{24}, v_{34}\}. \end{aligned}$$

Let now $P = [v_{12}, v_{13}, v_{14}, v_{23}, v_{24}]$, where the v_{ij}'s are suitably rescaled. Then we obtain

$$P^T\tilde{\omega}^1 P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \qquad P^T\tilde{\omega}^2 P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P^T\tilde{\omega}^3 P = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -a & 0 & 0 \end{bmatrix}, \quad P^T\tilde{\omega}^4 P = \begin{bmatrix} 0 & b & 0 & 1 & 0 \\ -b & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since

$$P^T\tilde{\omega}_5 P = \begin{bmatrix} 0 & b & 1 & 1 & 1 \\ -b & 0 & 1 & c & 0 \\ -1 & -1 & 0 & 0 & a \\ -1 & -c & 0 & 0 & 1 \\ -1 & 0 & -a & -1 & 0 \end{bmatrix}$$

has also rank 2 it must be $a = -b = c = 1$. Finally we have

$$\omega = \sum_{i=1}^4 \tilde{\omega}^i \tilde{f}^{\pi_i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tilde{f}^{\pi_1} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tilde{f}^{\pi_2} +$$
$$+ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix} \tilde{f}^{\pi_3} + \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tilde{f}^{\pi_4}$$

and the following multiplication table

$$\begin{aligned} &[f_4, f_5] = \tilde{f}^{\pi_1} \\ &[f_2, f_3] = \tilde{f}^{\pi_2} \\ &[f_1, f_3] = [f_1, f_5] = [f_3, f_5] = \tilde{f}^{\pi_3} \\ &-[f_1, f_2] = [f_1, f_4] = [f_2, f_4] = \tilde{f}^{\pi_4} \\ &[f_2, f_5] = [f_3, f_4] = 0. \end{aligned} \tag{20}$$

The normal form is reported in Section 6 equation (45).

Case 2)

Consider next the generic case where two solutions x^1, x^2 are complex conjugate and x^3, x^4, x^5 are real with $\Re(x^1), \Im(x^1), x^3, x^4$ and x^5, 4 by 4 linearly independent as points of $\mathbb{R}^4$. Then we can choose $\lambda_1, \dots, \lambda_5 \neq 0$ such that

$$\lambda_1 \Re(x^1) + \lambda_2 \Im(x^1) + \sum_{i=3}^{4} \lambda_i x^i = \lambda_5 x^5.$$

With this choice, $\tilde{\omega}^1 = \omega(\lambda_1 \Re(x^1))$, $\tilde{\omega}^2 = \omega(\lambda_2 \Im(x^1))$ and $\tilde{\omega}^i = \omega(\lambda_i x^i)$, for $i = 3, 4, 5$, are such that

$$\sum_{i=1}^{4} \tilde{\omega}^i = \tilde{\omega}^5.$$

Notice that

$$\tilde{\omega}_1 = \Re\left(\omega(\lambda_1 x^1)\right) \text{ and } \quad \tilde{\omega}_2 = \Im\left(\omega(\lambda_2 x^2)\right).$$

Now $\omega(\lambda_1 x^1)$ has rank 2 and, if v_1, v_2, v_3 are three independent vectors of the kernel, then, the conjugate vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3$, are independent vectors of the kernel of $\omega(\lambda_2 x^2)$.
Observe that for any $\alpha_1, \alpha_2, \alpha_2 \in \mathbb{C}^3$, $\omega(\lambda_1 x^1)(\sum \alpha_i v_i) = \omega(\lambda_2 x^2)\overline{(\sum \alpha_i v_i)} = 0$. On the other hand, there exist α_i, $i = 1, 2, 3$, such that $v = \sum \alpha_i v_i = \sum\left(\Re(\alpha_i)\Re(v_i) - \Im(\alpha_i)\Im(v_i)\right) + \imath \sum\left(\Re(\alpha_i)\Im(v_i) + \Im(\alpha_i)\Re(v_i)\right)$ is a vector with real coefficients, i.e. $\sum\left(\Re(\alpha_i)\Im(v_i) + \Im(\alpha_i)\Re(v_i)\right) = 0$. Therefore there exist $v \in \mathbb{R}^5$ such that $\omega(\lambda_1 x^1)v = \omega(\lambda_2 x^2)v = 0$ and, in particular, $\tilde{\omega}^1 v = \tilde{\omega}^2 v = 0$. Then we take v_2, v_3 and $v_1 = v$ as base for the kernel of $\omega(\lambda_1 x^1)$.

Let V_1 be the space generated by $\{v_1, \Re v_2, \Re v_3\}$ and V_3, V_4 the kernels of $\tilde{\omega}^3$, $\tilde{\omega}^4$ respectively. Then we denote, for $j = 3, 4$, $v_{1j} = V_1 \cap V_j$, and $\hat{v}_{1j}$ the vectors such that $\omega(\lambda_1 x^1)(v_{1j} + \imath \hat{v}_{1j}) = 0$. Finally, letting $P = (v_1\, v_{13}\, v_{14}\, \hat{v}_{13}\, \hat{v}_{14})$, we obtain

$$\begin{cases} \Re\left(\omega(\lambda_1 x^1)\right) v_{1j} = \Im\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} \\ \Re\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} = -\Im\left(\omega(\lambda_1 x^1)\right) v_{1j} \end{cases}$$

from which

$$\begin{aligned}
&\hat{v}_{1j}^T \Re\left(\omega(\lambda_1 x^1)\right) v_{1j} = \hat{v}_{1j}^T \Im\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} = 0, \\
&\hat{v}_{1i}^T \Im\left(\omega(\lambda_1 x^1)\right) v_{1j} = v_{1i}^T \Re\left(\omega(\lambda_1 x^1)\right) v_{1j} \\
&= v_{1i}^T \Im\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} = -\hat{v}_{1i}^T \Re\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j}, \\
&\hat{v}_{1i}^T \Im\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} = v_{1i}^T \Re\left(\omega(\lambda_1 x^1)\right) \hat{v}_{1j} \\
&= -v_{1i}^T \Im\left(\omega(\lambda_1 x^1)\right) v_{1j} = \hat{v}_{1i}^T \Re\left(\omega(\lambda_1 x^1)\right) v_{1j}.
\end{aligned}$$

Therefore,

$$P^T \Re\left(\omega(\lambda_1 x^1)\right) P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b \\ 0 & -a & 0 & -b & 0 \\ 0 & 0 & b & 0 & -a \\ 0 & -b & 0 & a & 0 \end{bmatrix}$$

and

$$P^T \Im\left(\omega(\lambda_1 x^1)\right) P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & a \\ 0 & b & 0 & -a & 0 \\ 0 & 0 & a & 0 & b \\ 0 & -a & 0 & -b & 0 \end{bmatrix}.$$

In analogous way we obtain a form for $\Re\left(\omega(\lambda_2 x^2)\right)$ and $\Im\left(\omega(\lambda_2 x^2)\right)$. Moreover, by suitably choosing the lenghts of the columns of P, we have:

$$P^T \tilde{\omega}^1 P = P^T \Re\left(\omega(\lambda_1 x^1)\right) P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix},$$

$$P^T \tilde{\omega}^2 P = P^T \Im\left(\omega(\lambda_2 x^2)\right) P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix},$$

$$P^T(\tilde{\omega}^3)P = \begin{bmatrix} 0 & 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & a_3 & a_4 \\ -a_1 & 0 & -a_3 & 0 & a_5 \\ -a_2 & 0 & -a_4 & -a_5 & 0 \end{bmatrix}$$

and

$$P^T(\tilde{\omega}^4)P = \begin{bmatrix} 0 & b_1 & 0 & b_2 & b_3 \\ -b_1 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ -b_2 & -b_4 & 0 & 0 & b_6 \\ -b_3 & -b_5 & 0 & -b_6 & 0 \end{bmatrix}.$$

Since $\tilde{\omega}^3$ and $\tilde{\omega}^4$ and $\tilde{\omega}^5 = \sum_i \tilde{\omega}^i$ have rank 2 we have $a_5 = a_1 a_4 - a_2 a_3$, $b_1 b_6 = b_2 b_5 - b_3 b_4$ and, $d_i(\tilde{\omega}^5) = 0$, for all $i = 1, \ldots, 5$, i.e.:

$$\begin{aligned} b_1 &= -\tfrac{b_2}{a_2}, \\ b_4 &= -\tfrac{(a_3-2)b_2}{a_2}, \\ b_5 &= -\tfrac{2a_2+a_4b_2}{a_2}. \end{aligned}$$

We now set, for all $i \leq j$, $\tilde{v}_{ij} = (P^T\tilde{\omega}^i P) \wedge (P^T \tilde{\omega}^j P)$. In particular we have that $\tilde{v}^{11} = \tilde{v}^{22} = [2, 0, 0, 0, 0]$ and $\tilde{v}^{33} = \tilde{v}^{44} = \tilde{v}^{12} = [0, 0, 0, 0, 0]$.

By choosing $P_1 = [\tilde{v}_{13}\tilde{v}_{14}\tilde{v}_{23}\tilde{v}_{24}\tilde{v}_{34}]$ (and suitably rescaling it) we obtain

$$(PP_1)^T\tilde{\omega}^1(PP_1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix},$$

$$(PP_1)^T\tilde{\omega}^2(PP_1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$(PP_1)^T\tilde{\omega}^3(PP_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$(PP_1)^T\tilde{\omega}^4(PP_1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, equation (2) gives the following multiplication table:

$$\begin{aligned}
&[f_3, f_5] = -[f_4, f_5] = f^{\pi_1}\\
&-[f_1, f_5] = [f_2, f_5] = f^{\pi_2}\\
&-[f_2, f_4] = f^{\pi_3}\\
&[f_1, f_3] = f^{\pi_4}\\
&[f_1, f_2] = -[f_3, f_4] = \tfrac{1}{2}(f^{\pi_1} + f^{\pi_2})\\
&[f_1, f_4] = -[f_2, f_3] = \tfrac{1}{2}(-f^{\pi_1} + f^{\pi_2}).
\end{aligned} \tag{21}$$

The normal form is given in Section 6, equation (46).

Case 3)

Consider, finally, the case where we have two distinct pairs of complex conjugate solutions and only one real solution. We denote these solutions by x^1, $x^2 = \bar{x}^1$ x^3, $x^4 = \bar{x}^2$ and $x_5 \in \mathbb{R}^4$. Moreover, generically, there exist $\lambda_1, \ldots, \lambda_5 \neq 0$ such that

$$\lambda_1 \Re(x^1) + \lambda_2 \Im(x^1) + \lambda_3 \Re(x^3) + \lambda_4 \Im(x^3) = \lambda_5 x^5,$$

hence, by denoting $y_1 = \lambda_1 + \imath\lambda_2$, $y_2 = \bar{y}_1$, $y_3 = \lambda_3 + \imath\lambda_4$, $y_4 = \bar{y}_3$ and $y_5 = \lambda_5$,

$$\Re(\bar{y}_1 x^1) + \Re(\bar{y}_3 x^3) = y_5 x^5.$$

Therefore,

$$\tilde{\omega}^1 + \tilde{\omega}^3 = \tilde{\omega}^5,$$

where

$$\begin{aligned}
\tilde{\omega}^1 &= \Re(\omega(\bar{y}_1 x^1)),\\
\tilde{\omega}^2 &= \Im(\omega(\bar{y}_1 x^1)),\\
\tilde{\omega}^3 &= \Re(\omega(\bar{y}_3 x^3)),\\
\tilde{\omega}^4 &= \Im(\omega(\bar{y}_3 x^3)),\\
\tilde{\omega}^5 &= \omega(y_5 x_5).
\end{aligned}$$

Moreover $\tilde{\omega}^1 + \imath\tilde{\omega}^2$ and $\tilde{\omega}^3 + \imath\tilde{\omega}^4$ are two complex, rank 2, antisymmetric matrices.

By the same arguments as for the previous case, there exist $v_1, v_2 \in \mathbb{R}^5$ such that $\tilde{\omega}^1 v_1 = \tilde{\omega}^2 v_1 = 0$ and $\tilde{\omega}_3 v_2 = \tilde{\omega}_4 v_2 = 0$. Complete v_1, v_2 to a base for $\mathbb{R}^5$. Then, in these coordinates we can write:

$$\tilde{\omega}^i = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\tilde{\omega}^i)_{22} \end{array}\right] \qquad i = 1, 2$$

and

$$\tilde{\omega}^i = \left[\begin{array}{c|c} (\tilde{\omega}^i)_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \qquad i = 3, 4,$$

where $(\tilde\omega^1)_{22}, (\tilde\omega^2)_{22}, (\tilde\omega^3)_{11}$ and $(\tilde\omega^4)_{11}$ are some 4×4 antisymmetric matrices with determinant 1. Indeed, since $\tilde\omega^1 + \imath\tilde\omega^2$ has rank 2, we can write

$$0 = (\tilde\omega^1 + \imath\tilde\omega^2)\wedge(\tilde\omega^1 + \imath\tilde\omega^2) = (\tilde\omega^1\wedge\tilde\omega^1 - \tilde\omega^2\wedge\tilde\omega^2) + 2\imath(\tilde\omega^1\wedge\tilde\omega^2),$$

which implies that

$$\begin{aligned} &\tilde\omega^1\wedge\tilde\omega^2 = 0 \quad \text{and} \\ &\tilde\omega^1\wedge\tilde\omega^1 = \tilde\omega^2\wedge\tilde\omega^2. \end{aligned} \tag{22}$$

Hence, from the second of equations (22), $\sqrt{\mathsf{det}\ ((\tilde\omega^1)_{22})} = \sqrt{\mathsf{det}\ ((\tilde\omega^2)_{22})}$. The same relation holds true for $(\tilde\omega^3)_{11}$ and $(\tilde\omega^4)_{11}$.

Now recall that $\tilde\omega^5 = \tilde\omega^1 + \tilde\omega^3$ is a real rank 2 antisymmetric matrix. Then

$$0 = \tilde\omega^5\wedge\tilde\omega^5 = \tilde\omega^1\wedge\tilde\omega^1 + \tilde\omega^3\wedge\tilde\omega^3 + 2\tilde\omega^1\wedge\tilde\omega^3,$$

hence

$$\tilde\omega^1\wedge\tilde\omega^3 = -\frac{1}{2}\left(\tilde\omega^1\wedge\tilde\omega^1 + \tilde\omega^3\wedge\tilde\omega^3\right). \tag{23}$$

Finally, denoting $v_{ij} = \tilde\omega^i\wedge\tilde\omega^j$, for all $i\le j$, we have that $v_{11} = v_{22}$, $v_{33} = v_{44}$, $v_{12} = 0$, $v_{34} = 0$ (by equations (22)) and $2v_{13} + v_{11} + v_{22} = 0$ (by equation (23)). Then the matrix $P = [v_{11}\, v_{14}\, v_{23}\, v_{24}\, v_{33}]$, suitably rescaled, transforms $\tilde\omega^i$ as follows:

$$P^T\tilde\omega^1 P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad P^T\tilde\omega^2 P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$P^T\tilde\omega^3 P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P^T\tilde\omega^4 P = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally we have the following multiplication table:

$$\begin{aligned} &-[f_4, f_5] = f^{\pi_1} \\ &[f_2, f_5] = [f_3, f_4] = f^{\pi_2} \\ &[f_1, f_4] = f^{\pi_3} \\ &-[f_1, f_3] = [f_2, f_4] = f^{\pi_4} \\ &[f_2, f_3] = -f^{\pi_1} + f^{\pi_3} \\ &[f_1, f_2] = [f_1, f_5] = [f_3, f_5] = 0. \end{aligned} \tag{24}$$

The normal form is given in Section 6, equation (47). $\square$

4 Cases with $r = 3$

We only have to consider the cases with

i) $d = 2$ and $m = 1$

ii) $d = 3$ and $m = 1$

iii) $d = 3$ and $m = 7$.

First we observe that **i)** corresponds to the Engel algebra: the growth vector is $(2, 3, 4)$. It is known that there is only one isomorphism class for this case. For completeness we report its normal form in Section 6, equation (48). For **ii)** and **iii)**, the following propositions allow us to reduce the analysis of **iii)** to that of **ii)**.

Proposition 12 *Any $E \in Gr_m(\mathcal{L}_3^3)$ can be identified with an m dimensional subspace of the space $\Gamma(3)$ of 3×3 matrices γ such that* $\mathsf{trace}\,(C\gamma) = 0$, *where* $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Proof. Fix f_1, f_2, f_3 generators for $\mathcal{L}_3^1$ and and let f^{π_h} be generators of $E \in \mathrm{Gr}_m(\mathcal{L}_3^3)$. Then we can write $[f_i, [f_j, f_k]] = \sum_{h=1}^m \gamma_{ijk}^h f^{\pi_h}$. Notice that, since $[f_1, [f_2, f_3] = [f_2, [f_1, f_3]] - [f_3, [f_1, f_2]]$, it holds that $\gamma_{123}^h = \gamma_{213}^h - \gamma_{312}^h$. By denoting

$$\gamma^h = \begin{bmatrix} \gamma_{112}^h & \gamma_{212}^h & \gamma_{312}^h \\ \gamma_{113}^h & \gamma_{213}^h & \gamma_{313}^h \\ \gamma_{123}^h & \gamma_{223}^h & \gamma_{323}^h \end{bmatrix}$$

we have that $\mathsf{trace}\,(C\gamma^h) = 0$ for all $h = 1, \ldots, m$. Moreover if we choose a different set of generators for E, that is $f^{\pi_h} = \sum_{i=1}^m x_{hi} \tilde{f}^{\pi_i}$, then $[f_i, [f_j, f_h]] = \sum_{i=1}^m \tilde{\gamma}^i \tilde{f}^{\pi_i}$, where $\tilde{\gamma}^i = \sum_{h=1}^m x_{hi} \gamma^h$, and $\mathsf{trace}\,(C\tilde{\gamma}^i) = \sum_h x_{hi}\,\mathsf{trace}\,(C\gamma^h) = 0$. Then E can be described by an m dimensional subspace in $\Gamma(3)$ generated by $\gamma^h,\ h = 1, \ldots, m$. $\square$

Proposition 13 *If $r = 3$ then the bi–dimension $(d, \ell_d^{r-1} + 1)$ is rigid if and only if the dual bi–dimension $(d, \ell_d^r - 1)$ is rigid.*

Proof. Now in $\Gamma(3)$ we define the bilinear symmetric product

$$\Gamma(3) \times \Gamma(3) \to \mathbb{R}$$

by $(\gamma, \eta) \mapsto \mathsf{trace}\ (\gamma\eta^T)$. Then if E is generated by $\gamma^h,\ h = 1, \dots, m$, we define $E^\perp$ to be the set of γ such that $(\gamma, \gamma^h) = 0$ for all $h = 1, , \dots, m$.

Consider now the maps

$$\begin{array}{rccc} \Psi : & \mathrm{GL}(\mathbb{R}^d) & \to & \mathrm{Gr}_m(\mathcal{L}_3^3) \\ & V & \mapsto & \Phi(V)(E) \\ \tilde{\Psi} : & \mathrm{GL}(\mathbb{R}^d) & \to & \mathrm{Gr}_{l_d(3)-m}(\mathcal{L}_3^3) \\ & V & \mapsto & \Phi(V)(E^\perp). \end{array}$$

We show that for all $V \in \mathrm{GL}(\mathbb{R}^d)$, $\tilde{\Psi}(V^{-T}) = (\Psi(V))^\perp$ so that a one to one correspondence between the image of Ψ and that of $\tilde{\Psi}$ is established and the proposition is proved.

The induced action of $\Phi(V)$ on $\gamma \in \Gamma(3)$ is computed as follows.

$$\begin{array}{l} \Phi(V)[f_i, [f_j, f_k]] = [\sum_{l=1}^3 v_{il} f_l, [\sum_{r=1}^3 v_{jr} f_r, \sum_{s=1}^3 v_{ks} f_s]] \\ = \sum_{l,s,r=1}^3 v_{il} v_{jr} v_{ks} [f_l, [f_r, f_s]] = \sum_{l=1}^3 v_{il} \left(\sum_{s,r=1}^3 v_{jr} v_{ks} [f_l, [f_r, f_s]] \right) \\ \qquad\qquad = \sum_{l=1}^3 v_{il} \left(\sum_{r<s} (v_{jr} v_{ks} - v_{js} v_{kr}) [f_l, [f_r, f_s]] \right), \end{array}$$

from which we write

$$\Phi(V)\gamma^h = CV^{-T}C^T\gamma^h V^T, \qquad h = 1, \dots, m.$$

Notice that $\mathsf{trace}\ (V^{-T}C^T\gamma V^T) = \mathsf{trace}\ (V^T V^{-T} C^T \gamma) = \mathsf{trace}\ (C^T\gamma)$. Therefore,

$$\Phi(V) : \Gamma(3) \mapsto \Gamma(3),$$

for all $V \in \mathrm{GL}(\mathbb{R}^3)$. Moreover

$$\begin{array}{rcl} (\Phi(V)\gamma, \Phi(V^{-T})\eta) & = & \mathsf{trace}\ ((CV^{-T}C^T\gamma V^T)(CVC^T\eta V^{-1})^T) \\ & = & \mathsf{trace}\ ((CV^{-T}C^T\gamma V^T)(V^{-T}\eta^T CV^T C^T)) \\ & = & \mathsf{trace}\ ((CV^T C^T)(CV^{-T}C^T)\gamma\eta^T) \\ & = & \mathsf{trace}\ (\gamma\eta^T) = 0 \end{array}$$

for all $\gamma \in E$ and $\eta \in E^\perp$. $\square$

We next analyse the case with $m = 1$.

Let $\tilde{\Phi}(V) : C^T\Gamma(3) \to C^T\Gamma(3)$, with $\tilde{\Phi}(V) = C^T \circ \Phi(V^T) \circ C$, i.e. $\tilde{\Phi}(V)(C^T\gamma) = C^T CV^{-1}C^T\gamma V = V^{-1}C^T\gamma V$. The dimension of the orbit of $\tilde{\Phi}$ and that of Φ coincide. We have the following:

Lemma 2 *The codimension of any orbit of $\tilde{\Phi}$ in $\Gamma(3)$ is greater or equal than 2.*

Proof. The characteristic polynomial of $C\gamma$ is an invariant of the action of $\tilde{\Phi}$; indeed

$$p(\lambda) = \mathsf{det}\,(V^{-1}C^T\gamma V - \lambda I_3) = \mathsf{det}\,(C^T\gamma - \lambda I_3).$$

Since $\mathsf{trace}\,(C^T\gamma) = 0$ we have that $p(\lambda)$ is determined by 2 coefficients and the codimension of any orbit of $\tilde{\Phi}$ is no less than 2. $\square$

From Proposition 13 and Lemma 2 it immediately follows:

Proposition 14 *The bi-dimensions* $(3, 7)$ *and* $(3, 13)$ *are not rigid.*

5 Cases with $r = 4$

For $r = 4$ we only have to consider the cases with $d = 2$ and $m = 1, 2$. For $d = 2$ and $r = 4$ we have $\ell_2(4) = 3$ brackets of degree 4 which are linearly independent with respect to the Jacobi identity:
$[f_1, [f_1, [f_1, f_2]]], [f_2, [f_1, [f_1, f_2]]]$ and $[f_2, [f_2, [f_1, f_2]]]$.

Let now $E \subset \mathrm{Gr}_m(\mathcal{L}_2^4)$ and $f^{\pi_h},\ h = 1, \ldots, m$ be generators of E. Then E can be identified with an m-dimensional space of 2×2 symmetric matrices.

Indeed, for all $l, s = 1, 2$ we can write $[f_l, [f_s, [f_1, f_2]]] = \sum_{h=1}^m q_{ls}^h f^{\pi_h}$. Let Q^h be the matrix with coefficients q_{ls}^h. Since

$$\lambda_1[f_1, [f_1, [f_1, f_2]]] + \lambda_2[f_2, [f_2, [f_1, f_2]]] = [f_2, [f_1, [f_1, f_2]]].$$

we have that Q^h are symmetric of order 2. Moreover if $f^{\pi_h} = \sum_{i=1}^m x_{hi}\tilde{f}^{\pi_i}$ then

$$[f_l, [f_s, [f_1, f_2]]] = \sum_{i=1}^m \tilde{q}_{ls}^i \tilde{f}^{\pi_i},$$

where $\tilde{Q}^i = \sum_{h=1}^m x_{hi}Q^h$.

Next we compute the induced action of $\Phi(V),\ V \in \mathbb{R}^2$ on the space $\mathcal{S}_m(2)$ of symmetric matrices of order 2 corresponding to E. From

$$\begin{aligned} \Phi(V)[f_l, [f_s, [f_i, f_j]]] &= [\textstyle\sum_r v_{lr} f_r, [\sum_m v_{sm} f_m,\ \mathsf{det}\,(V)[f_1, f_2]]] \\ &= \mathsf{det}\,(V)\,[\textstyle\sum_r v_{lr} f_r, [\sum_m v_{sm} f_m, [f_1, f_2]]] \\ &= \mathsf{det}\,(V) \textstyle\sum_{rm} v_{lr} v_{sm} [f_r, [f_m, [f_1, f_2]]] \end{aligned}$$

we obtain $\Phi(V)Q^h = \mathsf{det}\,(V)VQ^hV^T$. Therefore the degree 2 homogeneous polynomial

$$\mathsf{det}\,(\sum_h x_h Q^h) \tag{25}$$

in the m variables $x_1, \ldots, x_m$ is invariant by the action Φ up to a positive scalar multiplier.

For $m = 1$ we have the following:

Proposition 15 *The bi–dimension* $(2,6)$ *is rigid with two isomorphism classes distinguished by the sign of the determinant (25).*

Proof. For $m = 1$ equation (25) reduces to $\mathsf{det}\,(Q)$ whose sign is invariant by the action Φ. A generic symmetric form Q can either be sign definite or indefinite (corresponding respectively to $\mathsf{det}\,(Q) > 0$ or $\mathsf{det}\,(Q) < 0$). For each of these cases we will give the multiplication table thus showing that $(2,6)$ is rigid with two isomorphism classes.

$m = 1$, Q positive definite

Assume that Q is positive definite and of the form:

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then we have the following multiplication table:

$$\begin{aligned} [f_2,[f_1,[f_1,f_2]]] &= [0,1]Q[1,0]^T f^{\pi_1} = f^{\pi_1} \\ [f_1,[f_1,[f_1,f_2]]] &= [1,0]Q[1,0]^T f^{\pi_1} = 2f^{\pi_1} \\ [f_2,[f_2,[f_1,f_2]]] &= [0,1]Q[0,1]^T f^{\pi_1} = 2f^{\pi_1} \end{aligned} \tag{26}$$

The normal form is given in Section 6, equation (49).

$m = 1$, Q non definite

Assume that Q non definite and of the form:

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we have the following multiplication table

$$\begin{aligned} [f_1,[f_2,[f_1,f_2]]] &= [1,0]Q[0,1]^T f^{\pi_1} = f^{\pi_1} \\ [f_1,[f_1,[f_1,f_2]]] &= [1,0]Q[1,0]^T f^{\pi_1} = 0 \\ [f_2,[f_2,[f_1,f_2]]] &= [0,1]Q[0,1]^T f^{\pi_1} = 0 \end{aligned} \tag{27}$$

The normal form is given in Section 6, equation (27). □

For $m = 2$ we have the following

Proposition 16 *The bi–dimension* $(2,7)$ *is rigid with two isomorphism classes distinguished by the sign of the discriminant of the polynomial (25).*

Proof. (25) is a homogeneous polynomial of degree 2 in two variables whose coefficients are invariant by the action of Φ. Then the equation

$$\mathsf{det}\ (x_1 Q^1 + x_2 Q^2) = 0 \tag{28}$$

has two solutions that can either be real or complex conjugates. For each of these cases we will give the multiplication table thus showing that $(2, 7)$ is rigid with two isomorphism classes.

Remark. The sign of the poynomial (25) could serve as an extra invariant in the complex case, but a simple analysis shows that this sign is unavoidably negative.

$m = 2$**, real case**

Assume that equation (28) has real solutions. Under generic assumptions we can assume that they are distinct and that there exist Q^1 and Q^2, in the linear space $\mathcal{S}_2(2)$ of 2×2 symmetric matrices of order two corresponding to E, of corank 1 with transversal kernel. By letting p_1 and p_2 to be eigenvectors relative to the zero eigenvalue of Q^2 and Q^1 respcetively, we have that $P = [p_1, p_2]$, the matrix of columns p_1 and p_2, is such that

$$\tilde{Q}_1 = P^T(Q_1)P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\tilde{Q}_2 = P^T(Q_2)P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Finally, we obtain the following multiplication table:

$$\begin{aligned} [f_1, [f_1, [f_1, f_2]]] &= [1, 0](\tilde{Q}_1 f^{\pi_1} + \tilde{Q}_2 f^{\pi_2})[1, 0]^T = f^{\pi_1} \\ [f_2, [f_2, [f_1, f_2]]] &= [0, 1](\tilde{Q}_1 f^{\pi_1} + \tilde{Q}_2 f^{\pi_2})[0, 1]^T = f^{\pi_2} \\ [f_2, [f_1, [f_1, f_2]]] &= [0, 1](\tilde{Q}_1 f^{\pi_1} + \tilde{Q}_2 f^{\pi_2})[1, 0]^T = 0. \end{aligned} \tag{29}$$

The normal form is reported in Section 6, equation (51).

$m = 2$**, complex case**

If there is a pair of complex conjugate solution to equation (28) then we can assume that Q_1 and Q_2 in $\mathcal{S}_2(2)$ are complex conjugate. Let now p_1 and p_2 be such that $p = p_1 + \imath p_2 \in \mathsf{ker}\, Q_1$, i.e.

$$0 = (\Re(Q_1) + \imath\Im(Q_1))\, p = (\Re(Q_1)p_1 - \Im(Q_1)p_2) + \imath\, (\Re(Q_1)p_2 + \Im(Q_1)p_1)\,,$$

hence, by setting $P = [p_1, p_2]$ we obtain

$$\Re(\tilde{Q}_1) = P^T(\Re(Q_1))P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$\Im(\tilde{Q}_1) = P^T(\Im(Q_1))P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By denoting

$$\begin{aligned} f^{\pi_1} &= \tfrac{1}{2}\left(\tilde{f}^{\pi_1} + \imath \tilde{f}^{\pi_2}\right) \\ f^{\pi_2} &= \tfrac{1}{2}\left(\tilde{f}^{\pi_1} - \imath \tilde{f}^{\pi_2}\right), \end{aligned}$$

we have that $\tilde{Q}_1 f^{\pi_1} + \tilde{Q}_2 f^{\pi_2} = \Re(\tilde{Q}_1)\tilde{f}^{\pi_1} - \Im(\tilde{Q}_1)\tilde{f}^{\pi_2}$ and obtain the following multiplication table:

$$\begin{aligned} [f_1,[f_1,[f_1,f_2]]] &= [1,0]\,(\Re(\tilde{Q}_1)\tilde{f}^{\pi_1} - \Im(\tilde{Q}_1)\tilde{f}^{\pi_2})[1,0]^T = \tilde{f}^{\pi_1} + \tilde{f}^{\pi_2} \\ [f_2,[f_2,[f_1,f_2]]] &= [0,1](\Re(\tilde{Q}_1)\tilde{f}^{\pi_1} - \Im(\tilde{Q}_1)\tilde{f}^{\pi_2})[0,1]^T = -\tilde{f}^{\pi_1} - \tilde{f}^{\pi_2} \\ [f_2,[f_1,[f_1,f_2]]] &= [0,1](\Re(\tilde{Q}_1)\tilde{f}^{\pi_1} - \Im(\tilde{Q}_1)\tilde{f}^{\pi_2})[1,0]^T = \tilde{f}^{\pi_1} - \tilde{f}^{\pi_2} \end{aligned} \tag{30}$$

The normal form is reported in Section 6, equation (52). □

6 Normal Forms

To compute the normal form of a set of smooth vector fields $\mathcal{F}$, once known their multiplication table, it is sufficient to apply the Campbell–Hausdorff formula. Indeed, assume that $\mathcal{F} = \{f_1, \dots, f_d\}$ is regular at q_0 and Lie algebra Lie$\mathcal{F}$ is n-dimensional with the basis f^i, $i = 1, \dots, n$. Then the exponential mapping

$$\Phi \,:\, \sum_{i=1}^{n} x_i f^i \;\mapsto\; q_0 e^{\sum_{i=1}^{n} x_i f^i}$$

is smoothly invertible in a neighbourhood of $0 \in \text{Lie}\mathcal{F} = \mathbb{R}^n$ and defines local coordinates in a neighbourhood of q_0.

Let $q \mapsto q e^{tf^j}$, $t \in \mathbb{R}$, be the flow on M generated by the field f^j. If $q = \Phi\left(\sum_{i=1}^{n} x_i f^i\right)$, then

$$q e^{tf^j} = q_0 e^{\sum_{i=1}^{n} x_i f^i} e^{tf^j} = q_0 e^{\ln\; e^{\sum_{i=1}^{n} x_i f^i} e^{tf^j}} = \Phi\left(\ln\left(e^{\sum_{i=1}^{n} x_i f^i} e^{tf^j}\right)\right),$$

where the product under the logarithm is an element of the ‘abstract’ Lie group generated by Lie algebra $\mathcal{F}$. Hence the coordinate representation of the flow

$q \mapsto qe^{tf^j}$ is as follows:

$$\sum_{i=1}^{n} x_i f^i \;\mapsto\; \ln\left(e^{\sum_{i=1}^{n} x_i f^i} e^{tf^j}\right)$$

and the coordinate representarion of the vector field f^j as a vector function of $x = (x_1, \ldots, x_n)$ is:

$$x \mapsto \frac{\partial}{\partial t}_{|t=0} \ln\left(e^{\sum_{i=1}^{n} x_i f^i} e^{tf^j}\right).$$

By the Campbell–Hausdorff formula we can write

$$\begin{aligned}
&\ln(e^f e^{tf^j}) = \\
&f + tf^j + \tfrac{1}{2}[f, tf^j] + \tfrac{1}{12}([f,[f,tf^j]] - [tf^j,[f,tf^j]]) - \tfrac{1}{24}[f,[tf^j,[f,tf^j]]] + \ldots,
\end{aligned} \tag{31}$$

from which

$$\frac{d}{dt}_{|t=0} \ln(e^f e^{tf^j}) = f^j + \frac{1}{2}[f, f^j] + \frac{1}{12}[f,[f,f^j]] + \ldots. \tag{32}$$

Notice that in equation (31) the brackets of order 4 appear only as a $O(t^2)$ term. Hence in equation (32) the brackets of order 4 disappear. Substituting in (32) the expression of f gives:

$$\frac{\partial}{\partial t}_{|t=0} \ln(e^f e^{tf^j}) = f^j + \frac{1}{2}\sum_{i=1}^{n} x_i [f^i, f^j] + \frac{1}{12}\sum_{i,h=1}^{n} x_i x_h [f^i,[f^h,f^j]] + \ldots, \tag{33}$$

and, finally, substituting the expressions for $[f^i, f^j]$ and $[f^i,[f^h,f^j]]$ as in the multiplication tables, gives the expression for f^j in the coordinates $\frac{\partial}{\partial x_i}$ at the point x.

Next we give the resulting expressions of the f^j's, for each multiplication table. Consider first the rigid bi–dimension $(d, d+1)$ corresponding to $r = 2$ and $m = 1$. From the multiplication table given in (3), we have $f^i = f_i$ for $i = 1, \ldots, d$ and $f^{d+1} = [f_1, f_2]$. Then equation (33) reads:

$$\frac{\partial}{\partial t}_{|t=0} \ln(e^f e^{tf_j}) = f_j + \frac{1}{2}\sum_{i=1}^{d} x_i [f_i, f_j],$$

hence, in the coordinates $\frac{\partial}{\partial x_j}$ we have

$$f_j = \begin{cases} \frac{\partial}{\partial x_j} + \frac{1}{2}x_{j-1}\frac{\partial}{\partial x_{d+1}} & \text{if } j \text{ is even} \\ \frac{\partial}{\partial x_j} - \frac{1}{2}x_{j+1}\frac{\partial}{\partial x_{d+1}} & \text{if } j \text{ is odd.} \end{cases} \tag{34}$$

For the dual case, the multiplication table is given in (4). By choosing f^j, for $j = 1, \ldots, n$, as follows:

$$\begin{array}{l} f^i = f_i, \ i = 1, \ldots, d \\ f^{d-1+i} = [f_{2i-1}, f_{2i}], \ i = 2, \ldots \frac{\hat{d}}{2} \\ f^{id+\frac{\hat{d}}{2}-l(i)+k} = [f_i, f_k], \ k > i, \end{array}$$

where $l(i) = \frac{5}{4} + \frac{1}{4}(-1)^{i+1} + \frac{1}{2}i^2 + i$ and $\hat{d} = d$ if d is even or $\hat{d} = d - 1$ if d is odd, we have the corresponding normal form:

$$f_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \sum_{i=2}^{\frac{\hat{d}}{2}} \frac{\partial}{\partial x_{d-1+i}} - \sum_{i=2}^{\frac{\hat{d}}{2}} \frac{1}{2}x_i \frac{\partial}{\partial x_{d+\frac{\hat{d}}{2}-l(1)+i}}$$

$$f_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \sum_{i=2}^{\frac{\hat{d}}{2}} \frac{\partial}{\partial x_{d-i+1}} - \sum_{i=2}^{\frac{\hat{d}}{2}} \frac{1}{2}x_i \frac{\partial}{\partial x_{2d+\frac{\hat{d}}{2}-l(2)+i}}$$

$$f_j = \frac{\partial}{\partial x_j} + \sum_{i=1}^{j-2} \frac{x_i}{2} \frac{\partial}{\partial x_{id+\frac{\hat{d}}{2}-l(i)+j}} + \frac{x_{j-1}}{2} \frac{\partial}{\partial x_{d-\frac{j}{2}+1}} - \sum_{i=j+1}^{d} \frac{x_i}{2} \frac{\partial}{\partial x_{jd+\frac{\hat{d}}{2}-l(j)+i}}$$

if j is even or

$$f_j = \frac{\partial}{\partial x_j} + \sum_{i=1}^{j-1} \frac{x_i}{2} \frac{\partial}{\partial x_{id+\frac{\hat{d}}{2}-l(i)+j}} - \frac{x_{j+1}}{2} \frac{\partial}{\partial x_{d-\frac{j+1}{2}+1}} - \sum_{i=j+2}^{d} \frac{x_i}{2} \frac{\partial}{\partial x_{jd+\frac{\hat{d}}{2}-l(j)+i}}$$

if j is odd.

(35)

Consider now the bi–dimensions $(d, d+2)$ corresponding to $r = 2$ and $m = 2$. Rigid cases are for $d = 4, 6$ each with two isomorphism classes and for $d = 5, 7$ with one isomorphism class. For $d = 4$ and the multiplication table as in equation (6), by setting $f^i = f_i$ for $i = 1, \ldots, 4$, $f^5 = [f_1, f_2] = f^{\pi_1}$ and $f^6 = [f_3, f_4] = f^{\pi_2}$, we have that the normal form is:

$$\begin{array}{l} f_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_5} \\ f_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_5} \\ f_3 = \frac{\partial}{\partial x_3} - \frac{x_4}{2}\frac{\partial}{\partial x_6} \\ f_4 = \frac{\partial}{\partial x_4} + \frac{x_3}{2}\frac{\partial}{\partial x_6}. \end{array} \qquad (36)$$

For $d = 4$ and the multiplication table as in equation (7), by setting $f^i = f_i$ for $i = 1, \ldots, 4$, $f^5 = [f_1, f_3] = -[f_2, f_4] = f^{\pi_1}$ and $f^6 = [f_1, f_4] = [f_2, f_3] = f^{\pi_2}$, we have that the normal form is:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{x_3}{2}\tfrac{\partial}{\partial x_5} - \tfrac{x_4}{2}\tfrac{\partial}{\partial x_6}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{x_4}{2}\tfrac{\partial}{\partial x_5} - \tfrac{x_3}{2}\tfrac{\partial}{\partial x_6}\\
f_3 &= \tfrac{\partial}{\partial x_3} + \tfrac{x_1}{2}\tfrac{\partial}{\partial x_5} + \tfrac{x_2}{2}\tfrac{\partial}{\partial x_6}\\
f_4 &= \tfrac{\partial}{\partial x_4} - \tfrac{x_2}{2}\tfrac{\partial}{\partial x_5} + \tfrac{x_1}{2}\tfrac{\partial}{\partial x_6}.
\end{aligned} \tag{37}$$

For $d = 5$, considering the multiplication table (10), and setting $f^i = f_i$ for $i = 1, \dots, 5$, $f^6 = [f_1, f_2] = [f_3, f_5] = f^{\pi_1}$ and $f^7 = [f_3, f_4] = [f_1, f_5] = f^{\pi_2}$, we obtain

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{x_2}{2}\tfrac{\partial}{\partial x_6} - \tfrac{x_5}{2}\tfrac{\partial}{\partial x_7}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{x_1}{2}\tfrac{\partial}{\partial x_6}\\
f_3 &= \tfrac{\partial}{\partial x_3} - \tfrac{x_5}{2}\tfrac{\partial}{\partial x_6} - \tfrac{x_4}{2}\tfrac{\partial}{\partial x_7}\\
f_4 &= \tfrac{\partial}{\partial x_4} + \tfrac{x_3}{2}\tfrac{\partial}{\partial x_7}\\
f_5 &= \tfrac{\partial}{\partial x_5} + \tfrac{x_3}{2}\tfrac{\partial}{\partial x_6} + \tfrac{x_1}{2}\tfrac{\partial}{\partial x_7}.
\end{aligned} \tag{38}$$

For $d = 6$, with multiplication table as in (8), by setting $f^i = f_i$ for $i = 1, \dots, 6$, $[f_1, f_2] = f^7$ and $[f_3, f_4] = f^8$, we have:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_7}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_7}\\
f_3 &= \tfrac{\partial}{\partial x_3} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_8}\\
f_4 &= \tfrac{\partial}{\partial x_4} + \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_8}\\
f_5 &= \tfrac{\partial}{\partial x_5} + \tfrac{1}{2}x_6(\tfrac{\partial}{\partial x_7} - \tfrac{\partial}{\partial x_8})\\
f_6 &= \tfrac{\partial}{\partial x_6} - \tfrac{1}{2}x_5(\tfrac{\partial}{\partial x_7} - \tfrac{\partial}{\partial x_8}).
\end{aligned} \tag{39}$$

For $d = 6$ with multiplication table as in (9) and setting $f^i = f_i$ for $i = 1, \dots, 6$, $f^7 = [f_3, f_5] = -[f_4, f_6] = f^{\pi_1}$ and $f^8 = [f_3, f_6] = [f_4, f_5] = f^{\pi_2}$, we have:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2(\tfrac{\partial}{\partial x_7} - \tfrac{\partial}{\partial x_8})\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1(\tfrac{\partial}{\partial x_7} - \tfrac{\partial}{\partial x_8})\\
f_3 &= \tfrac{\partial}{\partial x_3} - \tfrac{1}{2}x_5\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_6\tfrac{\partial}{\partial x_8}\\
f_4 &= \tfrac{\partial}{\partial x_4} + \tfrac{1}{2}x_6\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_5\tfrac{\partial}{\partial x_8}\\
f_5 &= \tfrac{\partial}{\partial x_5} + \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_8}\\
f_6 &= \tfrac{\partial}{\partial x_6} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_8}.
\end{aligned} \tag{40}$$

Finally for $d = 7$, whose multiplication table is given in (11), by setting $f^i = f_i$ for $i = 1, \dots, 7$, $f^8 = [f_1, f_2] = [f_3, f_7] = [f_5, f_7] = f^{\pi_1}$ and $f^9 = [f_3, f_4] =$

$[f_1,f_7]=[f_6,f_7]=f^{\pi_2}$, we have the following normal form:

$$\begin{array}{l}
f_1=\frac{\partial}{\partial x_1}-\frac{1}{2}x_2\frac{\partial}{\partial x_8}-\frac{1}{2}x_7\frac{\partial}{\partial x_9}\\
f_2=\frac{\partial}{\partial x_2}+\frac{1}{2}x_1\frac{\partial}{\partial x_8}\\
f_3=\frac{\partial}{\partial x_3}-\frac{1}{2}x_7\frac{\partial}{\partial x_8}-\frac{1}{2}x_4\frac{\partial}{\partial x_9}\\
f_4=\frac{\partial}{\partial x_4}+\frac{1}{2}x_3\frac{\partial}{\partial x_9}\\
f_5=\frac{\partial}{\partial x_5}+\frac{1}{2}(x_6-x_7)\frac{\partial}{\partial x_8}-\frac{1}{2}x_6\frac{\partial}{\partial x_9}\\
f_6=\frac{\partial}{\partial x_6}-\frac{1}{2}x_5\frac{\partial}{\partial x_8}+\frac{1}{2}(x_5-x_7)\frac{\partial}{\partial x_9}\\
f_7=\frac{\partial}{\partial x_7}+\frac{1}{2}(x_3+x_5)\frac{\partial}{\partial x_8}+\frac{1}{2}(x_1+x_6)\frac{\partial}{\partial x_9}.
\end{array}\tag{41}$$

Next we consider the bi–dimensions $(d,d+3)$ corresponding to $r=2$ and $m=3$. We have seen that such bi–dimensions are rigid for $d=4$ and $d=5$ with two isomorphism classes.
For $d=4$ and multiplication table (16), by setting $f^i=f_i$ for $i=1,\dots,4$ and $f^5=f^{\pi_1}=-[f_1,f_2]-f^{\pi_3}$, $f^6=f^{\pi_2}=[f_3,f_4]+f^{\pi_3}$ and $f^7=[f_1,f_3]=[f_2,f_4]=f^{\pi_3}$, we obtain:

$$\begin{array}{l}
f_1=\frac{\partial}{\partial x_1}+\frac{1}{2}x_2\frac{\partial}{\partial x_5}+\frac{1}{2}(x_2-x_3)\frac{\partial}{\partial x_7}\\
f_2=\frac{\partial}{\partial x_2}-\frac{1}{2}x_1\frac{\partial}{\partial x_5}-\frac{1}{2}(x_1+x_4)\frac{\partial}{\partial x_7}\\
f_3=\frac{\partial}{\partial x_3}-\frac{1}{2}x_4\frac{\partial}{\partial x_6}+\frac{1}{2}(x_1+x_4)\frac{\partial}{\partial x_7}\\
f_4=\frac{\partial}{\partial x_4}+\frac{1}{2}x_3\frac{\partial}{\partial x_6}+\frac{1}{2}(x_2-x_3)\frac{\partial}{\partial x_7}.
\end{array}\tag{42}$$

For $d=4$ and multiplication table (18), by setting $f^i=f_i$ for $i=1,\dots,4$, $f^5=f^{\pi_1}$, $f^6=f^{\pi_2}$ and $f^7=[f_1,f_2]=[f_3,f_4]=f^{\pi_3}$, we obtain:

$$\begin{array}{l}
f_1=\frac{\partial}{\partial x_1}-\frac{1}{2}(x_3+x_4)\frac{\partial}{\partial x_5}+\frac{1}{2}(x_3-x_4)\frac{\partial}{\partial x_6}-\frac{1}{2}x_2\frac{\partial}{\partial x_7}\\
f_2=\frac{\partial}{\partial x_2}-\frac{1}{2}(x_3-x_4)\frac{\partial}{\partial x_5}-\frac{1}{2}(x_3+x_4)\frac{\partial}{\partial x_6}+\frac{1}{2}x_1\frac{\partial}{\partial x_7}\\
f_3=\frac{\partial}{\partial x_3}+\frac{1}{2}(x_1+x_2)\frac{\partial}{\partial x_5}-\frac{1}{2}(x_1-x_2)\frac{\partial}{\partial x_6}-\frac{1}{2}x_4\frac{\partial}{\partial x_7}\\
f_4=\frac{\partial}{\partial x_4}+\frac{1}{2}(x_1-x_2)\frac{\partial}{\partial x_5}+\frac{1}{2}(x_1+x_2)\frac{\partial}{\partial x_6}+\frac{1}{2}x_3\frac{\partial}{\partial x_7}.
\end{array}\tag{43}$$

For $d=5$ and multiplication table (19), by setting $f^i=f_i$ for $i=1,\dots,5$, $f^6=[f_2,f_5]=[f_3,f_4]=f^{\pi_1}$, $f^7=-[f_1,f_4]=-[f_3,f_5]=f^{\pi_2}$ and $f^8=[f_1,f_3]=-[f_2,f_3]=\mp[f_4,f_5]=f^{\pi_3}$, we have:

$$\begin{array}{l}
f_1=\frac{\partial}{\partial x_1}+\frac{1}{2}x_4\frac{\partial}{\partial x_7}-\frac{1}{2}x_3\frac{\partial}{\partial x_8}\\
f_2=\frac{\partial}{\partial x_2}-\frac{1}{2}x_5\frac{\partial}{\partial x_6}+\frac{1}{2}x_3\frac{\partial}{\partial x_8}\\
f_3=\frac{\partial}{\partial x_3}-\frac{1}{2}x_4\frac{\partial}{\partial x_6}+\frac{1}{2}x_5\frac{\partial}{\partial x_7}+\frac{1}{2}(x_1-x_2)\frac{\partial}{\partial x_8}\\
f_4=\frac{\partial}{\partial x_4}+\frac{1}{2}x_3\frac{\partial}{\partial x_6}-\frac{1}{2}x_1\frac{\partial}{\partial x_7}\pm\frac{1}{2}x_5\frac{\partial}{\partial x_8}\\
f_5=\frac{\partial}{\partial x_5}+\frac{1}{2}x_2\frac{\partial}{\partial x_6}-\frac{1}{2}x_3\frac{\partial}{\partial x_7}\mp\frac{1}{2}x_4\frac{\partial}{\partial x_8}.
\end{array}\tag{44}$$

We consider now the bi–dimension $(d,d+4)$ corresponding to $r=2$, $m=4$. This bi–dimension is rigid with three isomorphism classes for $d=5$. First consider the multiplication table as in (20) and set $f^i = f_i$ for $i = 1, \ldots, 5$, $f^6 = [f_4, f_5] = f^{\pi_1}$, $f^7 = [f_2, f_3] = f^{\pi_2}$, $f^8 = [f_1, f_3] = [f_1, f_5] = [f_3, f_5] = f^{\pi_3}$ and $f^9 = -[f_1, f_2] = [f_1, f_4] = [f_2, f_4] = f^{\pi_4}$. The normal form is

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}(x_3 + x_5)\tfrac{\partial}{\partial x_8} + \tfrac{1}{2}(x_2 - x_4)\tfrac{\partial}{\partial x_9} \\
f_2 &= \tfrac{\partial}{\partial x_2} - \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}(x_1 + x_4)\tfrac{\partial}{\partial x_9} \\
f_3 &= \tfrac{\partial}{\partial x_3} + \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}(x_1 - x_5)\tfrac{\partial}{\partial x_8} \\
f_4 &= \tfrac{\partial}{\partial x_4} - \tfrac{1}{2}x_5\tfrac{\partial}{\partial x_6} + \tfrac{1}{2}(x_1 + x_2)\tfrac{\partial}{\partial x_9} \\
f_5 &= \tfrac{\partial}{\partial x_5} + \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_6} + \tfrac{1}{2}(x_1 + x_3)\tfrac{\partial}{\partial x_8}.
\end{aligned} \tag{45}$$

Next, for the multiplication table as in (21), setting $f^i = f_i$ for $i = 1, \ldots, 5$, $f^6 = [f_3, f_5] = -[f_4, f_5] = f^{\pi_1}$, $f^7 = -[f_1, f_5] = [f_2, f_5] = f^{\pi_2}$, $f^8 = -[f_2, f_4] = f^{\pi_3}$ and $f^9 = [f_1, f_3] = f^{\pi_4}$, we have the following normal form:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{4}(x_2 - x_4)\tfrac{\partial}{\partial x_6} - \tfrac{1}{4}(x_2 + x_4 - 2x_5)\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_9} \\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{4}(x_1 - x_3)\tfrac{\partial}{\partial x_6} + \tfrac{1}{4}(x_1 + x_3 - 2x_5)\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_8} \\
f_3 &= \tfrac{\partial}{\partial x_3} + \tfrac{1}{4}(x_2 + x_4 - 2x_5)\tfrac{\partial}{\partial x_6} - \tfrac{1}{4}(x_2 - x_4)\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_9} \\
f_4 &= \tfrac{\partial}{\partial x_4} - \tfrac{1}{4}(x_1 + x_3 - 2x_5)\tfrac{\partial}{\partial x_6} + \tfrac{1}{4}(x_1 - x_3)\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_8} \\
f_5 &= \tfrac{\partial}{\partial x_5} + \tfrac{1}{2}(x_3 - x_4)\tfrac{\partial}{\partial x_6} - \tfrac{1}{2}(x_1 - x_2)\tfrac{\partial}{\partial x_7}.
\end{aligned} \tag{46}$$

Finally, for the multiplication table as in (24), setting $f^i = f_i$ for $i = 1, \ldots, 5$, $f^6 = -[f_4, f_5] = f^{\pi_1}$, $f^7 = [f_2, f_5] = [f_3, f_4] = f^{\pi_2}$, $f^8 = [f_1, f_4] = f^{\pi_3}$ and $f^9 = -[f_1, f_3] = [f_2, f_4] = f^{\pi_4}$, we obtain the normal form:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_8} + \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_9} \\
f_2 &= \tfrac{\partial}{\partial x_2} - \tfrac{1}{2}x_5\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_9} + \tfrac{1}{2}x_3(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_8}) \\
f_3 &= \tfrac{\partial}{\partial x_3} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_7} - \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_9} - \tfrac{1}{2}x_2(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_8}) \\
f_4 &= \tfrac{\partial}{\partial x_4} + \tfrac{1}{2}x_5\tfrac{\partial}{\partial x_6} + \tfrac{1}{2}x_3\tfrac{\partial}{\partial x_7} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_8} + \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_9} \\
f_5 &= \tfrac{\partial}{\partial x_5} - \tfrac{1}{2}x_4\tfrac{\partial}{\partial x_6} + \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_7}.
\end{aligned} \tag{47}$$

For $r = 3$ the only rigid case is the Engel algebra. The multiplication table is given by $f^i = f_i$, $i = 1, 2$, $f^3 = [f_1, f_2]$, $f^4 = [f_1, f^3] = [f_2, f^3]$. The normal form of $\mathcal{F}$ is:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 + x_1x_2 + x_2^2)\tfrac{\partial}{\partial x_4} \\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 - x_1x_2 - x_1^2)\tfrac{\partial}{\partial x_4}.
\end{aligned} \tag{48}$$

For $r = 4$ the only rigid bi–dimensions are $(2, 6)$ and $(2, 7)$, corresponding to $d = 2$, $m = 1$ and $d = 2$, $m = 2$ respectively. For $n = 6, 7$ we compute the brackets $[f, f_j]$ by setting $f^i = f_i$ for $i = 1, 2$ and $f^3 = [f_1, f_2]$, $f^4 = [f_1, f^3]$

and $f^5 = [f_2, f^3]$:

$$\begin{aligned}
[f, \quad f_1] &= \textstyle\sum_{i=1}^n x_i [f_i, f_1] \\
&= +x_2[f_2, f_1] + x_3[[f_1, f_2], f_1] + x_4[[f_1, [f_1, f_2]], f_1] + x_5[[f_2, [f_1, f_2]], f_1] \\
&= -x_2[f_1, f_2] - x_3[f_1, [f_1, f_2]] - x_4[f_1, [f_1, [f_1, f_2]]] - x_5[f_1, [f_2, [f_1, f_2]]], \\
[f, \quad f_2] &= \textstyle\sum_{i=1}^n x_i [f_i, f_2] \\
&= +x_1[f_1, f_2] + x_3[[f_1, f_2], f_2] + x_4[[f_1, [f_1, f_2]], f_2] + x_5[[f_2, [f_1, f_2]], f_2] \\
&= +x_1[f_1, f_2] - x_3[f_2, [f_1, f_2]] - x_4[f_2, [f_1, [f_1, f_2]]] - x_5[f_2, [f_2, [f_1, f_2]]], \\
[f, \quad [f, \quad f_1]] &= -x_2 \left(x_1[f_1, [f_1, f_2]] + x_2[f_2, [f_1, f_2]]\right) \\
&\quad -x_3 \left(x_1[f_1, [f_1, [f_1, f_2]]] + x_2[f_2, [f_1, [f_1, f_2]]]\right), \\
[f, \quad [f, \quad f_2]] &= +x_1 \left(x_1[f_1, [f_1, f_2]] + x_2[f_2, [f_1, f_2]]\right) \\
&\quad -x_3 \left(x_1[f_1, [f_2, [f_1, f_2]]] + x_2[f_2, [f_2, [f_1, f_2]]]\right).
\end{aligned}$$

According to the multiplication table (26) for $m = 1$, and setting $f^6 = [f_2, f^4] = f^{\pi_1}$, we then have:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}(x_2 \tfrac{\partial}{\partial x_3} + x_3 \tfrac{\partial}{\partial x_4} + (2x_4 + x_5)\tfrac{\partial}{\partial x_6}) \\
&\quad - \tfrac{1}{12}(x_2 x_1 \tfrac{\partial}{\partial x_4} + x_2^2 \tfrac{\partial}{\partial x_5} + (2x_3 x_1 + x_3 x_2)\tfrac{\partial}{\partial x_6}) \\
&= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2} x_2 \tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 + x_1 x_2)\tfrac{\partial}{\partial x_4} - \tfrac{1}{12} x_2^2 \tfrac{\partial}{\partial x_5} \\
&\quad - \tfrac{1}{12}(12x_4 + 6x_5 + 2x_1 x_3 + x_2 x_3)\tfrac{\partial}{\partial x_6} \\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}(x_1 \tfrac{\partial}{\partial x_3} - x_3 \tfrac{\partial}{\partial x_5} - (x_4 + 2x_5)\tfrac{\partial}{\partial x_6}) \\
&\quad + \tfrac{1}{12}(x_1^2 \tfrac{\partial}{\partial x_4} + x_1 x_2 \tfrac{\partial}{\partial x_5} - (x_3 x_1 + 2x_3 x_2)\tfrac{\partial}{\partial x_6}) \\
&= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2} x_1 \tfrac{\partial}{\partial x_3} + \tfrac{1}{12} x_1^2 \tfrac{\partial}{\partial x_4} - \tfrac{1}{12}(6x_3 - x_1 x_2)\tfrac{\partial}{\partial x_5} \\
&\quad - \tfrac{1}{12}(12x_5 + 6x_4 + 2x_2 x_3 + x_1 x_3)\tfrac{\partial}{\partial x_6}.
\end{aligned} \tag{49}$$

For the multiplication table (27), setting $f^6 = [f_1, f^5] = f^{\pi_1}$, we have instead the following normal form:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}(x_2\tfrac{\partial}{\partial x_3} + x_3\tfrac{\partial}{\partial x_4} + x_5\tfrac{\partial}{\partial x_6}) - \tfrac{1}{12}(x_2x_1\tfrac{\partial}{\partial x_4} + x_2^2\tfrac{\partial}{\partial x_5} + x_3x_2\tfrac{\partial}{\partial x_6})\\
&= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 + x_1x_2)\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}x_2^2\tfrac{\partial}{\partial x_5} - \tfrac{1}{12}(6x_5 + x_2x_3)\tfrac{\partial}{\partial x_6}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}(x_1\tfrac{\partial}{\partial x_3} - x_3\tfrac{\partial}{\partial x_5} - x_4\tfrac{\partial}{\partial x_6}) + \tfrac{1}{12}(x_1^2\tfrac{\partial}{\partial x_4} + x_1x_2\tfrac{\partial}{\partial x_5} - x_1x_3\tfrac{\partial}{\partial x_6})\\
&= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_3} + \tfrac{1}{12}x_1^2\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}(6x_3 - x_1x_2)\tfrac{\partial}{\partial x_5} - \tfrac{1}{12}(6x_4 + x_1x_3)\tfrac{\partial}{\partial x_6}.
\end{aligned} \tag{50}$$

For $m = 2$ and the multiplication table (29) and setting $f^6 = [f_1, f^4] = f^{\pi_1}$ and $f^7 = [f_2, f^5] = f^{\pi_2}$, we have

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}(x_2\tfrac{\partial}{\partial x_3} + x_3\tfrac{\partial}{\partial x_4} + x_4\tfrac{\partial}{\partial x_6}) - \tfrac{1}{12}(x_2x_1\tfrac{\partial}{\partial x_4} + x_2^2\tfrac{\partial}{\partial x_5} + x_3x_1\tfrac{\partial}{\partial x_6})\\
&= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 + x_1x_2)\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}x_2^2\tfrac{\partial}{\partial x_5} - \tfrac{1}{12}(6x_4 + x_1x_3)\tfrac{\partial}{\partial x_6}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}(x_1\tfrac{\partial}{\partial x_3} - x_3\tfrac{\partial}{\partial x_5} - x_5\tfrac{\partial}{\partial x_7}) + \tfrac{1}{12}(x_1^2\tfrac{\partial}{\partial x_4} + x_1x_2\tfrac{\partial}{\partial x_5} - x_3x_2\tfrac{\partial}{\partial x_7})\\
&= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_3} + \tfrac{1}{12}x_1^2\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}(6x_3 - x_1x_2)\tfrac{\partial}{\partial x_5} - \tfrac{1}{12}(6x_5 + x_2x_3)\tfrac{\partial}{\partial x_7}.
\end{aligned} \tag{51}$$

Finally, for the multiplication table (30) and setting $f^6 = f^{\pi_1}$ and $f^7 = f^{\pi_2}$, we have:

$$\begin{aligned}
f_1 &= \tfrac{\partial}{\partial x_1} + \tfrac{1}{2}(-x_2\tfrac{\partial}{\partial x_3} - x_3\tfrac{\partial}{\partial x_4} - x_4(\tfrac{\partial}{\partial x_6} + \tfrac{\partial}{\partial x_7}) - x_5(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}))\\
&+ \tfrac{1}{12}(-x_1x_2\tfrac{\partial}{\partial x_4} - x_2^2\tfrac{\partial}{\partial x_5} - x_1x_3(\tfrac{\partial}{\partial x_6} + \tfrac{\partial}{\partial x_7}) - x_2x_3(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}))\\
&= \tfrac{\partial}{\partial x_1} - \tfrac{1}{2}x_2\tfrac{\partial}{\partial x_3} - \tfrac{1}{12}(6x_3 + x_1x_2)\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}x_2^2\tfrac{\partial}{\partial x_5}\\
&- \tfrac{1}{12}(6x_4 + 6x_5 + x_1x_3 + x_2x_3)\tfrac{\partial}{\partial x_6} + \tfrac{1}{12}(-6x_4 + 6x_5 - x_1x_3 + x_2x_3)\tfrac{\partial}{\partial x_7}\\
f_2 &= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}(x_1\tfrac{\partial}{\partial x_3} - x_3\tfrac{\partial}{\partial x_5} - x_4(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}) - x_5(-\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}))\\
&+ \tfrac{1}{12}(x_1^2\tfrac{\partial}{\partial x_4} + x_1x_2\tfrac{\partial}{\partial x_5} - x_3x_1(\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}) - x_3x_2(-\tfrac{\partial}{\partial x_6} - \tfrac{\partial}{\partial x_7}))\\
&= \tfrac{\partial}{\partial x_2} + \tfrac{1}{2}x_1\tfrac{\partial}{\partial x_3} + \tfrac{1}{12}x_1^2\tfrac{\partial}{\partial x_4} - \tfrac{1}{12}(6x_3 - x_1x_2)\tfrac{\partial}{\partial x_5}\\
&- \tfrac{1}{12}(6x_4 - 6x_5 + x_1x_3 - x_2x_3)\tfrac{\partial}{\partial x_6} + \tfrac{1}{12}(6x_4 + 6x_5 + x_1x_3 + x_2x_3)\tfrac{\partial}{\partial x_7}.
\end{aligned} \tag{52}$$

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