Sub-Riemannian Objects: From Geodesics to Minimal Surfaces

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Outline

Sub-Riemannian Settings

* What is a sub-Riemannian manifold?
  Let M be a smooth manifold of dimension m endowed with a smooth distribution Δ(called horizontal bundle) of dimension k with $k < m$. Let Δ(H-bundle) equipped with an inner product g_c(called sub-Riemannian metric).

Sub-Riemannian Settings

* We call the triple (M, Δ, g_c) a sub-Riemannian(次黎曼) manifold with the sub-Riemannian structure (Δ, g_c).
* A sub-Riemannian structure on a manifold is a generalization of a Riemannian structure in that a metric is only defined on a **proper** vector sub-bundle of the tangent bundle to the manifold, rather than on the whole tangent bundle.
* As a result, the metric associated is **degenerate** on the whole tangent bundle.

Sub-Riemannian Settings

* A piecewise smooth curve $\gamma(t), t \in [a, b]$ in M is horizontal if $\gamma'(t) \in \Delta_{\gamma(t)}$ a.e. $t \in [a, b]$.
* We can define **distance** between two points just as in Riemannian case, except we are only allowed **to travel along the horizontal curves** between two points.

Sub-Riemannian Settings

* Let $\{X_1, ..., X_k\}$ be an orthonormal basis of Δ. If, at each point of M, $X_1, ..., X_k$ and all of their commutators span the tangent space of M at this point, we call Δ satisfies the **Hörmander's condition or Chow's condition**.

Sub-Riemannian Settings

* If M is connected and Δ satisfies the Chow's condition, the Chow Connectivity theorem asserts that there exists at least one piecewise smooth horizontal curve connecting two given points $\{p, q\}$ and thus (Δ, g_c) yields a metric d_c (called **Carnot-Caratheodory metric, or C-C metric**) as the infimum among the lengths of all horizontal curves joining p to q.

Examples of sub-Riemannain manifolds

* **An example of C-C spaces**
* **A Carnot group** is a connected Lie group G whose Lie algebra $\mathcal{G}$ is graded and r-nilpotent. That is, $V_1 = span\{X_1, ..., X_k\}, \mathcal{G} = V_1 \oplus V_2 \oplus ... \oplus V_r$, $[V_1, V_i] = V_{i+1} fori = 1, ..., r-1$, $[V_1, V_r] = \{0\}$.

Examples of sub-Riemannain manifolds

* **The topological dimension of** G:

$$rankLie\{X_1, ..., X_k\}$$

$$= dimG = \sum_{i=1}^{r} dimV_i.$$

* **The Hausdorff dimension** Q **of** G w.r.t. a sub-Riemannian metric:

$$\sum_{i=1}^{r} idimV_i.$$

Examples of sub-Riemannain manifolds

* In particular, **the Heisenberg group** H^n is the simplest Carnot group of step 2. The Heisenberg group H^n may be realized as $R^{2n} \times R = \{q = (z, t)\}(z = (x, y))$ with the group law:

$$(z, t) \circ (z', t') = (z + z', t + t' + (yx' - xy')).$$

* The vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t},$$

are left invariant and form a sub-tangent bundle(or a horizontal bundle) $\Delta = span\{X, Y\}$ in H^1.

* H^1 is a Lie group defined by a Lie algebra $\mathfrak{g}$ generated by three vector fields, $\{X, Y, T\}$ with one nontrivial bracket relation, namely $[X, Y] = T, T = -2\frac{\partial}{\partial t}$.

Examples of sub-Riemannain manifolds

* Actually the Heisenberg group H^n is the simplest, also most important, model of sub-Riemannian manifolds. Roughly speaking, Carnot groups, in particular the Heisenberg group H^n, play a role in sub-Riemannian manifolds as Euclidean spaces R^n do in Riemannian manifolds.

Backgrounds

* Generalization of Riemannian manifolds(limiting cases, singularities,...)—Sub-Riemannian manifolds;
* Geometric control systems;
* Nonholonomic mechanics;
* CR geometry, Contact structures;
* Hypeoellipitic operators (sub-elliptic equations);
* Image processing such as the non-commutative geometric analysis approach to clinical MRI, inpainting;
* ......

Mathematical Views

- sub-Riemannian structures may produce some phenomena which do not exist in Riemannian cases:

* a) The Hausdorff dimension is larger than the manifold dimension;
* b) The conjugate locus of a point contains that point. That is to say, the exponential map is never a local diffeomorphism in a neighborhood of the point at which it is based;

Mathematical Views

* c) There exist **singular geodesics**. That is, there are sub-Riemannian manifolds which admit minimizing geodesics that do not solve the geodesic equations as defined by the Hamiltonian H w.r.t. their h-bundles.

Mathematical Views

* d)There are some interesting facts:

–Balls in Carnot groups is not isoperimetric sets;

–Balls are only Lipschitz;

–Lack of symmetrization properties;

–The Brunn-Minkowski inequality fails;

–Bernstein‘s Theorem is not true;

–......

Shortest and Straightest

♣Sub-Riemannian geodesics(shortest curves)
A sub-Riemannian geodesic is a minimizer which is an absolutely continuous horizontal curve γ and is such that, for each t, there exists $\epsilon > 0$ so that γ minimizes the length between $\gamma(t_0)$ and $\gamma(t_1)$ whenever t_0, t_1 are in $(t - \epsilon, t + \epsilon)$.
♣ Nonholonomic geodesics(straightest curves)

Nonholonomic geodesics

* Nonholonomic geodesics are objects in the free nonholonomic mechanical problem in M. That is, the equations of motion of the nonholonomic free particle are given by the "geodesic" equations $D_{\dot\gamma}\dot\gamma(t) = 0$.
* An example of a mechanical system modeled by a nonholonomic Engle system is that of a coin rolling without slipping on the Euclidean plane.

Sub-Riemannian Geodesics

* Sub-Riemannian (Minimizing) Geodesics
* A minimizing geodesic $\gamma : [a, b] \to M$ in a sub-Riemannian manifold (M, Δ, g_c) is a shortest curve among the horizontal curves connecting $\gamma(a)$ and $\gamma(b)$.
* Problem A: Find out a minimizer of the following energy functional

$$E(\gamma) = \frac{1}{2} \int_a^b g_c(\gamma\prime, \gamma\prime) dt$$

under the constraints
(a) $\gamma\prime(t) \in \Delta$, a.e. $t \in [a, b]$;
(b) $\gamma\prime \in L^2[a, b]$ and $\gamma(a) = q_0, \gamma(b) = q_1$.

Normal and Abnormal

* Normal and Abnormal Geodesics
* The Hamiltonian
  For the basis $\{X_1, \cdots, X_k\}$ of Δ, define the matrix-valued function $g_{ij}(q) =< X_i(q), X_j(q) >, q \in M$, and let g^{ij} be the inverse matrix. Think of the X_i as fiber-linear functions on cotangent bundle:

$$X_i(q, p) = p(X_i(q)), q \in M, p \in T_q^* M.$$

The Hamiltonian H is defined as

$$H = \frac{1}{2} \sum g^{ij} X_i X_j,$$

which is a fiber-quadratic positive semi-definite form $T^* M \to R$ whose rank is k.

Normal and Abnormal

* Normal curves and Geodesics
  The projections of solutions of a Hamiltonian system, which is in a sense the Legendre transform of the inner product on Δ.
* A normal geodesic $\gamma(t)$: there exists a $(\gamma(t), p(t)) \in T^*M$,which satisfies the following Hamiltonian equations：

$$\dot{x}(t) = \frac{\partial H}{\partial p}, \ \dot{p}(t) = -\frac{\partial H}{\partial x}.$$

Normal and Abnormal

* Abnormal curves and Geodesics

- An abnormal geodesic $\gamma(t)$: $\gamma(t)$ is a minimizing curve, which is not the solution of Hamiltonian equations, that is, there is no $(\gamma(t), p(t)) \in T^*M$ satisfying the Hamiltonian equations.
- Abnormal geodesics depend only on the distribution D,and not on the metric g_c.

Normal and Abnormal

* Abnormal curves and Geodesics

Definition An abnormal curve is a horizontal curve which is the projection onto M of an absolutely continuous curve in the annihilator $\Delta^{\perp} \subset T^{*}M$ of Δ, with square integrable derivative, which does not intersect the zero section and whose derivative, whenever it exists, is in the kernel of the the canonical two form restricted to $\Delta^{\perp}$.

Normal and Abnormal

* Equivalent Definitions
  Let x be a curve in M and ζ be an absolutely continuous curve in T^*M which does not intersect the zero section and whose derivative is square integrable. Suppose that $x = \pi(\zeta)$ and x and ζ satisfy the above definition. Then, the following are equivalent to the above definition:
* x is an abnormal extremal in the sense of the Pontryagin maximum principal of control theory;
* ζ annihilates the image of the differential $d(end(x(t)))$ at each t, where *end* is the map associating to a curve its endpoint;
* x is horizontal, $\zeta \in \Delta^{\perp}$ and $\zeta(t) = (D\Phi_t^T)^{-1}\zeta(0)$ where Φ_t is any time dependent flow which generates the curve x.

Nonexistence

Assume that G is a Carnot group of dimension n and $D = \{e_1, e_2\}$ is a bracket generating left invariant distribution of G. Set $s_i(q) = \mathtt{dim} V_i(q)$. The integer list of $(s_1(q), ..., s_r(q))$ is called the type of Δ at the point q. We have the following theorem.

Theorem(Huang-Y). If the type of D is $(2, 1, \cdots, 1)$ or $(2, 1, \cdots, 1, 2)$, then there does not exist any strictly abnormal minimizer in G.

Corollary. If G is a Carnot group and $\dim G \leq 5$, there does not exist any strictly abnormal minimizers in G.

Existence

Theorem(Huang-Y). Assume G is a n-dimension Carnot group whose the rank two-distribution $D = \text{span}\{e_1, e_2\}$ satisfies the following bracket generating condition:

$$[e_1, e_2] = e_3, \cdots, [e_i, e_j] = e_{j+1}, \cdots, [e_1, e_m] = e_{m+1},$$

$$[e_2, e_m] = e_{m+2}, \quad [e_1, e_{m+1}] = e_{m+3}, \quad [e_k, e_l] = \alpha_{l_k} e_{l_k}.$$

where $i = 1$ or 2; $2 < j < m$; $k = 1, 2$; $l \geq m + 2$; and α_{l_k} are constants, the remains Lie brackets are zero, then there exist strictly abnormal extremals on G.

Existence

* **Remark 1.** When $n \geq 6$, we can construct a n-dimensional Carnot group G such that there exists strictly abnormal minimizers in G. For instance, the type of D is $(2, 1, \cdots, 1, 2, n_0, \cdots, n_a)$, where $n_0 \geq 1, a \in \mathbb{N}, n_i \geq 0$ for $i = 1, \cdots, a$. satisfies the conditions of the above Theorem.
* **Remark 2.** The proofs of above theorems are based on equations of normal and abnormal extremals.

Smoothness

- Since normal geodesics always differentiable, the question ”are geodesics always differentiable” reduces to ”are singular geodesics always differentiable”. Hamenstädt suggested to try to prove that singular geodesics is normal in a submanifold of M, then it imply that singular geodesics are differentiable.

* **Theorem(Tan K-H.-Y).** Let G be a Carnot Group, D is a bracket generating distribution, if step≤ 3, then all sub-Riemannian geodesics are smooth in G.

Smoothness

- We say that a rank 3 distribution D satisfies condition (B_2) if it satisfies the following bracket generating condition

$$[e_i, e_{2k}] = \alpha_{i(2k)(2k+2)} e_{2k+2}, \ \ [e_j, e_{2k+1}] = \alpha_{j(2k+1)(2k+3)} e_{2k+3},$$

for $k = 1, 2, 3, \cdots$, and $i, j = 1, 2, 3$, where $\alpha_{i(2k)(2k+2)}, \alpha_{j(2k+1)(2k+3)}$ are constants.

* **Theorem(Huang T-R.-Y).** Let G be a connected $(2m+1)$-dimensional Lie group and $D = span\{e_1, e_2, e_3\}$ is a bracket generating distribution which satisfies Condition(B_2), then all sub-Riemannian geodesics are smooth in G.

H-Harmonic Functions

- Let $(z,t)=(x,y,t)\in R^{2n+1}, x,y\in R^n$.
- Solutions to the following Kohn-Laplacian equation are called **H-harmonic functions**

$$-\Delta_H u = \Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial}{\partial t}(Pu) = 0,$$

where $\Delta_z = \sum_{i=1}^n (\frac{\partial^2}{\partial^2 x_i} + \frac{\partial^2}{\partial^2 y_i})$ and $Pu = \sum_{i=1}^n (y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i})$.

H-Harmonic Functions

- In another form

$$-\Delta_H u = div(A(z)\nabla),$$

where

$$A(z) = \begin{pmatrix} I_n & 0_n & (2y)^T \\ 0_n & I_n & (-2x)^T \\ 2y & -2x & 4|z|^2 \end{pmatrix}.$$

H-Harmonic Functions

- $\Delta_H u$ is degenerate.
- The operators Δ_H are hypoelliptic from Hormander's hypoellipticity theorem.
- The classical sub-elliptic theory due to G.B. Folland, L.P. Rothschild, E.M. Stein,...

H-Harmonic Functions

- The horizontal gradient of a function u

$$\nabla_{H^n} u = Xu = (X_1 u, \cdots, X_n u, X_{n+1} u, \cdots, X_{2n} u).$$

- The symmetrized second horizontal derivative matrix $(X^2 u)^*$

$$(X^2 u)^* = \frac{1}{2}(X_i X_j u + X_j X_i u).$$

H-Harmonic Functions

- For $1 < p < \infty$, we consider

$$-\Delta_H^p u = -X_i(|Xu|^{p-2} X_i u) = 0$$

This is the Euler equation of the sub-elliptic $p-$Dirichlet energy functional

$$\frac{1}{p}\int |Xu|^p.$$

- For $p = 2$, the Kohn-Laplace

$$-\Delta_H u = -\sum_{i=1}^{2n} X_i^2 u = 0.$$

We call the solutions to the equation H-Harmonic functions.

Nodal and singular sets

- The nodal set of a function u is defined by

$$N(u) = \{x : u(x) = 0\}.$$

- The singular set of a harmonic function u is defined by

$$S(u) = \{x : u(x) = 0, Du(x) = 0\}.$$

Nodal and singular sets

- To control growth of solutions $\rightleftarrows$ To control the geometry and topology of their level sets.
- Measure estimates of nodal sets.
- Geometric structure of singular sets.

Nodal and singular sets

- In 1979, Almgren first gave the definition of frequency for harmonic functions.
- in 1986, Garofalo and Lin established the monotonicity formula for frequencies and the doubling conditions for solutions of a class of uniformly elliptic linear PDEs.
- Main contributions: F-H. Lin, H.Donnelly, C.Fefferman, Q. Han, R. Hardt, ,.....

Frequency function and growth of H-harmonic functions

Definition

For every $r \in (0, R_0)$, let

$$H(r) = \int_{\partial B_r} u^2 \frac{\psi}{|\nabla \rho|} dH^{N-1},$$

and

$$D(r) = \int_{B_r} |\nabla_{\mathcal{H}} u|^2 dzdt.$$

The generalized Almgren's frequency of u on B_r is defined by

$$N(r) = \begin{cases} \frac{rD(r)}{H(r)}, & \text{if } H \neq 0, \\ 0, & \text{if } H = 0. \end{cases}$$

Measure Estimates

Theorem

(Tian-Y)*Suppose that u is a nontrivial $H-$harmonic function in $B_d(0,1)$ and $Pu = 0$. Then there exists positive constant $\widetilde{r} < 1$ depending only on Q, C_1 and g such that*

$$\mathcal{H}^{2n}\{p \in B_d(0,\widetilde{r}) : \quad u(p) = 0\} \leq CN(0, r_0) + C.$$

Here r_0 is as before and C is a positive constant depending on Q, C_1 and g, and

$$P = \sum_{j=1}^{n}(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}).$$

Sketch of Proof

Step1. We claim that

$$N(p, r) \leq CN(0, r_0) + C$$

for $r < cr_0$, where C and c are positive constants depending on Q, C_1 and g. This claim comes from Lemma of Changing Centers, the monotonicity formula of frequency and the doubling conditions.

Step 2. We first assume that

$$\fint_{B_d(0,r_0)} u^2 \psi dzdt = 1.$$

Under this assumption, The doubling condition implies that one can find p_j on the axis, and $p_j \in \partial B_d(0, \frac{r_0}{4})$, $j = 1, 2, \cdots, 2n + 1$,

$$\fint_{B_d(p_j, \frac{r_0}{16})} u^2 \psi dzdt \geq 4^{-CN(0,r_0)-C},$$

Finally one can show that there exists $\widetilde{p_j} \in B_d(p_j, \frac{r_0}{16})$ such that

$$|u(\widetilde{p_j})| \geq 2^{-CN(0,r_0)-C}.$$

Step 3. Define $f_j(\omega;\xi) = u(\widetilde{p_j} + \xi\omega)$ for ξ belongs to suitable interval and ω be any unit vector of $\mathbb{R}^{2n+1}$. Then f_j are all analytic with respect to ξ. Then we do the complexification of f_j. By using a theorem of H.Donnelly-C.Fefferman, we can have

$$\mathcal{H}^0\left\{|\xi| < \frac{5r_0}{8} : \quad u(\widetilde{p_j} + \xi\omega) = 0\right\} \leq CN(0, r_0) + C.$$

Step 4. From the integral geometric formula, the desired result can be derived.

Definition of Horizontal Singular Sets

Definition

Let u be a smooth function from H^n to $\mathbb{R}$. The horizontal singular set of u is defined as

$$\mathcal{S}(u) = \left\{ x \in H^n : u(x) = 0, \sum_{i=1}^{2n} |X_i u|^2 = 0 \right\}.$$

We also denote

$$\mathcal{S}_k(u) = \left\{ x \in H^n : X^\alpha u(x) = 0, \forall \alpha \in \bigcup_{m=0}^{k-1} \mathcal{I}_m, \exists \alpha_0 \in \mathcal{I}_k, X^{\alpha_0} u(x) \neq 0 \right\},$$

where

$$X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{2n}^{\alpha_{2n}} X_{2n+1}^{\alpha_{2n+1}},$$

$$\mathcal{I}_m = \left\{ \alpha = (\alpha_1, \cdots, \alpha_{2n}, \alpha_{2n+1}) : \quad \sum_{i=1}^{2n} \alpha_i + 2\alpha_{2n+1} = m \right\}$$

, and call it the $k-$horizontal singular set of u.

H–homogeneous polynomial and Horizontal Singular Set

Lemma

Let P be a nontrivial H–homogeneous polynomial of degree k. Then
(1)$\mathcal{S}_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and all points on t–axis are in $\mathcal{S}_k(P)$.
or
(2)$\mathcal{S}_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and t–axis is orthogonal to $\mathcal{S}_k(P)$. Moreover, in this case, the dimension of $\mathcal{S}_k(P)$ is at most n.

Sketch of Proof

We first prove the following five properties of $\mathcal{S}_k(P)$:
(1) $0 \in \mathcal{S}_k(P)$.
(2) $(z,t) \in \mathcal{S}_k(P)$ implies $\delta_\lambda((z,t)) \in \mathcal{S}_k(P)$, $\forall \lambda > 0$.
(3) $(z_1,t_1),(z_2,t_2) \in \mathcal{S}_k(P)$ implies $(z_1,t_1)\circ(z_2,t_2) \in \mathcal{S}_k(P)$.
(4) If $(z,t) \in \mathcal{S}_k(P)$, then $(-z,t) \in \mathcal{S}_k(P)$.
(5) If $(z,t) \in \mathcal{S}_k(P)$ for some $t > 0 (t < 0)$, then all points $(0,t)$ satisfying $t > 0 (t < 0)$ are in $\mathcal{S}_k(P)$. Moreover, the polynomial P is independent of t in this case.
Then by using this five properties we can get the desired result.

Geometric Structure

Theorem

(Tian-Y) *Let u be a nontrivial H–harmonic function in* $B_d(0,1)$*. Then the horizontal singular set in* $B_d(0,\frac{1}{2})$ *is a countable union of* C^1 *sub-manifolds in* $\mathbb{R}^{2n+1}$ *with dimension at most* $2n-1$*. Thus the horizontal singular set of u is at most* $(2n-1)$*–countably rectifiable.*

Sketch of Proof

Step 1. We first write $\mathcal{S}(u)$ as

$$\mathcal{S}(u) = \bigcup_{k\geq 2} \mathcal{S}_k(u).$$

Because u is a non-trivial $\mathbb{H}$–harmonic function on $\mathbb{H}^n$ and has the strong unique continuity property, that is a finite union.

Step 2. Do the Taylor extension of u at point z for $z \in \mathcal{S}_k(u)$. Then

$$u(z \circ p) = P_z(p) + \mathcal{O}(d^{k+1}(z^{-1} \circ p)).$$

Let

$$\overline{\mathcal{S}}^j_k(u) = \{z \in \mathcal{S}_k(u) : dim \mathcal{S}_k(P_z) = j, P_z \text{ is independent of } t\},$$

$$\widetilde{\mathcal{S}}^j_k(u) = \{z \in \mathcal{S}_k(u) : dim \mathcal{S}_k(P_z) = j, P_z \text{ depends on } t\},$$

$$\mathcal{S}^j_k(u) = \overline{\mathcal{S}}^j_k(u) \bigcup \widetilde{\mathcal{S}}^j_k(u)$$

Then we discuss $\overline{\mathcal{S}}^j_k(u)$ and $\widetilde{\mathcal{S}}^j_k(u)$ separately. First consider the case $z_0 \in \widetilde{\mathcal{S}}^j_k(u)$. Let $\{z_m\} \in \widetilde{\mathcal{S}}^j_k(u)$, $z_m \longrightarrow z_0$ as $m \longrightarrow \infty$. Through the same argument as in Euclidean case, we have

$$\lim_{m \longrightarrow \infty} Angle < line_{z_m, z_0}, \mathcal{S}_k(P_{z_0}) >= 0.$$

Step 3. Then consider the case $z_m, z_0 \in \overline{\mathcal{S}}_k^j(u)$. Through some similar but more complex argument, we can obtain the same result, i.e.,

$$\lim_{m \longrightarrow \infty} Angle < line_{z_m, z_0}, \mathcal{S}_k(P_{z_0}) >= 0.$$

Step 4. From the above result, we can say that $\overline{\mathcal{S}}^j_k(u)$ and $\widetilde{\mathcal{S}}^j_k(u)$ both are countable union of j–dimensional C^1–manifolds. Then

$$\mathcal{S}^j(u) = \bigcup_{k \geq 2} \mathcal{S}^j_k(u),$$

is a countable union of j–dimensional C^1–manifolds for $j = 0, 1, 2, \cdots, 2n - 1$. That is the result we need.

The Plateau Problem

* **Plateau Problem in** H^1
  Given a closed curve $\gamma \in H^1$, can one find a topologically two dimensional surface $\Sigma \subset H^1$ spanning γ which minimizes **an appropriate surface measure**?
* What is an appropriate surface measure in H^1?
* What is a minimal surface in H^1 with respect to the c-c metric?

The surface measures

♣ One natural measure for the generic two dimensional surface in H^1 is the three dimensional Hausdorff measure associated to d_c as every C^1 surface which has topological dimension two has Hausdorff dimension three w.r.t. d_c.(Gromov(96))

♣ (Pansu(89), Heinnonen(95), Franchi-Serapioni-Serra Cassano(99)) If Σ is a C^1 smooth surface in H^1 bounding an open set O, then,

$$\mathcal{H}^3_{cc}(\Sigma) = \mathcal{P}(O) = \int_\Sigma \frac{|N_0|}{|N_E|} d\mathcal{H}^2_E,$$

where N_0 is the horizontal normal, N_E is the Euclidean normal(identifying H^1 with R^3) and $\mathcal{H}^2_E$ is the 2-dimensional Hausdorff measure in R^3.

The surface energy

* If Σ is the graph of $u : \Omega \subset R^2 \to R$ and $u \in C^1(\Omega)$, then

$$\mathcal{P}(u) := \mathcal{H}^3_{cc}(\Sigma) = \int_\Omega |N_0| dxdy.$$

* Let $F(x, y, t) = u(x, y) - t$, then $|N_0| = ((XF)^2 + (YF)^2)^{1/2}$. The energy functional is defined by

$$E(u) = \int_\Omega |\nabla_H F| dxdy$$

for $u \in W^{1,p}$, where ∇_H denotes the horizontal gradient operator defined by (XF, YF). We note that $E(u) = \mathcal{P}(u)$ if $u \in C^1(\Omega)$.

The minimal surface equations: analytic version

* For above energy functional, the associated Euler-Lagrange equation is

$$-\nabla_H \cdot \frac{\nabla_H F}{|\nabla_H F|} = 0.$$

We call it the horizontal minimal surface equation ((HMSE) for brief). But what is the quantity of the left hand side of the above equation?

What is the quantity $H_{cc}(u)$?

♣ Define

$$H_{cc}(u) = \nabla_H \cdot \frac{\nabla_H F}{|\nabla_H F|} = div_H(\frac{\nabla_H F}{|\nabla_H F|}).$$

♣ What is $H_{cc}(u)$?

♣ Is there a counterpart of curvature description for minimal surfaces in sub-Riemannian manifolds?

Nonholonomic Connections and Horizontal Mean Curvatures

* **Nonholonomic Connections and Horizontal Mean Curvatures(Tan-Y)**
  A horizontal connection $D : \Gamma(\Delta) \times \Gamma(\Delta) \longrightarrow \Gamma(\Delta)$.
  A projection connection onto $\Gamma(\Delta)$ of a Levi-Civita connection induced by an extension g (a Riemannian metric on TM) of g_c.
* The nonholonomic connection D
  $\Longrightarrow$ the shape operator A
  $\Longrightarrow$ the eigenvalues of A
  $\Longrightarrow$ the horizontal mean curvature H.

The minimal surface equations: geometric version

* If Σ is the graph of $u : \Omega \subset R^2 \to R$, its horizontal mean curvature H is exactly $H_{cc}(u)$. That is, solutions of the horizontal minimal surface equation correspond to the graph with ”least” surface measures and the horizontal mean curvature equal to zero at each point on the graph.

The minimal surface equations

* More precisely, in nonparametric case, the horizontal minimal surface equation ((HMSE) is equivalent to the following PDE:

$$(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0,$$

  which is a degenerate (hyperbolic and elliptic) nonlinear PDE.
* S.D. Pauls: an existence theorem; continuous piecewise C^1 ”low regularity”.
* J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang: singular sets; a Bernstein-type theorem; a uniqueness theorem.
* N. Garofalo, S.D. Pauls: a Bernstein-type theorem; seed curves.

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A Bäcklund type transformation

Introduce the following operators:

$$\Box u = (u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy},$$
$$\bar{\Box}\varphi = \varphi_y^2 \varphi_{xx} - 2\varphi_x \varphi_y \varphi_{xy} + \varphi_x^2 \varphi_{yy}.$$

Theorem(Y-). There is a Bäcklund type transformation for the horizontal minimal surface equation

$$u = c_0 \varphi^{-1} + u_1$$

where u, φ and u_1 satisfy

$$\Box u = 0, \quad \Box u_1 = 0, \quad \bar{\Box}\varphi = 0$$

and

$$\frac{\varphi_x}{\varphi_y} = \frac{u_{1x} - y}{u_{1y} + x}.$$

Bateman's equation

- It is interesting to note that

$$\overline{\Box}\varphi = 0$$

is so called Bateman's equation which first appeared in a 1929 paper on hydrodynamics.

Theorem.(1) The Bateman equation is invariant under not only Euclidean transformations on the co-ordinates x, t but also under the full general linear group $GL(2, R)$. In addition the invariant property is true under replacement of any solution φ by any arbitrary twice differentiable function $F(\varphi)$.
(2) Any smooth, real-valued function homogeneous of degree one in the derivatives φ_x and φ_t, and with arbitrary dependence on φ, works as a Lagrangian for the Bateman equation.

Solutions to HMSEs

Corollary. The solutions of

$$x^2\varphi_{xx} + 2xy\varphi_{xy} + y^2\varphi_{yy} = 0, \tag{1}$$

$$x\varphi_x + y\varphi_y = 0 \tag{2}$$

are the solutions of $\Box u = 0$.
We remark that (1) and (2) have infinite number of solutions, for instance, $\varphi(x,y) = F(\frac{y}{x})$ satisfies (1) and (2) for arbitrary twice differential F.

Solutions to HMSEs

* Introducing

$$w(x,y) = \frac{u_y + x}{u_x - y},$$

we observe that the H-minimal surface equation turns out to be

$$\frac{\partial w}{\partial y} = w\frac{\partial w}{\partial x}. \tag{3}$$

Since (3) obviously implies

$$\frac{\partial}{\partial y}(w^k) = \frac{\partial}{\partial x}(\frac{k}{k+1}w^{k+1}).$$

We known that (3) has an infinite number of conservation laws and the general solution in an implicit form is as follows:

$$w = F(x + yw), \text{for arbitrary} F.$$

Solutions to HMSEs

Theorem(Y-). (1)If, for an arbitrary F, u is a solution of

$$\frac{u_y + x}{u_x - y} = F\left(\frac{xu_x + yu_y}{u_x - y}\right),$$

then u is also a solution of the H-minimal surface equation.
(2)The H-minimal surface equation has the Lagrangian of the form

$$\mathcal{L} = \sqrt{(u_x - y)^2 + (u_y + x)^2}.$$

That is, the H-minimal surface equation can be expressed as a conservation law:

$$\partial_x\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \partial_y\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) = 0.$$

Solutions to HMSEs

* If we apply the following Legendre transform to $\Box u = 0$:

$$\xi = u_y, \eta = u_x,$$

$$\omega(\xi, \eta) = y\xi + x\eta - u,$$

we can check that

Theorem(Y-). The H-minimal surface equation is invariant under the Legendre transform, that is,

$$\Box \omega = 0.$$

Thank you!