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ON SUB-RIEMANNIAN CAUSTICS AND WAVE FRONTS FOR CONTACT DISTRIBUTIONS IN THE THREE-SPACE

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Abstract. In a number of previous papers of the first and third authors, caustics, cut-loci, spheres, and wave fronts of a system of sub-Riemannian geodesics emanating from a point q_0 were studied. It turns out that only certain special arrangements of classical Lagrangian and Legendrian singularities occur outside q_0. As a consequence of this, for instance, the generic caustic is a globally stable object outside the origin q_0.

Here we solve two remaining stability problems.

The first part of the paper shows that in fact generic caustics have moduli at the origin, and the first module that occurs has a simple geometric interpretation.

On the contrary, the second part of the paper shows a stability result at q_0. We define the "big wave front": it is the graph of the multivalued function arclength $\to$ wave-front reparametrized in a certain way. This object is a three-dimensional surface that also has a natural structure of the wave front. The projection of the singular set of this "big wave front" on the 3-dimensional space is nothing else but the caustic. We show that in fact this big wave front is Legendre-stable at the origin.

1. Introduction

Sub-Riemannian geometry studies the interrelation between a Riemannian structure on a manifold M and a completely nonintegrable distribution Δ over M.

Here we restrict ourselves to the case where Δ is a *contact structure*.

In this case, sub-Riemannian geodesics are (smooth) length extremals chosen among the (smooth) curves, whose tangent vectors belong to Δ.

As in Riemannian geometry, the following facts hold (see, e. g., [2]):

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Geodesics are locally C^0-optimal. The system of sub-Riemannian geodesics emanating from a distinguished point $q_0 \in M$ defines the exponential mapping $\mathcal{E}_{q_0} : U \subset T^*_{q_0}M \to M$, where U is an open neighborhood of 0. The first point on a geodesic that is a singular value of $\mathcal{E}_{q_0}$ (the first conjugate point) is a point at which the geodesic is no longer locally C^0-optimal. It corresponds to the conjugate time (arclength), along the geodesic.

It turns out that in contrast to Riemannian geometry, this conjugate time is not uniformly bounded from below, and as a consequence, the first caustic (the union of all these first singular values) has always the point q_0 in its closure.

As a consequence, the study of this first caustic becomes partly a local problem.

Generic singularities of Riemannian caustics are classical objects (going back to Huygens and Newton). Now they represent the most popular application of singularity theory. They are governed by the singularities of projections of smooth Lagrangian submanifolds of the cotangent bundle on the base [17].

The first example of caustic for generic sub-Riemannian exponential mapping was described only recently [1]. In the 3-dimensional case, a complete generic classification of caustics of germs of contact sub-Riemannian metrics was given [4].

In this paper, we restrict to germs at *generic points* of these generic contact 3-dimensional sub-Riemannian metrics. In that case, the germ of the corresponding caustic Σ has the following asymptotic form near the base point q_0 (Fig. 1).

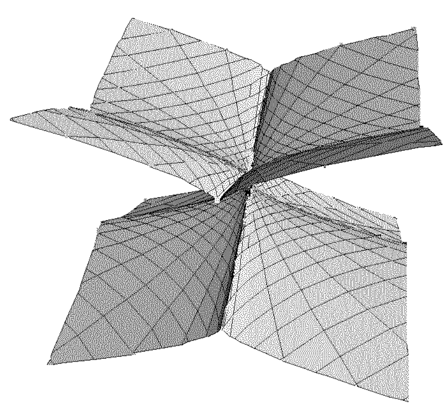

Fig. 1. Generic sub-Riemannian caustic

The plane parallel to the distribution plane Δ_0 at the origin intersects the caustic by an astroid (closed curve with four cusps). This section becomes smaller and smaller as the plane tends to the origin.

The four cuspidal edges of each half (upper and lower with respect to Δ_0) of the caustic actually form two singular curves diffeomorphic to semicubic parabolas. The tangent directions of these edges at the origin coincide with the direction of the kernel line of the exterior derivative of the distribution contact form.

Outside the origin, the singularities of the caustic (regular points and cuspidal edges) are structurally stable (in fact, they fall among classical well-known Lagrangian singularities in dimension 3).

We prove the two following facts:

(1) The germ of Σ at the origin is not structurally stable (for different contact sub-Riemannian systems, the corresponding germs are not diffeomorphic), and we describe its principal module (Theorem 2.5).

(2) There exists a singular 3-dimensional submanifold W in $M \times \mathbb{R}$ (called the big wave front of the sub-Riemannian contact system). The projection of the singular locus of W on M coincides with the first caustic. Roughly speaking, W is the graph of the sub-Riemannian distance function reparametrized in a certain way.

In fact, W is the wave front of a Legendre submanifold $\mathcal{L}_*$ in $PT^*(M \times \mathbb{R})$ formed by the exponential trajectories with the reparametrized time. Moreover, $\mathcal{L}_*$ turns out to be a proper submanifold with only one singular point.

We prove that in contract to the caustic itself, this "big wave front" W is structurally stable (Theorem 2.7).

This means that the Legendrian germ $\mathcal{L}_*$ is stable with respect to contactomorphisms of $PT^*(M \times \mathbb{R})$ preserving the projection on the base.

2. Preliminaries and results

2.1. Hamiltonian, normal form and invariants. In the contact case, the sub-Riemannian geodesics are projections of the trajectories of a Hamiltonian vector field on the cotangent bundle T^*M on the manifold M.

The corresponding Hamiltonian function is a quadratic form with respect to impulses (coordinates on the fibers) depending on the base point. This quadratic form is everywhere degenerate (while this is not the case in Riemannian geometry), and its kernel coincides with the annihilator of Δ.

The following theorem was proved in [4].

Let a germ of contact sub-Riemannian structure at q_0 in M defined by its Hamiltonian H be given.

Theorem 2.1. *There is a local coordinate system (x, y, w) near the point $q_0 \in M$, called a "normal" coordinate system, such that $q_0 = 0$ and H can be written as*

$$H_{\beta,\gamma} = \frac{1}{2}\left(p_x + \beta y(p_x y - p_y x) + p_w \frac{\gamma}{2} y\right)^2 + \\ + \frac{1}{2}\left(p_y - \beta x(p_x y - p_y x) - p_w \frac{\gamma}{2} x\right)^2, \tag{2.1}$$

where $p = (p_x, p_y, p_w)$ are conjugate coordinates (impulses) on the fibers of T^*M (the Liouville form is $p_x\, dx + p_y\, dy + p_w\, dw$).

Moreover, the functions $\beta(x, y, w)$ and $\gamma(x, y, w)$ satisfy the boundary conditions

$$\begin{gathered} \beta(0,0,w) = 0, \qquad \gamma(0,0,w) = 1, \\ \frac{\partial \gamma}{\partial x}(0,0,w) = 0, \quad \frac{\partial \gamma}{\partial y}(0,0,w) = 0. \end{gathered} \tag{2.2}$$

Normal coordinate systems are such that $\Delta(0) = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}_{|0}$. They are unique up to a constant rotation of the variables (x, y).

Remark 2.1. We choose the notation (x, y, w) instead of (x, y, z) for normal coordinates because the letter z will be used for $x + iy$ according to the fact that $\Delta(q)$ has a natural complex structure I given by $\mathrm{Vol}_g(u, v) = g(I(u), v)$.

A careful examination of the formulas (2.1) and (2.2) shows that they are invariant under the action of constant rotations at the origin.

Lemma 2.2. *A rotation R_θ of the variables (x, y) maps β and γ into $\widetilde{\beta} = \beta \circ R_{-\theta}$, and $\widetilde{\gamma} = \gamma \circ R_{-\theta}$.*

Let us work in a normal coordinate system (x, y, w), and define the following quantities: $\beta^k(x, y) = \frac{\partial^k \beta}{\partial w^k}\Big|_{w=0}$ and $\gamma^k(x, y) = \frac{\partial^k \gamma}{\partial w^k}\Big|_{w=0}$. Denote by β_l^k and γ_l^k the lth differentials of β^k and γ^k w.r.t (x, y) at $(x, y) = 0$. These quantities β_l^k and γ_l^k are homogeneous polynomials of degree l on $\Delta^*(0)$ and can be canonically identified with l-covariant symmetric tensors over $\Delta(0)$.

Lemma 2.2 has the following important consequence.

Lemma 2.3. *The tensors β_l^k and γ_l^k do not depend on the given normal coordinate system.*

This process allows us to define l-covariant tensor fields, still denoted by β_l^k and γ_l^k, which are invariants of the sub-Riemannian structure.

2.2. Decomposition of tensors. Denote by $\odot^k\Delta^*$ the tensor bundle of symmetric k-covariant tensors over Δ. Assume that $q_0 \in M$ is fixed. Then the metric subbundle Δ of TM has the structural group SO(2), and $\odot^k_{q_0}\Delta^*$ naturally becomes an SO(2)-module.

This SO(2)-module can be decomposed into real isotypic (that are in fact irreducible) components, leading to corresponding decompositions of the bundles $\odot^k\Delta^*$:

$$\odot^k\Delta^* = \bigoplus_{j\in\mathbb{N}} (\odot^k\Delta^*)^j, \tag{2.3}$$

where the irreducible component $(\odot^k_{q_0}\Delta^*)^j$ corresponds to the character $e^{ji\varphi}$ of SO(2).

Then, according to these decompositions, we have

$$\begin{cases} \beta^k_l(q_0) = \sum\limits_{r\in\mathbb{N}}\beta^k_{l,r}(q_0), \\ \gamma^k_l(q_0) = \sum\limits_{r\in\mathbb{N}}\gamma^k_{l,r}(q_0). \end{cases} \tag{2.4}$$

The following facts hold:

- in (2.4), all the terms are zero for $r > l$;
- in (2.3), the dimension corresponding to the jth component is always 2, except for $j = 0$, where it is 1 (the zero character);
- in (2.4), the sum should be taken over odd terms r if l is odd, and over even terms r if l is even;
- decomposition (2.4) is just the (finite) Fourier series of β^k_l (resp. γ^k_l) as an homogeneous polynomial restricted to the unit circle. The term corresponding to $r = l$ is the highest harmonic.

The most important components in this decomposition are $\gamma^0_{2,2}$ and $\gamma^0_{3,3}$.:

$$\begin{aligned} \gamma^0_2 &= \gamma^0_{2,0} + \gamma^0_{2,2}, \\ \gamma^0_3 &= \gamma^0_{3,1} + \gamma^0_{3,3}. \end{aligned} \tag{2.5}$$

γ^0_2 is a quadratic and γ^0_3 is a cubic. Let us explicit decomposition (2.5). Here $\gamma^0_2 = Q(dx, dy)$ is a quadratic form in dx, dy, $\gamma^0_{2,0}$ is the trace of this form with respect to the metric g on Δ, and $\gamma^0_{2,2} = 0$ iff the discriminant of the quadratic form Q vanishes (Q is "umbilic"). The component γ^0_3 is a cubic form in dx, dy: $\gamma^0_{3,1} = L(dx, dy)(dx^2 + dy^2)$, where L is linear, $L = \Re(a.dz)$, $\gamma^0_{3,3} = \Re(b.dz^3)$, a and b are complex numbers, and $dz = dx + i\,dy$.

Remark 2.2. For a generic contact sub-Riemannian metric, the tensor field $\gamma^0_{2,2}$ vanishes on a smooth curve (possibly empty) in M (i. e., Q is umbilic on this curve). The generic points studied in this paper form the complement of this curve in M.

Assumption. From now on, $\gamma_{2,2}^0(q_0) \neq 0$, and we chose the normal coordinates for $\gamma_{2,2}^0(q_0) = A(dx^2 - dy^2)$, where A is real > 0. Then, up to a rotation by $k\pi$ of the variables (x, y), the normal coordinates at q_0 are uniquely determined. In these coordinates, $\gamma_{3,3}^0(q_0) = \Re(b.dz^3)$ and $b = Be^{i\varphi_0}$.

In fact, *the main invariant in this study will just be* $\tan\varphi_0$. If a rotation of π is applied to (x, y), this invariant is not changed.

Remark 2.3. When β and $\gamma - 1$ vanish identically, the metric is the so-called "Heisenberg sub-Riemannian metric". If $\mathbb{R}^3$ is equipped with the structure of the Heisenberg group $\mathcal{H}$, then this metric is a (unique up to conjugation) left-invariant sub-Riemannian structure over $\mathcal{H}$. Everything in the following will be computed as a perturbation of this case.

From now on, the considerations are local and M is an open neighbourhood of $q_0 = 0$ in $\mathbb{R}^3$.

2.3. Exponential mapping, conjugate time and first caustic. Consider the intersection C_0 of the level hypersurface $H^{-1}\left(\dfrac{1}{2}\right)$ of the Hamiltonian function H with the fiber $T_0^*\mathbb{R}^3$ over the origin. If (x, y, w) are normal coordinates and (p_x, p_y, p_w) are the dual coordinates, this intersection is the standard cylinder $C_0 = \left\{(p_x, p_y, p_w)\middle| p_x^2 + p_y^2 = 1\right\}$ in the p-space over the origin.

Denote by L the Lagrangian submanifold swept by all the trajectories of the Hamiltonian vector field emanating (at time zero) from the points of C_0. Coordinates on C_0 and values s of time (arclength) along each trajectory parametrize L.

The restriction of the projection $(p, q) \mapsto q$ to L is the *exponential mapping* $\mathcal{E}_0 : C_0 \times \mathbb{R} \to \mathbb{R}^3$. Its singular values form the caustic. Projections of the first critical points on each trajectory form the *first caustic* Σ.

It is easy to integrate explicitly the Hamiltonian system in the Heisenberg case: the computation of the Hamiltonian flow is equivalent to the solution of a linear differential equation.

In this case, the caustic coincides with the w-axis as is well known (see [7]).

In our previous papers (see, e. g., [4]), the explicit computation of the exponential mapping $\mathcal{E}_0$ was done in normal coordinates (x, y, w) and dual coordinates (p_x, p_y, p_w) in C_0.

We have $T_0^*\mathbb{R}^3 = \Delta^*(0) \oplus \Delta^0(0)$, where $\Delta^*(0)$ is the dual of $\Delta(0)$ and $\Delta^0(0)$ is the annihilator of $\Delta(0)$. If g denotes the metric on Δ, the metric induced by g on Δ^* is just $2H$ (twice the Hamiltonian restricted to Δ^*). In the dual coordinates, this metric is $p_x^2 + p_y^2$, and it defines an angular coordinate φ in $\Delta^*(0)$. φ is the angle variable in the chosen dual normal coordinates (p_x, p_y). p_w, being the coordinate dual to w (coordinate on $\Delta^0(0)$) $\left(\varphi, \rho = \dfrac{1}{p_w}\right)$, define coordinates on a neighborhood of infinity in C_0.

Making the time reparametrization $s = \rho t$ (s is the arclength and t is the "new time") it turns out that the following assertion holds.

Lemma 2.4 (see [4]**).** *The exponential mapping $\mathcal{E}_0$ is a smooth mapping of the variables ρ, t, and φ even at $\rho = 0$.*

This lemma is a key point. It allows us to compute the exponential mapping as Taylor series in ρ at $\rho = 0$ (see [4]). This Taylor series has the following form:

$$\mathcal{E}_0(\rho, t, \varphi) = \rho\mathcal{E}^{-1}(t, \varphi) + \rho^2\mathcal{E}^0(t, \varphi) + \rho^3\mathcal{E}^1(t, \varphi) + ...,$$

and $\mathcal{E}_\mathcal{H} = \rho\mathcal{E}^{-1}(t, \varphi) + \rho^2\mathcal{E}^0(t, \varphi)$ is just the exponential mapping associated with the Heisenberg subriemannian metric:

$$\begin{aligned} x_\mathcal{H} &= 2\rho\cos\left(\varphi - \frac{t}{2}\right)\sin\frac{t}{2}, \\ y_\mathcal{H} &= 2\rho\sin\left(\varphi - \frac{t}{2}\right)\sin\frac{t}{2}, \\ w_\mathcal{H} &= \frac{\rho^2}{2}(t - \sin t). \end{aligned} \tag{2.6}$$

The first conjugate time in the Heisenberg case appears for $t = \pm 2\pi$ (a direct computation). As we said before, the first caustic in this case is the w-axis (the upper part corresponds to $\rho > 0, t = 2\pi$, the lower part to $\rho > 0, t = -2\pi$).

The caustic in the general case will also fall into two pieces corresponding to $t \simeq 2\pi$ and $t \simeq -2\pi$.

Remark 2.4 (*on orientation*). The contact structure defines an orientation on $\mathbb{R}^3$, but we have no canonical orientation on Δ. The choice of an orientation on Δ is equivalent to that of the direction $w > 0$ transversal to Δ. The change of this orientation exchanges the two parts of the caustic. In the same way, the invariants $\gamma^k_{l,r}$ defined above are invariant with respect to the orientation if k is even, but are covariant if k is odd. In this study, as we said, $k = 0$ only occurs. We refer to [4] to see how the change of orientation can affect the caustic.

From now on, we fix an orientation on Δ and study only the upper part of the caustic corresponding to $w > 0$ and $t \simeq 2\pi$, $s \simeq 2\pi\rho = 2\pi/p_w$.

It can be shown that in an appropriate neighborhood of the first singular locus S (in the source space of $\mathcal{E}_0$) outside the origin, the mapping η,

$$\eta : (\rho, \varphi, t) \longmapsto (h, \varphi, \theta),$$

where $h = \sqrt{\dfrac{w}{\pi}}$ and $\theta = \dfrac{s}{h}$, is a smooth change of coordinates.

In these coordinates, the exponential mapping takes the suspended form

$$\mathcal{E}_0(h, \varphi, \theta) = h\mathcal{E}^{-1}(\varphi, \theta, h) + \sum_{i=1}^{N} h^{i+2}\mathcal{E}^i(\varphi, \theta) + O(h^{N+3}), \qquad (2.7)$$

where

$$\begin{aligned} \mathcal{E}^{-1} &= (\mathcal{E}_x^{-1}(\varphi, \theta), \mathcal{E}_y^{-1}(\varphi, \theta), h), \\ \mathcal{E}^i &= (\mathcal{E}_x^i, \mathcal{E}_y^i, 0), \ i = 1, ..., N. \end{aligned}$$

and $O(h^{N+3})$ is a smooth mapping whose components belong to the ideal generated by h^{N+3}.

Explicit formulas for $N = 2$ in (2.7) were computed in our previous papers (see, e. g., [8], Lemma 4.9, p. 387). This approximation only $N = 2$ is sufficient for the considerations in this paper.

From these expressions, one can easily compute the first caustic Σ:

$$\begin{array}{rcl} (\mathbb{R} \times S^1) & \longrightarrow & \mathbb{R}^3, \\ (h, \varphi) & \longmapsto & (x(h, \varphi) + \widetilde{x}(h, \varphi), y(h, \varphi) + \widetilde{y}(h, \varphi), w(h)), \end{array}$$

where

$$\begin{aligned} x(h, \varphi) &= -2h^3 A(3\cos\varphi + \cos 3\varphi) - \frac{15}{2}h^4 B(2\sin(\varphi_0 + 2\varphi) + \\ &\quad + \sin(\varphi_0 + 4\varphi)) + 5C_1 h^4, \\ y(h, \varphi) &= 2h^3 A(3\sin\varphi - \sin 3\varphi) - \frac{15}{2}h^4 B(2\cos(\varphi_0 + 2\varphi) - \\ &\quad - \cos(\varphi_0 + 4\varphi)) + 5C_2 h^4, \\ w(h) &= h^2. \end{aligned} \qquad (2.8)$$

A, B, φ_0 are defined above, and $\widetilde{x}$, $\widetilde{y}$ are smooth functions in $h\cos\varphi$, $h\sin\varphi$, of order $O(h^5)$.

Remark. This latter fact that $\widetilde{x}$ and $\widetilde{y}$ are smooth functions in $h\cos\varphi$, $h\sin\varphi$ is a consequence of Lemma 2.4 above.

2.4. The extended wave front and the big wave front. We define the extended exponential mapping $\widetilde{\varepsilon}$:

$$\begin{aligned} \widetilde{\varepsilon}: C_+ \times \mathbb{R} &\to \mathbb{R}^4 = \{(x,y,w,s)\} \\ (p_w, \varphi, s) &\mapsto (\varepsilon(p_w,\varphi,s), s), \end{aligned} \tag{2.9}$$

and denote by V its image.

Here C_+ is the upper half part of the cylinder $C_0 = H^{-1}\left(\frac{1}{2}\right) \cap T_0^*\mathbb{R}^3$ and s is the time (or arclength) along a geodesic.

This 3-dimensional submanifold V is the graph of the multivalued time (or arclength) function. It is the projection on $\mathbb{R}^4$ of the Legendrian submanifold $\mathcal{L} \subset PT^*\mathbb{R}^4$, formed by the projectivized (with respect to the fibers) trajectories of the Hamiltonian system in $T^*(\mathbb{R}^3 \times \mathbb{R})$ emanating from $C_+ \times \{0\}$. Its sections by the hyperplanes $s = \text{const}$ are the momentary sub-Riemannian wave fronts (equidistant surfaces). We call it the *extended wave front*.

The image of the projection of the singular points of V (we forget the time value) on $\mathbb{R}^3$ form the complete caustic of the system. It consists of an infinite number of components, and *the closure of each of them contains the origin*. The first component of Σ was described above (Fig. 1).

The components fall into two series, and all components of the first series are similar to the first one. The second series is different and is already stable outside the origin, at the level of the Heisenberg sub-Riemannian metric.

At any time s, the wave front meets all of these components (for small enough values of $\rho = \dfrac{1}{p_w}$). We consider the first series only.

In the Heisenberg approximation, the conjugate time corresponding to the kth component in this series is $2k\pi\rho$. There is no hope to have diffeomorphism structural stability of the (instantaneous) wave fronts or of the extended wave front V with respect to generic perturbations of the system. Hence it seems natural to blow up the manifold V with respect to the time axis to distinguish the limits (as $p_w \to +\infty$) of V-singular points corresponding to different components of the caustic.

To do this, we consider the diffeomorphism of the open upper half-space $\mathbb{R}^4_+$ (defined by the inequality $w > 0$), $\Xi : (x,y,w,s) \mapsto (x,y,w,\theta)$, where θ has been defined above $\left(\theta = \dfrac{s}{h}\right)$. The image $W = \Xi(V)$ is the graph of the reparametrized time function. We call it the *big wave front*.

Obviously $W \cap \mathbb{R}^4_+$ is diffeomorphic to $V \cap \mathbb{R}^4_+$.

2.5. Statement of the results.

2.5.1. *The caustics under the action of diffeomorphisms.*

Theorem 2.5 (module for the caustics). *If the germs at the origin of two caustics Σ determined by two pairs of functions (β, γ) are diffeomorphic, then the corresponding values of $\pm\varphi_0 + \dfrac{k\pi}{2}$ should coincide provided that $A \neq 0$ and $B \neq 0$.*

Hence this expression provides the main (of lowest degree) module of the caustics under the action of diffeomorphisms.

2.5.2. *The stability of the big wave fronts.*

Proposition 2.6. *Let W_1 be the closure of $\Xi(V \cap \mathbb{R}^4_+)$ near the point $(0,0,0,2\pi)$. The projections of the singular points of W_1 to $\mathbb{R}^3$ form the first caustic of the system.*

Theorem 2.7 (stability of big wave fronts). *The germs $W_1(0,0,0,2\pi)$ are diffeomorphic for all generic sub-Riemannian systems.*

Remark 2.5. Of course, the same result holds for equivalent blowing-ups $\theta' = \dfrac{s}{h} f$, where f is any regular (nonzero) germ of function in x, y, w.

Remark 2.6. This implies that the corresponding Legendrian germs $\mathcal{L}_* \subset PT^*\mathbb{R}^4$ are equivalent with respect to the contactomorphisms preserving the fibration $PT^*\mathbb{R}^4 \to \mathbb{R}^4$.

3. Proof of Theorem 2.5

The proof of the theorem follows from the elementary geometry of $\Sigma^{1,1}$, the locus of singular points of the caustic.

Lemma 3.1. *If $A \neq 0$, then $\Sigma^{1,1}$ is diffeomorphic to the union of two semicubic parabolas Γ_x and Γ_y belonging to coordinate planes and defined by the equations*

$$\Gamma_x = \{(x,y,w) \mid x = 0, y^2 = w^3\},$$
$$\Gamma_y = \{(x,y,w) \mid y = 0, x^2 = w^3\}.$$

Proof. See Remark 4.2. $\square$

Hence the set $\Sigma^{1,1}$ (being a set of cuspidal edges) of a generic caustic has no module. However, these cuspidal edges are equipped with a *framing*: at each point of a cuspidal edge, there is a well-defined tangent plane to the caustic. We will prove that the orbits under diffeomorphisms of these framed pairs of cuspidal edges are distinguished by the above-mentioned invariant $\tan\varphi_0$.

Consider a germ of a smooth mapping $f : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ with singular point at the origin such that in some (and, therefore, in any) local coordinates q in $\mathbb{R}^3$, the mapping has the form $f : \sigma \mapsto a\sigma^2 + b\sigma^3 + \ldots$, where the dots mean a mapping with zero 3-jet at the origin and the vectors a and b are linearly *independent* (in particular $a \neq 0$).

The image of such a germ is called a *cusp*. Any cusp Γ is actually a germ of a plane curve. There exists a smooth surface S containing Γ. In the tangent space at the origin, there is a well-defined tangent flag to the cusp Γ. It consists of a plane T_Γ tangent to any ambient surface S that contains Γ and of a semiline l_Γ that is a tangent cone (limit of secants) to Γ at the origin ($l_\Gamma \subset T_\Gamma$).

Lemma 3.2. *The union of two cusps Γ_1 and Γ_2 with common singular point, common tangent direction l, and pairwise transversal tangent planes T_{Γ_1} and T_{Γ_2}, is diffeomorphic to the standard pair of cusps from Lemma* 3.1.

Proof. Take a pair of ambient smooth surfaces S_1 and S_2 such that $\Gamma_1 \subset S_1$ and $\Gamma_2 \subset S_2$. Using an appropriate germ of diffeomorphism, one can rectify these surfaces, mapping them into coordinate planes $\widetilde{S_1} = \{x = 0\}$ and $\widetilde{S_2} = \{y = 0\}$. The positive direction of the w-axis becomes a tangent direction to the images $\widetilde{\Gamma_1}$ and $\widetilde{\Gamma_2}$ of the initial cusps. The union of a plane cusp and a smooth curve passing through the cusp singular point at its tangent direction is diffeomorphic to a standard collection (say to the one defined by the equation $y(y^2 - w^3) = 0$ in the y, w-plane). This union is the zero level of a germ of a function on the plane with an isolated *simple* singularity of type E_7 (see [6]).

Therefore, there exists a germ $\mathcal{D}_1$ of the diffeomorphism of the coordinate plane (y, w), which maps $\widetilde{\Gamma_1}$ to the cusp $y^2 - w^3 = 0$ and preserves the w-axis. Hence it has the following form $\mathcal{D}_1 : (y, w) \mapsto (yY(y, w), W_1(y, w))$ with smooth functions Y and W_1.

Moreover, the zero-level set $y(y^2 - w^3) = 0$ of the normalized E_7 singularity is invariant under the action of phase flows of the Euler vector field $3y\dfrac{\partial}{\partial y} + 2w\dfrac{\partial}{\partial w}$ multiplied by an arbitrary function of y and w. An appropriate choice of such a factor as a function in w only allows us to find a diffeomorphism $\mathcal{D}_1$ with trivial action on the w-axis (that is $W_1(0, w) = w$).

Similarly, there exists a germ $\mathcal{D}_2$ of the diffeomorphism of the coordinate plane (x, w), rectifying the pair $\widetilde{\Gamma_2}$ and having the form $\mathcal{D}_2 : (x, w) \mapsto (xX(x, w), W_2(x, w))$ with $W_2(0, w) = w$.

Now, obviously, the germ

$$\mathcal{D} : (x, y, w) \mapsto (xX(x, w), yY(y, w), W_1(y, w) + W_2(x, w) - w),$$

is a germ of a diffeomorphism of $(\mathbb{R}^3, 0)$. It preserves the coordinate planes $x = 0$ and $y = 0$. Its restrictions to these planes coincide with the actions of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Hence it normalizes both cusps. The lemma is proved. □

Remark 3.1. The first jets at the origin of a diffeomorphisms that preserve the normalized variety $\Sigma^{1,1}$ (from Lemma 3.1) form a one-dimensional subgroup G_1 of $\mathrm{Gl}(3, \mathbb{R})$. G_1 contains only linear transformations of the form $(x, y, w) \mapsto \big(kx, ky, k^{\frac{2}{3}} w\big)$.

Actually, a linear transformation preserving the coordinate planes $x = 0$ and $y = 0$ is of the form $(x, y, w) \mapsto (ax, by, cw + dx + ey)$. The diffeomorphisms with such a linear part preserve $\Sigma^{1,1}$ if $a = m^3$, $b = m^3$, and $c = m^2$ for a certain $m \in \mathbb{R}$ and $d = e = 0$.

A *framing of a cusp* $f : \mathbb{R} \to \mathbb{R}^3$ sets in correspondence the plane $U_\sigma \subset T_{f(\sigma)}\mathbb{R}^3$, which contains the tangent direction to the cusp and depends smoothly on σ, to a point $\sigma \in \mathbb{R}$. In other words a*framing* is a smooth mapping $\nu : \mathbb{R} \to PT^*\mathbb{R}^3$ into the projective cotangent bundle of $\mathbb{R}^3$ (which is the space of all planes in the tangent spaces to $\mathbb{R}^3$) such that $\pi \circ \nu = f$ and $\left\langle \overline{\nu}, \dfrac{df}{d\sigma} \right\rangle = 0$. Here π is the natural projection $\pi : PT^*\mathbb{R}^3 \to \mathbb{R}^3$, $\overline{\nu}$ is any cotangent vector, the projectivization of which is ν, and $\langle , \rangle$ is the natural pairing between cotangent and tangent vectors.

Since the projection of the mapping ν on the base has a singular (cusp) point at the origin, the velocity $\dfrac{d\nu}{d\sigma}$ at this point is a vertical tangent vector of $T(PT^*\mathbb{R}^3)$: it is tangent to the fiber $PT_0^*\mathbb{R}^3$.

A framing $\nu(\sigma)$ of a cusp Γ is called a *tangent framing* if $\nu(0)$ is the plane T_Γ tangent to the cusp at the singular point.

Proposition 3.3. *The velocity* $\dfrac{d\nu(\sigma)}{d\sigma}$ *at the singular point of a tangent framing is a tangent vector (in the tangent plane* T_Γ*) to the projective line* $P_1 \subset P_0T^*\mathbb{R}^3$ *consisting of all planes that contain the tangent direction* l_Γ *to the cusp at the origin.*

Proof. Since the statement is coordinate-free, it is sufficient to prove it using appropriate coordinates. Let the cusp be given by the parametrization $x = k\sigma^4 + ...$, $y = \sigma^3 + ...$, $w = \sigma^2 + ...$ The affine chart consisting of the planes transversal to the x-axis in $PT_0^*\mathbb{R}^3$ can be identified with the space of forms $dx - (a\,dy + b\,dw) = 0$. The tangent plane T_Γ at the origin is the form $dx = 0$. Therefore, the coordinates (a, b, x, y, w) parametrize $PT^*\mathbb{R}^3$ near T_Γ.

Assume that the framing is given by a nonzero vector field of the form $u(\sigma)\dfrac{\partial}{\partial x} + v(\sigma)\dfrac{\partial}{\partial y}$ (that is by the direction of the intersection of the framing with a plane parallel to the $w = 0$ plane) and by the direction:

$$(4k\sigma^2 + ...)\frac{\partial}{\partial x} + (3\sigma + ...)\frac{\partial}{\partial y} + (2 + ...)\frac{\partial}{\partial w},$$

tangent to the cusp.

Then the coefficients $a(\sigma)$ and $b(\sigma)$ of the corresponding form satisfy the linear system

$$\begin{cases} u(\sigma) - av(\sigma) = 0, \\ (4k\sigma^2 + ...) - a(3\sigma + ...) - b(2 + ...) = 0. \end{cases}$$

Since the framing is tangent, $u(\sigma)$ vanishes at the origin and $v(\sigma)$ does not vanish. We find now that $a = \dfrac{u}{v}$, $b = 2k\sigma^2 + ... - \left(\dfrac{3}{2}\sigma + ...\right)\dfrac{u}{v}$. Thus, the first jet of $b(\sigma)$ vanishes at the origin. Therefore, in our coordinate system (a, b, x, y, w), the derivative $\dfrac{d\nu}{d\sigma}\Big|_{\sigma=0} = \dfrac{da}{d\sigma}\Big|_{\sigma=0}\dfrac{\partial}{\partial a} + \dfrac{db}{d\sigma}\Big|_0\dfrac{\partial}{\partial b} + \dfrac{dx}{d\sigma}\Big|_0\dfrac{\partial}{\partial x} + \dfrac{dy}{d\sigma}\Big|_0\dfrac{\partial}{\partial y} + \dfrac{dw}{d\sigma}\Big|_0\dfrac{\partial}{\partial w}$ takes the form $\dfrac{d\nu}{d\sigma}\Big|_{\sigma=0} = \dfrac{da}{d\sigma}\Big|_0\dfrac{\partial}{\partial a}$. Hence it represents the tangent vector to the coordinate line $b = 0$ in the affine chart. Since this line represents the projective line P_1, the proposition is proved. $\square$

Denote by Λ a pair of cusps Γ_1 and Γ_2 from Lemma 3.2 and equip each of them with tangent framings ν_1 and ν_2.

Consider the projective line $P_1(\Lambda) \subset PT_0^*\mathbb{R}^3$ of the planes containing the tangent direction $l(\Lambda)$ of the cusps at the singular point.

Proposition 3.4. *The assignment of a Riemannian metric ρ_0 on $P_1(\Sigma^{1,1})$ uniquely determines a Riemannian metric ρ_Λ on $P_1(\Lambda)$ compatible with the action of diffeomorphisms on the framed pairs of cusps: if a diffeomorphism $\mathcal{D}$ maps Λ_1 into Λ_2 then its adjoint projectivized transformation at the origin maps ρ_{Λ_1} into ρ_{Λ_2}.*

Proof. Choose an arbitrary metric ρ_0 on $P_1(\Sigma^{1,1})$ (here $\Sigma^{1,1}$is the standard pair of cusps of Lemma 3.1). According to Remark 3.1, the diffeomorphisms which preserve $\Sigma^{1,1}$ act trivially on $P_1(\Sigma^{1,1})$; thus, there is a unique projective transformation $P_r : P_1(\Lambda) \to P_1(\Sigma^{1,1})$ adjoint to any diffeomorphism mapping Λ into $\Sigma^{1,1}$. Transporting ρ_0 by P_r^{-1}, one obtains the required metric. $\square$

Two parametrizations $f_1(\sigma)$ and $f_2(\sigma)$ of a pair of cusps Λ are called *congruent* if they determine equal tangent vectors $\dfrac{d^2 f_1}{d\sigma^2} = \dfrac{d^2 f_2}{d\sigma^2}$ at a common singular point.

Lemma 3.5. *Fix a system of metrics ρ_Λ as above. The ratio of lengths of the velocities $\left.\dfrac{d\nu_1}{d\sigma}\right|_0$ and $\left.\dfrac{d\nu_2}{d\sigma}\right|_0$ (the tangent vectors to $P_1(\Lambda)$ at two points $T_0\Gamma_1$ and $T_0\Gamma_2$) computed for congruent parametrizations is an invariant of the action of diffeomorphisms on framed pairs of cusps.*

Proof. This follows immediately from the intrinsic (coordinate free) definitions of all the objects and from the fact that the choice of another congruent parametrization produces a simultaneous scaling of the framing velocities $\dfrac{d\nu_i}{d\sigma}$. $\square$

To prove Theorem 2.5, first observe that we have a special framing for $\Sigma^{1,1}$ defined by Σ itself: Σ has tangent planes at points of $\Sigma^{1,1}$. It remains only to compute the invariant ratio for this special framing. For this purpose, we use parametrization (2.8) of the first caustic.

Assume that $A \neq 0$ and $B \neq 0$.

First, we note that a diffeomorphism of the form $(x, y, w) \mapsto (x+aw^2, y+aw^2, w)$ eliminates the terms with coefficients c_1 and c_2 of the normal form (2.8). A scaling $(x, y, w) \mapsto (kx, ky, w)$ normalizes the coefficient A for an appropriate constant k. Finally, a scaling of the form $(x, y, w, h) \mapsto (xl^3, yl^3, wl^2, hl)$ with an appropriate constant l normalizes B. All these transformations do not change the tangent lines and planes to the singular locus $\Sigma^{1,1}$.

Hence, using the complex variable $z = x + iy$, the transformed caustic obtains the following asymptotics (up to the order 4 in h):

$$z = h^3(3e^{-i\varphi} + e^{3i\varphi}) + ih^4(2e^{-(2\varphi+\varphi_0)i} - e^{(4\varphi+\varphi_0)i}),$$
$$w = h^2.$$

The intersection of the cuspidal edges with the plane $w = h^2$ is determined by the equation

$$\frac{dz}{d\varphi} = 3h^3 i\big(e^{3i\varphi} - e^{-i\varphi}\big) + 4h^4\big(e^{(4\varphi+\varphi_0)i} + \mathrm{e}^{-(2\varphi+\varphi_0)i}\big) = 0.$$

This expression provides the following four approximate (up to the first order in h) cusp solutions:

$$\begin{aligned}\varphi_1 &= \frac{2}{3}\cos(\varphi_0)h,\\ \varphi_2 &= \frac{\pi}{2} - \frac{2}{3}\sin(\varphi_0)h,\\ \varphi_3 &= \pi - \frac{2}{3}\cos(\varphi_0)h,\\ \varphi_4 &= \frac{3\pi}{2} + \frac{2}{3}\sin(\varphi_0)h.\end{aligned}$$

The first and third values (as well as second and fourth) correspond to the different halfs of the same cusp.

The direction fields $\overline{\nu}$ (scaled in order to be nonzero at the origin) corresponding to the intersection of the framing with the planes $w = \text{const}$ are given by the directions of

$$\frac{1}{h^3}\frac{d^2z}{d\varphi^2} = -3\big(3e^{3i\varphi} + e^{-i\varphi}\big) + i8h\big(2e^{(4\varphi+\varphi_0)\mathrm{i}} - e^{-(2\varphi+\varphi_0)i}\big),$$

counted at the critical points φ_i $(i = 1, ..., 4)$.

Taking into account the affine terms in h only, we find that

$$\begin{aligned}\nu(\varphi_1) &= -\left((1+2h\sin\varphi_0)\frac{\partial}{\partial x} + \frac{2}{3}h\cos\varphi_0\frac{\partial}{\partial y}\right),\\ \nu(\varphi_2) &= \frac{2}{3}h\sin\varphi_0\frac{\partial}{\partial x} + (1+2h\cos\varphi_0)\frac{\partial}{\partial y},\\ \nu(\varphi_3) &= \left((1-2h\sin\varphi_0)\frac{\partial}{\partial x} - \frac{2}{3}h\cos\varphi_0\frac{\partial}{\partial y}\right),\\ \nu(\varphi_4) &= \frac{2}{3}h\sin\varphi_0\frac{\partial}{\partial x} - (1-2h\cos\varphi_0)\frac{\partial}{\partial y}.\end{aligned}$$

They are the directions of the *tangent framings* (tending to coordinate planes as $h \to 0$).

Finally, we find the derivatives $\left.\dfrac{d\nu_1}{dh}\right|_0 = \dfrac{2}{3}\cos\varphi_0$ for the cusp corresponding to φ_1 and φ_3, and $\left.\dfrac{d\nu_2}{dh}\right|_0 = \dfrac{2}{3}\sin\varphi_0$ for the other.

Therefore the invariant ratio is $|\tan(\varphi_0)|$. Different values of it correspond to nondiffeomorphic caustics.

Here, for ρ_0, we have used the metric defined by $\left(\dfrac{da}{1+a^2}\right)^2$ in the affine charts of Proposition 3.3.

Remark 3.2.

(1) Actually, the number of moduli is infinite. The tangent space to the orbit under the diffeomorphisms of a *generic* caustic has infinite codimension in the tangent space to the space of caustics.
(2) According to [18], the duals to the tangent spaces of caustics are the projectivized kernels of the corresponding Lagrangian projection. Thus, the main invariant has an easy interpretation in terms of the derivatives of the vertical directions in $T^*\mathbb{R}^3$ tangent to the critical locus of the exponential mapping.
(3) We have skipped the action of the finite group of square symmetries acting by permutations of cusps and their halfs. It reduces slightly the range of the orbits under the action of diffeomorphisms. Taking this group into account, the module is $\tan\varphi_0/\{\pm; 1/.\}$, that is, $|\log|\tan\varphi_0||$.

4. Proof of Theorem 2.7

4.1. Generating families. Using constructions of generating families of Lagrangian and Legendrian varieties (see [17], [19]), we reduce the problem to that of the stability of discriminants of a certain family of functions depending on parameters within a special class of deformations.

A family $F(u,q)$, $u \in N$, of functions on a manifold N with parameters $q \in \mathbb{R}^n$ is called a *generating family* for a Lagrangian variety $L \subset T^*\mathbb{R}^n$ (the symplectic structure $dp \wedge dq$ is standard) if

$$L = \left\{ (p,q) \middle| \exists u : \frac{\partial F(u,q)}{\partial u} = 0,\ p = \frac{\partial F}{\partial q} \right\}.$$

A family of functions $G(u,q)$ is called a *generating family of Legendre variety* $\mathcal{L}$ in $PT^*\mathbb{R}^n$ if

$$\mathcal{L} = \left\{ \big([p], q\big) \mid \exists u : G(u,q) = 0,\ \frac{\partial G(u,q)}{\partial u} = 0,\ [p] = \left[\frac{\partial G}{\partial q}\right] \right\},$$

where $[p]$ denotes the projectivization of the covector p in $T^*_{\cdot}\mathbb{R}^n$.

Let $u = (p_w, \varphi, s) \in C_+ \times \mathbb{R}$, $(P(u), Q(u)) \in T^*\mathbb{R}^3$ be the corresponding point of the Lagrangian manifold, $L : Q(u) = \mathcal{E}_0(p_w, \varphi, s)$ be the exponential mapping, and let $P(u)$ be the adjoint impulse.

Proposition 4.1. *The family $F(u,q) = P(u)(q - Q(u)) + s$ is a generating family of the restriction of L to an open neighborhood U of the set $\{p_w s = 2\pi,\ x = y = w = 0\}$.*

Proof. Let γ be a path on N joining the point u with a distinguished point u_0. Consider the following function $A(u,q)$ defined up to a constant:

$$A(u,q) = \int_\gamma (q - Q(u))\, dP(u) = P(q-Q) + \int_\gamma P\, dQ.$$

Since L is contained in the level $H^{-1}(\frac{1}{2})$ of the Hamiltonian H and H is homogeneous of degree 2 w.r.t. the impulses, we have $\int_\gamma PdQ = 2Hs = s$. Hence $A(u,q) = F(u,q)$.

The differential of A with respect to u vanishes on L : $d_u A = (q - Q(u))\, dP(u)$. The relation $p = \dfrac{\partial F}{\partial q}$ holds on L. In the Heisenberg case, the Jacobian $\det\Big(\dfrac{\partial P}{\partial u}\Big)$ does not vanish for any $p_w s$ sufficiently close to 2π. This completes the proof. $\square$

Consider the Hamiltonian function as a function on $T^*(\mathbb{R}^3 \times \mathbb{R})$ independent of the last variable. Any of its regular level surfaces in $T^*(\mathbb{R}^3 \times \mathbb{R})$ (equipped with coordinates p, q, s, H and the Liouville form $p\,dq - 2H\,ds$) is a contact 7-dimensional manifold. It is locally isomorphic to an affine chart E of the projectivization $PT^*(\mathbb{R}^3 \times \mathbb{R})$ determined by the value $H = \mathrm{const}$ of one of the coordinates on the fiber.

Coordinates on C_+ are $\upsilon = (p_w, \varphi)$. The points $\Big(P(\upsilon), Q(\upsilon), s, \dfrac{1}{2}\Big)$ form a Legendre submanifold $\mathcal{L}$ in E. The projection of $\mathcal{L}$ on $\mathbb{R}^3 \times \mathbb{R}$ is the *extended wave front* V of the exponential mapping.

The previous proposition implies the following assertion.

Proposition 4.2. *The family $G(\upsilon,s,q) = P(\upsilon,s)(q - Q(\upsilon,s))$ of functions of the variables υ with parameters q,s, is a Legendre generating family of $\mathcal{L}$. In particular the extended wave front V is a component of the discriminant variety*:

$$\widetilde{V} = \big\{(q,s) \mid \exists \upsilon : G(\upsilon,s,q) = 0, d_\upsilon G(\upsilon,s,q) = 0\big\}$$

of the family G.

The couples $(\Psi, \mathcal{D})$ formed by a nonzero germ of the function $\Psi(\upsilon,s,q)$, and a germ of diffeomorphism $\mathcal{D} : (\upsilon,s,q) \mapsto (\widetilde{\upsilon}(\upsilon,s,q), \widetilde{s}(s,q),\ \widetilde{q}(s,q))$ acts on the space of germs of families G. The couple $(\Psi, \mathcal{D})$ transforms G into the germ of $\Psi \cdot G(\mathcal{D}^{-1})$. This action is said to be *contact*. The transformed family $\Psi \cdot G(\mathcal{D}^{-1})$ is a generating family of a Legendrian germ, whose wave front is the image of the germ V_G under the diffeomorphism $\mathcal{D}' : (s,q) \mapsto (\widetilde{s}, \widetilde{q})$.

Moreover, the stability under diffeomorphisms of a germ of wave front is equivalent to the stability with respect to the contact action of the corresponding generating family (provided that the latter is almost everywhere Morse in the sense of [17], [20]).

Outside the fiber $PT_0^*\mathbb{R}^4$ over the origin, the variety $\mathcal{L}$ is smooth. Its generating family G is locally contact equivalent to a family of the form $\widetilde{G}(u,q)+s$. The germ of $\mathcal{L}$ at a nonzero point is defined by the equations $\widetilde{G}(u,q)=-s$ and $\dfrac{\partial \widetilde{G}}{\partial u}=0$. Thus s is a regular function on $\mathcal{L}$. Therefore, singular points of the projection of $\mathcal{L}$ on $\mathbb{R}^3\times\mathbb{R}$ coincide with singular points of the projection to $\mathbb{R}^3$ only. This proves Proposition 2.6.

We will use again extensively the time reparametrization $t=p_w s=\dfrac{s}{\rho}$ of Lemma 2.4. It is equivalent to the consideration of the "cylindrical" projection of C_+ on the plane $B_0=\{p_w=1\}\subset T_0^*\mathbb{R}^3$ along the radial rays. It maps the point $p=(\cos\varphi,\sin\varphi,p_w)$ to the point $n=(n_x,n_y,1)\in B_0$. Here $nx=\dfrac{\cos\varphi}{p_w}$, $n_y=\dfrac{\sin\varphi}{p_w}$, and $\rho=\dfrac{1}{p_w}=\sqrt{(n_x)^2+(n_y)^2}$.

4.2. The Heisenberg case. In the Heisenberg case, the time (arclength) to the first conjugate point along the trajectory equals $2\pi\rho$, and its w-coordinate equals $\pi\rho^2$.

From now on, we will consider only germs of extended wave fronts along the subset $Z=\left\{w-\dfrac{s^2}{4\pi}=0\right\}\subset\mathbb{R}^4$.

The Heisenberg case corresponds to $\beta=\gamma=0$. The generating family of $\mathcal{L}$ has the form

$$G(n_x,n_y,s,x,y,w)=\frac{1}{\rho}\left(w+x\left(n_x\frac{1+\cos t}{2}+n_y\frac{\sin t}{2}\right)+y\left(-n_x\frac{\sin t}{2}+\right.\right.$$
$$\left.\left.+\,n_y\frac{1+\cos t}{2}\right)-\frac{\rho^2}{2}(t+\sin t)\right).$$

In the new variables $u=P_x(n_x,n_y,t)\rho$ and $v=P_y(n_x,n_y,t)\rho$, this family takes the following form:

$$G(u,v,s,x,y,w)=-\frac{\cos\frac{t}{2}}{r}\left(w+ux+vy-\frac{1}{2}r^2\frac{t+\sin t}{\cos^2\frac{t}{2}}\right),$$

where $r=\sqrt{u^2+v^2}$. It will be useful to reparametrize again the time axis. Let $\tau=\dfrac{s}{r}=-\dfrac{t}{\cos\frac{t}{2}}$. This formula determines a regular function $t=t(\tau)$ near the point $\tau=2\pi$, $t=2\pi$. Let $T(\tau)=\dfrac{t(\tau)+\sin(t(\tau))}{1+\cos(t(\tau))}$. Dropping the nonzero factor, we reduce the generating family to the form

$$G = \frac{1}{r}\big(w + xu + yv - r^2 T(\tau)\big). \tag{4.1}$$

4.3. Reduction to a prenormal form. Let (u, v) be local coordinates on the plane. The germ $B(\tau, u, v)$ at the point $(2\pi, 0, 0)$ is said to be *degenerate*, if for any τ close to 2π, we have $B(\tau, 0, 0) = 0$ and $\frac{\partial B}{\partial u}\Big|_{(\tau,0,0)} = \frac{\partial B}{\partial v}\Big|_{(\tau,0,0)} = 0$. The germ is said to be *typical* if, moreover, the quadratic form $d^2_{u,v}B$ is not umbilic at the origin (not proportional to $u^2 + v^2$).

Actually, we will prove that the change of variables $(u_x, u_y) \mapsto \left(\frac{P_x}{P_w}, \frac{P_y}{P_w}\right)$, which is regular near $\tau = 2\pi$, maps the generating family into a that of a certain "prenormal form," which is obtained by using a degenerate additive perturbation B of $T(\tau)$ in (4.1).

Lemma 4.3 (prenormal form for generating families). *The germ of the extended wave front V of a sub-Riemannian system with smooth functions β and γ is diffeomorphic to the discriminant variety of one of the following germs of generating family at the origin*:

$$G(u, v, \tau, x, y, w) = \frac{1}{r}\Big(w + xu + yv - r^2\big(T(\tau) + B(\tau, u, v)\big)\Big).$$

Here $(u, v) \in (\mathbb{R}^2, 0)$, $\tau = \frac{s}{r}$, $r = \sqrt{u^2 + v^2}$, and B is degenerate.

Denote by $\mathcal{O}_q$ the local ring of germs at the origin of smooth functions of $q \in \mathbb{R}^n$.

Let $\Phi_1(q), ..., \Phi_k(q)$ be a set of smooth functions vanishing at the origin, and let $\nu = \Sigma \nu_i \frac{\partial}{\partial q_i}$ be a germ of smooth vector field such that $\nu(0) = 0$.

Denote by $\mathcal{D}^t_\nu(q)$ the germ of phase flow diffeomorphism at 0 generated by ν. Denote by $\mathcal{O}_q\{\Phi_j\}$ the ideal in $\mathcal{O}_q$ generated by Φ_j. We also identify the space $\mathcal{O}_q$ with the subspace of $\mathcal{O}_{q,q'}$ of functions, not depending on the additional variables q'.

Proposition 4.4. *If $L_\nu \Phi_i \in \mathcal{O}_q\{\Phi_j\}$, then $\Phi_i \circ \mathcal{D}^t_\nu \in \mathcal{O}_q\{\Phi_j\}$.*

Proof. Obvious. □

Let $\Phi_1, ..., \Phi_k$ satisfy the claim of Proposition 4.4. Denote by χ the set $\big\{q \mid \Phi_2 = ... = \Phi_k = 0\big\}$ and by χ_{t_0} the germ at the point $(0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ of the set $\big\{(\mathcal{D}^t_\nu(\chi), t)\big\}$.

Assume that χ contains a germ of curve passing through the origin and parametrized by a smooth mapping $\Gamma : (\mathbb{R}, 0) \to (\chi, 0)$ such that the orders of the first nonzero jet at the origin of the functions $\Phi_1 \circ \Gamma$ and $\Phi_1 \circ \mathcal{D}_\nu^{t_0} \circ \Gamma$ on the real line coincide and are finite.

Proposition 4.5. *There exist smooth germs $V_i(q,t)$ such that $\Phi_i - \Phi_1 V_i$ $(i = 2, ..., k)$ vanish on χ_{t_0}.*

Proof. The above proposition implies the existence of decompositions $\Phi_i(q) = \Sigma_{j=1}^k K_{ij}(q,t)\Phi_j \circ \mathcal{D}_\nu^{-t}(q)$ with smooth functions K_{ij}. By the definition of χ_{t_0}, at a point $(q,t) \in \chi_{t_0}$, the functions $\Phi_j \circ \mathcal{D}_\nu^{-t}$ vanish for $j = 2, \dots, k$. Restrict the first of the above decompositions to the curve Γ:

$$\Phi_1 \circ \mathcal{D}_\nu^{t_0} \circ \Gamma(\varepsilon) = K_{11}(\mathcal{D}_\nu^{t_0} \circ \Gamma(\varepsilon), t_0)\Phi_1 \circ \Gamma(\varepsilon).$$

The equality of orders w.r.t. ε of the functions $\Phi_1 \circ \mathcal{D}_\nu^{t_0} \circ \Gamma$ and $\Phi_1 \circ \Gamma$, implies $K_{11}(\mathcal{D}_\nu^{t_0} \circ \Gamma(\varepsilon), t_0) \neq 0$. Thus, $\Phi_i - \Phi_1 \dfrac{K_{i1}(q,t)}{K_{11}(q,t)}$ belongs to $\mathcal{O}_{q,t}\{\Phi_j \circ \mathcal{D}_\nu^{-t}\}$, where $j \geq 2$ and, therefore, vanishes on χ_{t_0}. $\square$

Lemma 4.6. *The collection of functions on $T^*\mathbb{R}^3$: $\Phi_1 = p_x^2 + p_y^2$; $\Phi_2 = x^2 + y^2$; $\Phi_3 = xp_x + yp_y$; $\Phi_4 = xp_y - yp_x$; $\Phi_5 = w$; $\Phi_6 = p_w - 1$, satisfies the conditions of Propositions* 4.4 *and* 4.5 *with respect to the $H_{\beta,\gamma}$-Hamiltonian vector field near the point $(p_x, p_y, p_w, x, y, w) = (0,0,1,0,0,0)$ and the reparametrized time moment $t = 2\pi$.*

Proof. The Hamiltonian function $H_{\beta,\gamma}$ actually belongs to the ideal $\mathcal{J}_\Phi = \mathcal{O}\{\Phi_1, ..., \Phi_6\}$ generated by these functions

$$H_{\beta,\gamma} = \frac{1}{2}\bigg(\Phi_1 + \frac{1}{4}P_w^2\gamma^2\Phi_2 + 2\beta\Phi_4 + \beta^2\Phi_2\Phi_4 + \beta\gamma\Phi_2\Phi_4 P_w + \frac{1}{2}\gamma P_w \Phi_4\bigg).$$

Hence the Poisson brackets of $H_{\beta,\gamma}$ with any of Φ_i belongs to this ideal.

For example,

$$[\Phi_4, H_{\beta,\gamma}] = \left(\frac{1}{2}p_w^2\Phi_2 + \frac{1}{2}\Phi_4\Phi_5 + \beta\Phi_2\Phi_4\right)\left(\frac{\partial\gamma}{\partial y}x - \frac{\partial\gamma}{\partial x}y\right) + \\ + \left(p_w\gamma\Phi_2\Phi_4 + \Phi_4^2 + 2\beta\Phi_2\Phi_4\right)\left(\frac{\partial\beta}{\partial y}x - \frac{\partial\beta}{\partial x}y\right).$$

This implies that Proposition 4.4 holds.

The Heisenberg approximation shows that near the value $\tau = 2\pi$, $\det \dfrac{\partial(P_x, P_y)}{\partial(n_x, n_y)} \neq 0$ on $\chi_{2\pi}$. Hence, on the Lagrangian submanifold L, which is exactly the set $\mathcal{D}_H^s(\chi_0)$ the curve required in Proposition 4.5 exists for this collection of functions. $\square$

Now, Proposition 4.5 implies the following proposition.

Proposition 4.6. *On the image of the Hamiltonian phase flow of the initial plane $p_w = 1$, all the functions Φ_i are divisible by $\Phi_1 = P_x^2 + P_y^2$.*

In particular, we see that the restriction of the Hamiltonian function to $\mathcal{L}$ is divisible by Φ_1:

$$H_{\beta,\gamma}|_{\mathcal{L}} = (P_x^2 + P_y^2)K, \quad K \neq 0.$$

The explicit formula of the generating family $G = \dfrac{1}{\sqrt{2H}}(wP_w + xP_x + yP_y - WP_w - XP_x - YP_y)$ is as follows:

$$G = \frac{f_1}{\sqrt{\Phi_1}}\left(w + x\frac{P_x}{P_w} + y\frac{P_y}{P_w} - f_2\Phi_1\right),$$

with some nonzero functions f_1 and f_2 of the variables P_x, P_y, t.

Near the value $t = 2\pi$, the functions $\dfrac{P_x}{P_w}$ and $\dfrac{P_y}{P_w}$ can be taken as new coordinates u and v on the fibers B of the source space of the family G.

Since $P_w \neq 0$, the function Φ_1 is equivalent to $u^2 + v^2 : \Phi_1 = g(u^2 + v^2)$, $g \neq 0$ on $\mathcal{L}$. The family G takes now the form

$$G = \frac{1}{r}\big(w + ux + vy - r^2 f(t, u, v)\big).$$

In the Heisenberg case, $f_{\mathcal{H}} = \dfrac{t + \sin t}{1 + \cos t}$ and does not depend on u and v.

The explicit forms of the Poisson brackets $[\Phi_i, H_{\beta,\gamma}]$ imply that in the general case, the functions $P_wW + P_xX + P_yY - r^2 f_{\mathcal{H}}$ and $H_{\beta,\gamma} - \dfrac{r^2}{\cos^2(t/2)}$ restricted to $\mathcal{L}$ have a zero 3-jet at the origin with respect to the initial coordinates n_x, n_y (or equivalently with respect to u, v) for any fixed value of t. Thus, $f = f_{\mathcal{H}} + \widetilde{B}(t, u, v)$ with a degenerate $\widetilde{B}$. The corresponding substitution of time $\tau = \dfrac{t\rho}{r} = t\sqrt{\dfrac{2H_{\beta,\gamma}}{r^2}} = -\dfrac{t}{\cos(t/2)}\sqrt{1 + A}$ with a certain degenerate $A(t, u, v)$ provides a solution of the form $t = t_{\mathcal{H}}(\tau) + C(\tau, u, v)$, where C is also degenerate. Therefore, in variables τ, u, and v, the family takes the required prenormal form.

4.4. End of the proof of Theorem 2.7. We complete the proof of Theorem 2.7 in two steps: reducing the prenormal generating family to the *circular* form (Lemma 4.8) and then proving its stability (Lemma 4.10).

The reduced problem turns out to be close to the geometrical problems "on vanishing flattenings" studied by V. Arnold's Paris school [22], [21].

Let $G(u, v, \theta, x, y)$ be a smooth germ at the origin of a family of functions on the Euclidean plane, $(u, v) \in (\mathbb{R}^2, 0)$, with parameters $(\theta, x, y) \in \mathbb{R}^3$.

Denote by G_w the restriction of G of the circle $S(u,v,w) = u^2+v^2-w = 0$, $w > 0$ to this plane (u,v), and by L_φ the differentiation $\dfrac{\partial}{\partial\varphi} = -u\dfrac{\partial}{\partial v} + v\dfrac{\partial}{\partial u}$ along the circle.

The set V of parameters (θ, x, y, w) such that the restriction G_w has a critical point with zero critical value, is called the *bifurcation diagram* of the pair (G, S) of germs of functions.

The union of this 3-dimensional submanifold $V = \{(\theta,x,y,w) \mid \exists u,v : G_w = 0,\ L_\varphi G_w = 0\}$, with the flat additional component $w = 0$ forms the *bifurcation diagram* of the complete intersection $\{G = 0\} \cap \{S = 0\}$, that is, the set of parameters for which this system of equations has a critical solution.

We consider:

(a) parameter depending families of germs $\mathcal{D}$ of diffeomorphisms and covering diffeomorphisms $\widetilde{\mathcal{D}}$ (via the coordinate projection π forgetting u and v):

$$\begin{array}{ccc} \mathbb{R}^2\times\mathbb{R}^4 & \xrightarrow{\mathcal{D}} & \mathbb{R}^2\times\mathbb{R}^4 \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{R}^4 & \xrightarrow{\widetilde{\mathcal{D}}} & \mathbb{R}^4 \end{array}\quad,\qquad \widetilde{\mathcal{D}}\circ\pi = \pi\circ\mathcal{D},$$

and

(b) multiplications of a pair $\begin{pmatrix} G \\ S \end{pmatrix}$ of functions by germs of 2×2 non-degenerate matrices M (with functional coefficients).

These transformations $(M, \mathcal{D})$ form a *contact group*. It acts on the space of pairs (G, S) according to the formula $\begin{pmatrix} G \\ S \end{pmatrix} \mapsto M \begin{pmatrix} G\circ\mathcal{D} \\ S\circ\mathcal{D} \end{pmatrix}$, whence the diffeomorphism $\widetilde{\mathcal{D}}$ maps the bifurcation diagram of the initial pair to that of the transformed one.

Definition 1. The family G is called *circular* if it has the following special form:

$$G = xu + yv + (u^2+v^2)F(u,v,\theta,x,y), \tag{4.2}$$

where $F(0) = 0$, $\dfrac{\partial F}{\partial u}\Big|_0 = \dfrac{\partial F}{\partial v}\Big|_0 = 0$, and $\dfrac{\partial F}{\partial \theta}\Big|_0 \neq 0$.

A circular family is called *generic* if the second differential at the origin of $F\mid_{\theta,x,y=0}$ is not umbilic.

Lemma 4.8 (suspension). *The germ of the big wave front V of a prenormal generating family G (Lemma 4.3) is diffeomorphic to the (non-trivial) component of the bifurcation diagram of a certain circular family.*

Proof. The variety V is defined by the set of equations

$$\begin{aligned} &\text{(i)} && G = 0, \\ &\text{(ii)} && \frac{\hat{\partial}}{\partial u} G = 0, \\ &\text{(iii)} && \frac{\hat{\partial}}{\partial v} G = 0. \end{aligned} \tag{4.3}$$

Here $\dfrac{\hat{\partial}}{\partial u}$ means $\dfrac{\partial}{\partial u}\Big|_{s=\text{const}} = \dfrac{\partial}{\partial u}\Big|_{\tau=\text{const}} + \dfrac{\partial \tau}{\partial u}\Big|_{s=\text{const}}.\dfrac{\partial}{\partial \tau}$ and $\dfrac{\hat{\partial}}{\partial v}$ is defined similarly.

Items (ii) and (iii) of (4.3) imply $r\dfrac{\hat{\partial}}{\partial r}G = 0$, where $r\dfrac{\hat{\partial}}{\partial r} = u\dfrac{\hat{\partial}}{\partial u} + v\dfrac{\hat{\partial}}{\partial v}$. This last equation has a simple form:

$$0 = r\frac{\hat{\partial}}{\partial r}G = \frac{1}{r}\big(-w + r^2\big(\widetilde{T}(\tau) + \widetilde{B}\big)\big), \tag{4.4}$$

where $\widetilde{B} = \widetilde{B}(u, v, \tau)$ is a smooth degenerate function and $\widetilde{T}(\tau) = \dfrac{\partial T}{\partial \tau}\tau - T$. Note that $\widetilde{T}(2\pi) = \pi$ and $\dfrac{\partial \widetilde{T}}{\partial \tau}\Big|_{2\pi} = 4\pi^3$.

Near the origin, (4.4) yields $w = \pi r^2(1 + B)$; therefore, the value of $l = \sqrt{\dfrac{w}{\pi r^2}}$ is bounded.

Substitute now the expression $s = \theta\sqrt{\dfrac{w}{\pi}}$ in Eq. (4.4). We have $\tau = \dfrac{s}{r} = \theta l$, and this equation takes the form of the set of equations

$$w = \pi l^2 r^2, \quad \pi r^2(\widetilde{T} + \widetilde{B}) = w$$

or

$$r = \sqrt{\frac{w}{\pi l^2}}, \quad l^2 = \widetilde{T} + \widetilde{B}.$$

Computations show that the equation $l^2 = \widetilde{T} + \widetilde{B}$ has a real smooth solution of the form $l = C(\theta) + \widetilde{A}(u, v, \theta, w)$, where the smooth function $\widetilde{A}$ is degenerate and $C(2\pi) = 1$.

Obviously, the mapping Sus : $(u, v) \mapsto (lu, lv)$ is a germ of diffeomorphism of the plane (for any parameter set close to the origin). As new variables we take $\overline{u} = lu$ and $\overline{v} = lv$ and set $\overline{G} = G_s \circ (\mathrm{Sus})^{-1}$.

As we have seen system (4.3) implies:

$$\frac{w}{\pi} = l^2 r^2 = l^2(u^2 + v^2) = \overline{u}^2 + \overline{v}^2. \tag{4.5}$$

In the new variables, system (4.3) is equivalent to the system $\overline{G} = 0$, $\dfrac{\partial \overline{G}}{\partial \overline{u}} = 0$, $\dfrac{\partial \overline{G}}{\partial \overline{v}} = 0$ or (outside the origin) to the system

$$\overline{G} = 0, \quad \overline{r}\frac{\partial \overline{G}}{\partial \overline{r}} = 0, \quad \frac{\partial \overline{G}}{\partial \overline{\varphi}} = 0.$$

Since the diffeomorphism Sus is close to the identity near the origin, Eq. (4.5) is independent of the system formed by the first and third equations, $\overline{G} = 0$, $\dfrac{\partial \overline{G}}{\partial \overline{\varphi}} = 0$. This implies that the germ of big wave front V (the subset formed by the points (x, y, w, s) such that $\left(x, y, w, \dfrac{s\sqrt{\pi}}{\sqrt{w}}\right)$ are close to $(0, 0, 0, 2\pi)$) is diffeomorphic to the bifurcation diagram of the pair $\{\overline{G}, w - \overline{u}^2 - \overline{v}^2\}$.

The family takes the form

$$\begin{aligned}\overline{G} &= \frac{l\sqrt{\pi}}{\sqrt{w}}\Big(w + \frac{x\overline{u}}{l} + \frac{y\overline{v}}{l} - \frac{w}{\pi l^2}(T + B)\Big) = \\ &= \frac{\sqrt{\pi}}{\sqrt{w}}\big(x\overline{u} + y\overline{v} - (\overline{u}^2 + \overline{v}^2)E(\theta, \overline{u}, \overline{v})\big),\end{aligned}$$

where E is again a sum of a degenerate function E_* and a function $E_0(\theta)$ such that $E_0(2\pi) = 0$ and $\left.\dfrac{\partial E_0}{\partial \theta}\right|_{2\pi} \neq 0$.

The multiplication by $\dfrac{1}{\sqrt{w}}$ only adds a trivial component to the bifurcation diagram.

Therefore, the pair $(\overline{G}, S)$ is contact equivalent to the circular pair

$$\{x\overline{u} + y\overline{v} + (\overline{u}^2 + \overline{v}^2)E, S\}.$$

The lemma is proved. □

Remark 4.1. Note that in (4.2) the addition to G of a function divisible by $S = w - \overline{u}^2 - \overline{v}^2$ does not change the bifurcation diagram.

Proposition 4.9. *A generic circular family corresponds to a generic sub-Riemannian system* ($\gamma^0_{2,2} \neq 0$).

Proof. It is possible to get the result by a direct computation: using asymptotic formulas for $\mathcal{E}_0$ (with $\gamma^0_{2,2} \neq 0$), one can compute the initial generating family, and then follow all the transformations described above keeping the terms of order 2 in u, v for the degenerate germ B in Lemma 4.3.

We will avoid these calculations, and base our reasoning only on asymptotics (2.8) of the caustic.

The reduction of the generating family to the circular form

$$G = xu + yv + (u^2 + v^2)(E_0(\theta) + E_*(u, v, \theta)),$$

consists only of contact equivalences, corresponding to the trivial action on the parameters. We have also preserved the simple form of the initial arclength $s = \theta\sqrt{\dfrac{w}{\pi}}$.

Therefore, the caustic of a sub-Riemannian system coincides with the set of (x, y, w) such that there exist θ, u, v satisfying the system of equations

$$G = 0, \quad \frac{\partial}{\partial\varphi}G = 0, \quad \frac{\partial^2}{\partial\varphi^2}G = 0, \quad w = u^2 + v^2.$$

Since $\left.\dfrac{dE_0}{d\theta}\right|_{2\pi} \neq 0$, the first and third equations

$$\begin{aligned} xu + yv + (u^2 + v^2)(E_0 + E_*) &= 0, \\ -yu + xv + (u^2 + v^2)\frac{\partial^2}{\partial\varphi^2}E_* &= 0, \end{aligned}$$

allow us to express θ as a function of $u, v : \theta = 2\pi + \Psi(u, v)$. Here the function Ψ is degenerate.

The substitution of $\theta = 2\pi + \Psi(u, v)$ in the second and third equations yields:

$$\begin{aligned} -vx + uy &= (u^2 + v^2)\frac{\partial}{\partial\varphi}E_*, \\ xu + yv &= (u^2 + v^2)\frac{\partial^2}{\partial\varphi^2}E_*. \end{aligned}$$

Hence

$$\begin{aligned} x &= -v\frac{\partial}{\partial\varphi}E_* + u\frac{\partial^2}{\partial\varphi^2}E_*, \\ y &= u\frac{\partial}{\partial\varphi}E_* + v\frac{\partial^2}{\partial\varphi^2}E_*. \end{aligned}$$

Assuming that E_* has only an umbilic second order term in u, v when $\theta = 2\pi$, we get

$$x = \Psi_1(u, v), \quad y = \Psi_2(u, v), \quad w = u^2 + v^2,$$

and the 3-jets at the origin of Ψ_1 and Ψ_2 vanish. Therefore, the caustic has asymptotic $x, y \sim O(w^2)$ as $w \to 0$. This contradicts the asymptotics (2.8). Hence E_* is not umbilic for a generic sub-Riemannian system. $\square$

Lemma 4.10. *All generic circular germs* (*at the origin*) *are contact equivalent. Hence their bifurcation diagrams are diffeomorphic. Moreover, there exists a diffeomorphism preserving the w-coordinate function.*

Proof. The Moser homotopy method prescribes to join an arbitrary generic family G with a standard one, for instance, with

$$G_0 = xu + yv + r^2(\pm\theta + (u^2 - v^2)), \tag{4.6}$$

by an arc G_ε, $\varepsilon \in [0, 1]$ in the space $\mathcal{G}$ of generating families: $G_\varepsilon \mid_{\varepsilon=0} = G_0$, $G_\varepsilon \mid_{\varepsilon=1} = G$, and to look for an ε-family K_ε of contact equivalences (of the required form), which maps G_ε into G_0.

Note that $\mathcal{G}$ has two connected components corresponding to signs $\pm$ in formula (4.6) for G_0. But the reversion of the "time" θ makes these two distinguished classes equivalent.

The contact equivalence K_ε consists of a pair (due to Remark 4.1) of smooth germs of functions $A \neq 0$, B of the variables ε, u, v, θ, x, y, w and of a family of diffeomorphisms $\mathcal{D}_\varepsilon : (u, v, \theta, x, y, w) \mapsto (U, V, \Xi, X, Y, w)$ the U, V components of which depend on all 7 variables, while the remaining ones depend only on parameters $\varepsilon, \theta, x, y, w$ (and the w-component is preserved). Of course, $\mathcal{D}_\varepsilon$ also preserves the system of circles $r^2 - w = 0$.

Thus, expecting to satisfy the equation $AG_\varepsilon \circ \mathcal{D}_\varepsilon + B(r^2 - w) = G_0$, we differentiate it w.r.t. ε:

$$\begin{aligned} A \circ \mathcal{D}_\varepsilon^{-1} \frac{\partial G_\varepsilon}{\partial \varepsilon} + \frac{\partial A}{\partial \varepsilon} \circ \mathcal{D}_\varepsilon^{-1} G_\varepsilon + A \circ \mathcal{D}_\varepsilon^{-1} L_\nu G_\varepsilon + \\ + \frac{\partial B}{\partial \varepsilon} \circ \mathcal{D}_\varepsilon^{-1}(r^2 - w) = 0. \end{aligned}$$

Here the phase flow of the vector field ν is the family $\mathcal{D}_\varepsilon$. Hence we try to find ν of the form

$$\nu = f_1\left(-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right) + g_\theta \frac{\partial}{\partial \theta} + g_x \frac{\partial}{\partial x} + g_y \frac{\partial}{\partial y},$$

whose components g_θ, g_x, and g_y are functions of the parameters, f_1, $f_2 = \dfrac{1}{A}\dfrac{\partial A}{\partial \varepsilon} \circ \mathcal{D}_\varepsilon^{-1}$, $f_3 = \dfrac{1}{A}\dfrac{\partial B}{\partial \varepsilon} \circ \mathcal{D}_\varepsilon^{-1}$ are functions of all 7 variables such that the homological equation

$$-\frac{\partial G_\varepsilon}{\partial \varepsilon} = L_\nu G_\varepsilon + f_2 G_\varepsilon + f_3(r^2 - w),$$

is satisfied.

We prove the solvability of this equation below (Proposition 4.11).

Moreover, decomposition (4.7) below of the germ $-\dfrac{\partial G_\varepsilon}{\partial \varepsilon}$ provides a vector field ν which vanishes at the origin for any ε and, therefore, determines a germ of flow. This completes the proof of the lemma and hence Theorem 2.7. □

Remark 4.2. Theorem 2.7 implies that the singular loci $\Sigma^{(1)}$ of generic big wave fronts are diffeomorphic. Their inverse images in the (u, v, θ, x, y, w)-space (as well as the corresponding subsets of the Legendre varieties), are diffeomorphic as well. The singular loci $\Sigma^{(1,1)}$ of $\Sigma^{(1)}$ are also diffeomorphic. They are determined by the equations $G = \dfrac{\partial}{\partial \varphi} G = \dfrac{\partial^2}{\partial \varphi^2} G = \dfrac{\partial^3}{\partial \varphi^3} G$ in terms of the circular families G.

For the standard family G_0, the inverse image of the singular locus $\Sigma^{(1,1)}$ consists of two smooth curves. Their projections to the (x, y, w)-space form two cusps on the (upper) half of the caustic. Hence this hods for arbitrary generic systems and implies Lemma 3.1.

Fix a certain value ε_0 of ε, and denote by $\mathcal{O}$ the ring of germs of smooth functions of the variables $\varepsilon, u, v, \theta, x, y, w$, at the point $(\varepsilon_0, 0, 0, 0, 0, 0, 0)$. Denote by $\mathcal{O}_*$ the ring of germs at $(\varepsilon_0, 0, 0, 0, 0)$ of functions of the parameters $\varepsilon, \theta, x, y, w$.

Also, we recall that $\dfrac{\partial G_\varepsilon}{\partial \varepsilon} \in \mathcal{O}\{r^2\}$ for any ε_0.

Proposition 4.11 (infinitesimal circular stability). *Any germ $f \in \mathcal{O}\{r^2\}$ has the decomposition*

$$f = f_1 \frac{\partial}{\partial \varphi} G_\varepsilon + f_2 G_\varepsilon + f_3(r^2 - w) + g_1 \frac{\partial}{\partial \theta} + g_2 \frac{\partial}{\partial x} + g_3 \frac{\partial}{\partial y}, \qquad (4.7)$$

where $f_1, f_2, f_3 \in \mathcal{O}$, $g_1, g_2, g_3 \in \mathcal{O}_*$, and $\dfrac{\partial}{\partial \varphi} = -v \dfrac{\partial}{\partial u} + u \dfrac{\partial}{\partial v}$.

Proof. The fact that the right-hand side terms of (4.7) does not belong to the ideal $\mathcal{O}\{r^2\}$ causes an obstruction to the immediate application of the Malgrange preparation theorem. We need certain tricks to avoid it.

Denote by $\mathcal{O}_0$ the ring of germs at the origin of functions in u, v only and by $[\varphi]$ the restriction of $\varphi \in \mathcal{O}$ to the coordinate subspace $\varepsilon = \theta = x = y = w = 0$.

An appropriate rotation of coordinates in the u, v plane (it does not affect the form of equations) allows us to write $G_\varepsilon = xu + yv + r^2(a(u^2 - v^2) + br^2 + ...)$, $a \neq 0$, and respectively, $\dfrac{\partial}{\partial\varphi} G_\varepsilon = -xv + yu - r^2(4auv + ...)$. Here and below the dots mean smooth functions Ψ the restrictions $[\Psi]$ of which belong to $\mathcal{M}_0^3$, where $\mathcal{M}_0$ is the maximal ideal of $\mathcal{O}_0$.

Observe that the functions

$$\delta_1 = uG_\varepsilon - v\frac{\partial}{\partial\varphi}G_\varepsilon = x(u^2 + v^2) + r^2 a(u^3 - u^2 v + 4uv^2) + br^2 u + ...,$$

and

$$\delta_2 = vG_\varepsilon + u\frac{\partial}{\partial\varphi}G_\varepsilon = y(u^2 + v^2) + r^2 a(u^2 v - v^3 - 4u^2 v) + br^2 v + ...,$$

are divisible by r^2 in the algebra $\mathcal{O}$.

Since modulo the ideal $\mathcal{O}\{r^2\}$, $u^2 \equiv -v^2$, $u^2 v \equiv -v^3$, $v^2 u \equiv -u^3$ in the space $\mathcal{O}_0$, modulo the ideal $\mathcal{O}_0\{r^2\}$, we obtain

$$\left[\frac{\delta_1}{r^2}\right] = a(u^3 - uv^2 + 4uv^2) + br^2 u + ... \equiv -2au^3 + ...,$$
$$\left[\frac{\delta_2}{r^2}\right] = a(u^2 v - v^3 - 4u^2 v) + br^2 v + ... \equiv 2av^3 + ...$$

Therefore, $\mathcal{M}_0^3 \subset \mathcal{I}_0 + \mathcal{M}_0^4$, where $\mathcal{I}_0 = \mathcal{O}_0\Big\{\left[\frac{\delta_1}{r^2}\right], \left[\frac{\delta_2}{r^2}\right], r^2\Big\}$. Nakayama's lemma implies now that $\mathcal{M}_0^3 \subset \mathcal{I}_0$, and consequently, the class of $1, u, v, u^2 - v^2, uv$ generate the $\mathbb{R}$-module $\mathcal{O}_0/\mathcal{I}_0$.

The Malgrange preparation theorem applied to the above module and the mapping $(\varepsilon, u, v, \theta, x, y, w) \mapsto \Big(\frac{\delta_1}{r^2}, \frac{\delta_2}{r^2}, r^2 - w, \varepsilon, \theta, x, y, w\Big)$ implies that any 5 function germs from $\mathcal{O}$ whose 2-jets of restrictions $[\ ]$ span the space of affine and nonumbilic quadratic functions form a basis of the $\mathcal{O}_*$-module $\mathcal{O}/\mathcal{I}$, where $\mathcal{I} = \mathcal{O}\Big\{\frac{\delta_1}{r^2}, \frac{\delta_2}{r^2}, r^2 - w\Big\}$. In particular, the functions

$$1, u, v, e_1 = \frac{1}{r^2}\big(G_\varepsilon - xu - yv\big) \sim a(u^2 - v^2),$$
$$e_2 = \frac{1}{r^2}\left(\frac{\partial G_\varepsilon}{\partial\varphi} yu - xv\right) \sim axy,$$

form such a basis. In other words,

$$\mathcal{O} = \mathcal{O}\left\{\frac{\delta_1}{r^2}, \frac{\delta_2}{r^2}, r^2 - w\right\} + \mathcal{O}_*\{1, u, v, e_1, e_2\}. \tag{4.8}$$

The multiplication by r^2 yields

$$\mathcal{O}\{r^2\} = \mathcal{O}\{\delta_1, \delta_2, (r^2 - w)r^2\} + \mathcal{O}_*\{r^2, r^2u, r^2v, r^2e_1, r^2e_2\}. \tag{4.9}$$

Since $\delta_1, \delta_2 \in \mathcal{J} = \mathcal{O}\Big\{G_\varepsilon, \dfrac{\partial G_\varepsilon}{\partial \varphi}, r^2 - w\Big\}$, modulo this ideal, $r^2 \equiv w$, $r^2e_1 \equiv -xu - yv$, $r^2e_2 \equiv uy - vx$. Hence, we obtain

$$\begin{aligned} &\mathcal{O}_*\{r^2, r^2u, r^2v, r^2e_1, r^2e_2\} + \mathcal{J} = \\ &= \mathcal{O}_*\{w, uw, vw, -xu - yv, uy - xv\} + \mathcal{J} \subset \mathcal{O}_*\{w, u, v\} + \mathcal{J}. \end{aligned} \tag{4.10}$$

On the other hand, the definition of a *circular family* implies $\mu_1 = \dfrac{\partial G_\varepsilon}{\partial x} = u + r^2\varphi_1$, $\mu_2 = \dfrac{\partial G_\varepsilon}{\partial y} = v + r^2\varphi_2$, $\mu_3 = \dfrac{\partial G_\varepsilon}{\partial \theta} = r^2(c + \varphi_3)$, where $c \in \mathbb{R}^*$ and $\varphi_1, \varphi_2, \varphi_3$, vanish on the line $u = v = \theta = x = y = w = 0$.

Decomposing these germs φ_i according to (4.8), modulo the ideal $\mathcal{I}$, we get

$$\varphi_i = \xi_{1i} + \xi_{2i}u + \xi_{3i}v + \xi_{4i}e_1 + \xi_{5i}e_2,$$

$i = 1, 2, 3$, where $\xi_{ji} \in \mathcal{O}_*$. Since $\dfrac{\delta_1}{r^2}, \dfrac{\delta_2}{r^2}$ and $r^2 - w$ belong to $\mathcal{M}$, the germs ξ_{1i} are forced to belong to $\mathcal{M}_*$.

According to (4.9), modulo the ideal $\mathcal{I}$, we obtain

$$\begin{aligned} \mu_1 &\equiv u + w\xi_{11} + uw\xi_{21} + vw\xi_{31} - (xu + yv)\xi_{41} + (uy - vx)\xi_{51}, \\ \mu_2 &\equiv v + w\xi_{12} + uw\xi_{22} + vw\xi_{32} - (xu + yv)\xi_{42} + (uy - vx)\xi_{52}, \\ \mu_3 &\equiv cw + w\xi_{13} + uw\xi_{23} + vw\xi_{33} - (xu + yv)\xi_{43} + (uy - vx)\xi_{53}, \end{aligned}$$

or equivalently

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \equiv M \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the entries of the matrix M are germs in $\mathcal{O}_*$.

Near the origin in the (θ, x, y, w)-space (and for an arbitrary ε), this matrix is close to the diagonal one $\mathrm{diag}(1, 1, c)$. Therefore, it is invertible.

Hence $\mathcal{O}_*\Big\{\dfrac{\partial G_\varepsilon}{\partial x}, \dfrac{\partial G_\varepsilon}{\partial y}, \dfrac{\partial G_\varepsilon}{\partial \theta}\Big\} + \mathcal{J} = \mathcal{O}_*\{u, v, w\} + \mathcal{J}$. Combining this with the relation (4.10), we obtain the required decomposition (4.7):

$$\mathcal{O}(r^2) = \mathcal{O}\Big\{G_\varepsilon, \frac{\partial}{\partial \varphi}G_\varepsilon, r^2 - w\Big\} + \mathcal{O}_*\Big\{\frac{\partial G_\varepsilon}{\partial x}, \frac{\partial G_\varepsilon}{\partial y}, \frac{\partial G_\varepsilon}{\partial \theta}\Big\}. \qquad \square$$

References

1. A. A. Agrachev, Methods of control theory in nonholonomic geometry. *Proceedings ICM*-94, *Zürich, Birkhäuser*, 1995.
2. ———, Exponential mappings for contact sub-Riemannian structures. *J. Dynam. Control Syst.* **2** (1996), No. 3, 321–358.
3. A. A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka, Sub-Riemannian sphere in Martinet flat case. (to appear in COCV S1, 377–448, Dec. 1997).
4. A. A. Agrachev, H. Chakir, and J. P. Gauthier, Sub-Riemannian metrics on $\mathbb{R}^3$. *Proc. Canad. Math. Soc.* **25** (1988), 29–77.
5. A. A. Agrachev, H. Chakir, J. P. Gauthier, and I. Kupka, Generic singularities of sub-Riemannian metrics on $\mathbb{R}^3$. *C. R. Acad. Sci., sér.* 1 **322** (1996), 377–384.
6. V. Arnold, A. Varchenko, and S. Goussein-Zadé, Singularités des applications différentiables. *Mir, Moscow*, French translation, 1986.
7. C. Carathéodory, Calculus of variations and partial differential equations of the first order. *Chelsea Publishing Company, New York*, 10016 (3rd edition), 1989.
8. H. Chakir, J. P. Gauthier, and I. Kupka, Small sub-Riemannian balls on $\mathbb{R}^3$. *J. Dynam. Control Syst.* **2** (1996), No. 3, 359–421.
9. M. Chyba, Exponential mapping, spheres, and wave fronts in sub-Riemannian geometry: the Martinet case. PHD thesis, *University of Burgandy*, 1997.
10. M. Golubitskii and V. Guillemin, Stable mappings and their singularities. *Springer Verlag, New York*, 1973.
11. S. Kobayashi and K. Nomizu, Foundations of differential geometry. *John Wiley*, 1969.
12. I. Kupka, Géométrie sous-Riemannienne. *Séminaire Bourbaki,* 48*ème année*, No. 817, 1995–96.
13. J. Mather, Stability of C^∞ mappings. I–VI, *Ann. Math.* **87**, 89–104, 1968; **89**, 254–291, 1969; *Publ. Sci. IHES* **35**, 127–156, 1969; **37**, 223–248, 1970; *Advances in Maths*, **4**, 301–335, 1970; *Lect. Notes Math.* **192**, 207–253, 1971.
14. R. Montgommery, Isholonomic problems and some applications. *Commun. Math. Physics* **128** (1990), 565–592.
15. L. Pontriaguine, V. Boltianski, R. Gamkrelidzé, and E. Mitchenko, La théorie Mathématique des Processus Optimaux. *Mir, Moscow*, French translation, 1974.
16. H. Whitney, On singularities of mappings of euclidian spaces. I–II, In: *Symposium Internacional de Topologia Algebraica, Mexico City*, 1958; *Ann. Math.*, Vol. 62, No. 3, 1955.

17. V. I. Arnold, Singularities of caustics and wavefronts. *Kluwer Acad. Math. and its Appl.* (1990), p. 259.
18. J. W. Bruce and V. M. Zakalyukin, On the geometry of caustics. *Amer. Math. Soc. Trans.* (2), **180** (1997), 1–11.
19. V. M. Zakalyukin, Generating ideals of Lagrangian varieties. *Adv. Soviet Math.* **1** (1990), 201–210.
20. ———, Reconstruction of fronts and caustics depending on parameters, versality of mappings. *J. Soviet Math.* **27** (1984), 2785–2811.
21. R. Uribe-Vargas, On the higher dimensional four-vertex theorem. *C. R. Acad. Sci. Paris*, sér. 1 **321** (1997), 1353–1358.
22. M. Garay, Vanishing flattering points and generalised Plücker formula. *C. R. Acad. Sci. Paris*, sér. 1 **323** (1996), 901–906.

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