ON SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

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We show that if Φ is an arbitrary countable set of continuous functions of n variables, then there exists a continuous, and even infinitely smooth, function $\psi(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n) \not\equiv g[\varphi(f_1(x_1), \ldots, f_n(x_n))]$ for any function φ from Φ and arbitrary continuous functions g and f_i, depending on a single variable.

1°. In k-valued logics Slupecki's criterion (see [1], for example), which gives a necessary and sufficient condition for the completeness of systems containing all functions of a single variable, is widely known.

In the generalization of this criterion to a countably valued logic (see [2]) it was found that if the function $\varphi(x_1, \ldots, x_n)$, together with all functions of a single argument, forms a complete system, then an arbitrary function $\psi(x_1, \ldots, x_n)$ can be obtained in the form of a superposition relative to φ of order not higher than the second, wherein only functions depending on a single variable are used, i.e., in the form

$$\psi(x_1, \ldots, x_n) = g_0(\varphi(g_1[\varphi(f_{11}(x_1), \ldots, f_{1n}(x_n))], \ldots, g_n[\varphi(f_{n1}(x_1), \ldots, f_{nn}(x_n))])).$$

Analogous problems for continuous functions are also of no small interest inasmuch as A. N. Kolmogorov [3] showed that an arbitrary function $\psi(x_1, \ldots, x_n)$, continuous on the unit n-dimensional cube, can be written in the form

$$\psi(x_1, \ldots, x_n) = \sum_{m=1}^{2n+1} g_m\left[\sum_{l=1}^{n} f_{ml}(x_l)\right],$$

where $g_m \in C[0, 1]$, $f_{ml} \in C[0, 1]$ for $l = 1, \ldots, n$; $m = 1, \ldots, 2n+1$, i.e., as a completely specific superposition of continuous functions of a single argument and addition (in Kolmogorov's construction the functions f_{ml} do not even depend on ψ).

2°. In this note we establish a negative result connected with the possibility of representing continuous functions in the form of superpositions of bounded order.

Let $E = [0, 1]$,* $n \geq 2$; E^n is the n-dimensional unit cube. Let $C(E^n)$ be the space of all real functions continuous on E^n. For any function φ from $C(E^n)$ let S_φ denote the set

$$\{s(x_1, \ldots, x_n) : s(x_1, \ldots, x_n) \in C(E^n) \,\&\, (\exists g, f_1, \ldots, f_n)(g \in C(E), f_i \in C(E), i = 1, \ldots, n) \,\&$$
$$\&\, s(x_1, \ldots, x_n) \equiv g[\varphi(f_1(x_1), \ldots, f_n(x_n))]\}.$$

The following theorem is valid.

<u>THEOREM.</u> There exists a nondenumerable set of functions Ψ such that $\Psi \subset C(E^n)$ and such that for any function $\varphi \in C(E^n)$ the set $\Psi \cap S_\varphi$ is at most denumerable.

*We can also take $E = (-\infty, +\infty)$.

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For the proof of this theorem we require the following two lemmas.

LEMMA 1. Let $[a_0, b_0]$ be an interval and let $\Delta^k = [a_1, b_1] \times \ldots \times [a_k, b_k]$ be a k-dimensional cube, where $a_i \in E$, $b_i \in E$, $a_i \neq b_i$ for $i = 0, 1, \ldots, k$. There exists a nondenumerable set $\{I_\alpha\}$ of compacta such that for any α the compactum I_α is denumerable, is contained in $[a_0, b_0]$, and, if $\alpha_1 \neq \alpha_2$, then $I_{\alpha_1} \times \Delta^k$ is not homeomorphic to $I_{\alpha_2} \times \Delta^k$.

Proof. Let α be an ordinal number, and let us denote by $W(\alpha)$ a topological Hausdorff space, the elements of which are all the ordinal numbers β such that $\beta \leq \alpha$, and the topology is induced by intervals of the form (β_1, β_2). Assume now that $\alpha < \omega_1$. Then

1) $W(\alpha)$ is a compactum, since from an arbitrary infinite set in $W(\alpha)$ we can select a monotonically increasing sequence, and an arbitrary monotonically increasing sequence in $W(\alpha)$ is convergent.

2) The compactum $W(\alpha)$ is homeomorphic to some compactum I_α, where $I_\alpha \subset [a_0, b_0]$.

Actually, the compactum $W(\alpha)$ is of measure zero since it consists of at most a denumerable number of points and is therefore homeomorphic to some closed subset of the Cantor perfect set (see, for example, [4]) and, hence, also of an interval.

To complete the proof of the lemma it remains to show that for any $\alpha < \omega_1$, we can find a γ such that $\alpha < \gamma < \omega_1$ and $W(\gamma) \times \Delta^k$ is not homeomorphic to any $W(\beta) \times \Delta^k$ for $\beta \leq \alpha$. Let $\gamma = \alpha \cdot \omega$; then if $W^{(\nu)}(\alpha) \neq \phi$ [$W^{(\nu)}(\alpha)$ is the ν-th derivative of the space $W(\alpha)$], then $\gamma \in W^{(\nu+1)}(\gamma)$. Consequently, if $W(\alpha) \times \Delta^k$ has exactly ν nonempty derivatives, then $W(\alpha) \times \Delta^k$ has at least $\nu + 1$ nonempty derivatives. Hence no subspace of the compactum $W(\alpha) \times \Delta^k$ is homeomorphic to $W(\gamma) \times \Delta^k$. Thus the lemma is proved.

Let $\Delta^{n-1} = [a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]$ and $\tilde{\Delta}^{n-1} = [a_2, b_2] \times \ldots \times [a_n, b_n]$ be n-dimensional cubes, where $a_i \in E$, $b_i \in E$, $a_i \neq b_i$ for $i = 1, \ldots, n$. We consider the set

$$T^n = (x_1^0 \times \tilde{\Delta}^{n-1}) \cup (\Delta^{n-1} \times x_n^0), \tag{1}$$

where $x_1^0 \in (a_1, b_1)$, $x_n^0 \in [a_n, b_n]$. It is not hard to see that T^n is a continuum (a connected compact set).

LEMMA 2. In E^n there exists an at most denumerable set of pairwise nonintersecting continua of the form (1).

In the case $n = 2$ this lemma is the well-known statement that we can locate on a plane an at most denumerable set of pairwise nonintersecting continua having the form of the letter "T."

3°. Let $f: E^n \to E^n$, where $f(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$ and the f_i are strictly monotonic functions from $C(E)$ $(i = 1, \ldots, n)$. Then if $M = M_1 \times \ldots \times M_n$, where $M_i \subset E (i = 1, \ldots, n)$, it follows that $f(M) = f_1(M_1) \times \ldots \times f_n(M_n)$, moreover, if $M_i = [a_i, b_i] \subset E$, then $f_i(M_i) = [f(a_i), f(b_i)]$. From this it follows that if T^n is a set of the form (1), then $f(T^n)$ is also a set of the form (1).

4°. We proceed, finally, to the basic formulation. Let $A = \{x = (x_1, \ldots, x_n)\colon\ x \in E^n \,\&\, x_1 = \ldots = x_n = x\}$. Further, let $\Delta^n = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \operatorname{int}(E^n \setminus A)$, where $a_i \in E$, $b_i \in E$, $a_i \neq b_i$ $(i = 1, \ldots, n)$. Let

$$\Delta^{n-2} = [a_2, b_2] \times \ldots \times [a_{n-1}, b_{n-1}],$$
$$\Delta^{n-1} = [a_1, b_1] \times \Delta^{n-2}, \quad \tilde{\Delta}^{n-1} = \Delta^{n-2} \times [a_n, b_n].$$

Using Lemma 1, we construct a nondenumerable set of compacta $\{I_\alpha\}$ such that $I_\alpha \subset [a_1, b_1]$ and, if $\alpha_1 \neq \alpha_2$, then $I_{\alpha_1} \times \Delta^{n-2}$ is not homeomorphic to $I_{\alpha_2} \times \Delta^{n-2}$. We introduce the notation: $B_\alpha = (I_\alpha \times \tilde{\Delta}^{n-1}) \cup (\Delta^{n-1} \times x_n^0)$, where $x_n^0 \in [a_n, b_n]$. It is obvious that the B_α are continua; $B_\alpha \subset \Delta^n \subset \operatorname{int}(E^n \setminus A)$ and, if $\alpha_1 \neq \alpha_2$, then B_{α_1} is not homeomorphic to B_{α_2} since the set of branching points* of B_α coincides with $I_\alpha \times \Delta^{n-2}$. We now define functions ψ_α, belonging to $C(E^n)$, in the following way:

$$\psi_\alpha(x) = \frac{\rho(A, x)}{\rho(A, x) + \rho(B_\alpha, x)}$$

(where ρ represents distance in E^n).

*We say that x is a branching point of the set B_α if $x \in B_\alpha$ and, for any closed neighborhood V of x, the intersection $V \cap B_\alpha$ is always nonhomeomorphic to E^{n-1} and B_α is locally connected at the point x.

It is clear that $A = \{x\colon x \in E^n \,\&\, \psi_\alpha(x) = 0\}$ is the level set of the function ψ_α corresponding to zero. $B_\alpha = \{x\colon x \in E^n \,\&\, \psi_\alpha(x) = 1\}$ is the level set corresponding to one. Let $\Psi = \{\psi_\alpha\}$ be a nondenumerable set of functions from $C(E^n)$.

Assume now that $\varphi \in C(E^n)$ and $\psi_\alpha \in S_\varphi \cap \Psi$, then $\psi_\alpha(x_1, \ldots, x_n) \equiv g_\alpha[\varphi(f_1^\alpha(x_1), \ldots, f_n^\alpha(x_n))]$, where $g_\alpha \in C(E)$, $f_i^\alpha \in C(E)$ $(i = 1, \ldots, n)$.

Let us suppose that some one of the functions f_i^α $(1 \le i \le n)$ is not strictly monotonic on E; assume, for definiteness, that it is f_1^α. Then there exist $x', x'' \in E$, such that $x' \ne x''$ and $f_1^\alpha(x') = f_1^\alpha(x'')$. If we let $f^\alpha = (f_1^\alpha, \ldots, f_n^\alpha)$, then $f^\alpha(x', x', \ldots, x') = f^\alpha(x'', x', \ldots, x')$. Consequently, $\psi_\alpha(x', x', \ldots, x') = \psi_\alpha(x'', x', \ldots, x')$. But $(x', x', \ldots, x') \in A$ and $(x'', x', \ldots, x') \in E^n \setminus A$. We have come to a contradiction with the fact that A is a level set of the function ψ_α. Thus, f_i^α is strictly monotonic on E for $i = 1, \ldots, n$. Consequently, f^α maps E^n homeomorphically onto $f^\alpha(E^n)$.

We prove that the continuum $f^\alpha(B_\alpha)$ is a connected component of some level set of the function φ.

1) It is obvious that $f^\alpha(B_\alpha) \subset \operatorname{int}(f^\alpha(E^n))$. It is therefore sufficient to show that $f^\alpha(B_\alpha)$ is, in fact, a level set of the function $\tilde{\varphi} = \varphi|_{f^\alpha(E^n)}$.

2) Let M_α be a level set of the function $\tilde{\varphi}$ such that $f^\alpha(B_\alpha) \cap M_\alpha \ne \phi$. Let us suppose that M_α does not appear in $f^\alpha(B_\alpha)$. Then there exist points $x \in B_\alpha$, $\bar{x} \in E^n \setminus B_\alpha$, such that $f^\alpha(x) \in M_\alpha$ and $f^\alpha(\bar{x}) \in M_\alpha$, consequently, $\varphi(f^\alpha(x)) = \varphi(f^\alpha(\bar{x}))$, and, hence, also $\psi_\alpha(x) = \psi_\alpha(\bar{x})$. But the last equation contradicts the fact that B_α is a level set of the function ψ_α. Thus $M_\alpha \subseteq f^\alpha(B_\alpha)$.

3) If the closed set M_α appears in $f^\alpha(B_\alpha)$, then $f^\alpha(E^n) \setminus M_\alpha$ is connected. Actually, this follows from the fact that $E^n \setminus B_\alpha$ is connected, B_α is a set nowhere dense in E^n, and f^α is a homeomorphism.

It follows from 2) and 3) that if M_α is a level set of the function $\tilde{\varphi}$ such that $M_\alpha \cap f^\alpha(B_\alpha) \ne \phi$, then either

$$\varphi(M_\alpha) = \max_{x \in f^\alpha(E^n)} \varphi(x), \quad \text{or} \quad \varphi(M_\alpha) = \min_{x \in f^\alpha(E^n)} \varphi(x).$$

That is, the continuous function φ assumes on the continuum $f^\alpha(B_\alpha)$ at most two values, consequently, it assumes exactly one value. Using 2) and 1), we find that $f^\alpha(B_\alpha)$ is a connected component of some level set of the function φ. Therefore, if $\alpha_1 \ne \alpha_2$ and $\psi_{\alpha_1}, \psi_{\alpha_2} \in S_\varphi \cap \Psi$, then either $f^{\alpha_1}(B_{\alpha_1}) \cap f^{\alpha_2}(B_{\alpha_2}) = \phi$, or $f^{\alpha_1}(B_{\alpha_1}) = f^{\alpha_2}(B_{\alpha_2})$. But B_{α_1} is not homeomorphic to B_{α_2}, while f^{α_1} and f^{α_2} are homeomorphic, consequently, $f^{\alpha_1}(B_{\alpha_1}) \cap f^{\alpha_2}(B_{\alpha_2}) = \phi$.

We note also that each B_α contains a continuum of the form (1). Then if $\psi_\alpha \in S_\varphi \cap \Psi$, then (see 3°) $f^\alpha(B_\alpha)$ also contains a continuum of the form (1). Therefore, using Lemma 2, we find that the set $S_\varphi \cap \Psi$ is at most denumerable. This completes the proof of the theorem.

Remark. From a corresponding theorem of Whitney [5] it follows that for an arbitrary closed set B, lying in E^n, there exists a function $\eta_B(x)$, nonnegative and infinitely differentiable in E^n, such that B is the level set of the function $\eta_B(x)$ corresponding to zero.

If in the definition of the function $\psi_\alpha(x)$ we replace $\rho(A, x)$ by $\eta_A(x)$ and $\rho(B_\alpha, x)$ by $\eta_{B_\alpha}(x)$, then the proof goes through without any changes and we may assume that all the functions of the set Ψ are infinitely differentiable.

COROLLARY 1. Assume again that $\{\varphi_i\}_{i=1}^{+\infty} = \Phi \subset C(E^n)$, then for an arbitrary function $F(x)$ from $C(E^n)$ and an arbitrary neighborhood of the function F in $C(E^n)$, it follows that the intersection of this neighborhood with the set $C(E^n) \setminus S_\Phi$ is nondenumerable, i.e., an arbitrary element of the space $C(E^n)$ is a condensation point of the set $C(E^n) \setminus S_\Phi$.

Proof. Suppose that $F \in C(E^n)$, $\varepsilon > 0$. For arbitrary $\delta > 0$ we denote $[0, \delta]$ by E_δ, and we choose a $\delta_0 > 0$, such that if $x \in E_{\delta_0}^n$, then $|F(x) - F(0)| < \varepsilon/2$.

For any x from $E_{\delta_0}^n$ we now define the function $\psi_\alpha^{F,\varepsilon}(x)$ as follows: $\psi_\alpha^{F,\varepsilon}(x) = F(0) + \frac{\varepsilon}{2}\psi_\alpha\left(\frac{x}{\delta_0}\right)$. It is then obvious that $\max_{x \in E_{\delta_0}^n} |F(x) - \psi_\alpha^{F,\varepsilon}(x)| < \varepsilon$. We extend the function $\psi_\alpha^{F,\varepsilon}$ onto the whole of E^n, so that the last inequality is maintained.

Let $\Psi^{F,\varepsilon} = \{\psi_\alpha^{F,\varepsilon}\}$. then the set $S_\Phi \cap \Psi^{F,\varepsilon}$ is at most denumerable. We can prove this fact in the same way we proved the theorem, the only difference being that, here and there, instead of f_i^α we need to consider $f_i^\alpha |_{E^n_{\delta_0}}$, and instead of $\varphi |_{f^\alpha(E^n)}$ to consider $\varphi |_{f^\alpha(E^n_{\delta_0})}$. The Corollary 2 now follows immediately from the non-denumerability of the set $\Psi^{F,\varepsilon}$.

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